Inequalities for General Chord-Integrals of Radial Blaschke-Minkowski Homomorphisms

Juan Zhang, Weidong Wang*

Abstract—**In this paper, we establish some inequalities for general chord-integrals of radial Blaschke-Minkowski homomorphisms. As the applications, some inequalities for chord-integrals of intersection bodies are obtained.**

Index Terms—**general chord-integral, radial Blaschke-Minkowski homomorphism, Brunn-Minkowski type inequality, intersection body.**

I. Introduction

LET $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbf{R}^n$. S^{n-1} denotes the unit sphere and $V(M)$ denotes the n-dimensional volume of the body M. For the standard unit ball B, its volume is written as $V(B)=\omega_n$.

If M is a compact star shaped (with respect to the origin) in $\mathbf{R}^n$, then its radial function, $\rho_M=\rho(M,\cdot):\mathbf{R}^n\setminus\{0\}\to[0,\infty)$, is defined by (see [10])

$$\rho(M,x)=\max\{\lambda\geq 0:\lambda x\in M\},\quad x\in\mathbf{R}^n\setminus\{0\}.$$

If ρ_M is positive and continuous, M will be called a star body (respect to the origin). Two star bodies M and N are said to be dilates (of one another) if $\rho_M(u)/\rho_N(u)$ is independent of $u\in S^{n-1}$.

For $M,N\in\mathcal{S}_o^n$ and $\lambda,\mu\geq 0$ (not both zero), the radial Minkowski combination, $\lambda M\tilde{+}\mu N$, of M and N is given by (see [11])

$$\rho(\lambda M\tilde{+}\mu N,\cdot)=\lambda\rho(M,\cdot)+\mu\rho(N,\cdot),\tag{1.1}$$

where λM denotes the radial Minkowski scalar multiplication.

For $M,N\in\mathcal{S}_o^n$ and $\lambda,\mu\geq 0$ (not both zero), the radial Blaschke combination, $\lambda\circ M\check{+}\mu\circ N$, of M and N is given by (see [11])

$$\rho(\lambda\circ M\check{+}\mu\circ N,\cdot)^{n-1}=\lambda\rho(M,\cdot)^{n-1}+\mu\rho(N,\cdot)^{n-1},\tag{1.2}$$

where $\check{+}$ denotes the radial Blaschke addition, $\lambda\circ M$ denotes the radial Blaschke scalar multiplication and $\lambda\circ M=\lambda^{\frac{1}{n-1}}M$. When $\lambda=\mu=1$, $M\check{+}N$ is called radial Blaschke sum.

Lutwak [20] introduced the notion of mixed width-integrals of convex bodies in 1977. Motivated by Lutwak's ideas, the notion of mixed chord-integrals of star bodies was recently defined by Lu [18]: For $M_1,\ldots,M_n\in\mathcal{S}_o^n$, the mixed chord-integral, $C(M_1,\ldots,M_n)$, is defined by

$$C(M_1,\ldots,M_n)=\frac{1}{n}\int_{S^{n-1}}c(M_1,u)\ldots c(M_n,u)dS(u),$$

where $dS(u)$ is the $(n-1)$-dimensional volume element on S^{n-1} and $c(M,u)$ denotes the half chord of M in the direction u, namely, $c(M,u)=\frac{1}{2}\rho(M,u)+\frac{1}{2}\rho(M,-u)$. Thus, the mixed chord-integral is a map $\underbrace{\mathcal{S}_o^n\times\ldots\times\mathcal{S}_o^n}_{n}\to\mathbf{R}$. It is positive, continuous and multilinear with respect to radial Minkowski combinations, positively homogeneous and monotone under set inclusion. Star bodies $M_1,\ldots,M_n$ are said to have a similar chord if there exist constants $\lambda_1,\ldots,\lambda_n>0$ such that $\lambda_1 c(M_1,u)=\ldots=\lambda_n c(M_n,u)$ for all $u\in S^{n-1}$.

In 2016, Feng and Wang (see [8]) generalized the definition of mixed chord-integrals to the general mixed chord-integrals of star bodies: For $M_1,\ldots,M_n\in\mathcal{S}_o^n$ and $\tau\in(-1,1)$, the general mixed chord-integral, $C^{(\tau)}(M_1,\ldots,M_n)$, is defined by

$$C^{(\tau)}(M_1,\ldots,M_n)=\frac{1}{n}\int_{S^{n-1}}c^{(\tau)}(M_1,u)\ldots c^{(\tau)}(M_n,u)dS(u),\tag{1.3}$$

where

$$c^{(\tau)}(M,u)=f_1(\tau)\rho(M,u)+f_2(\tau)\rho(M,-u),\tag{1.4}$$

for all $u\in S^{n-1}$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are given by

$$f_1(\tau)=\frac{(1+\tau)^2}{2(1+\tau^2)},\quad f_2(\tau)=\frac{(1-\tau)^2}{2(1+\tau^2)}.\tag{1.5}$$

Clearly, $f_1(\tau)+f_2(\tau)=1$.

Together with (1.5), the case $\tau=0$ in (1.3) is just Lu's mixed chord-integral $C(M_1,\ldots,M_n)$. Similarly, the general mixed chord-integral, $C^{(\tau)}(M_1,\ldots,M_n):\mathcal{S}_o^n\times\ldots\times\mathcal{S}_o^n\to\mathbf{R}$, is also positive, continuous and multilinear with respect to radial Minkowski combinations, positively homogeneous and monotone under set inclusion. Star bodies $M_1,\ldots,M_n$ are said to have a similar general chord if there exist constants $\lambda_1,\ldots,\lambda_n>0$ such that $\lambda_1 c^{(\tau)}(M_1,u)=\ldots=\lambda_n c^{(\tau)}(M_n,u)$ for all $u\in S^{n-1}$. They are said to have a joint constant general chord if the product $c^{(\tau)}(M_1,u)\ldots c^{(\tau)}(M_n,u)$ is constant for all $u\in S^{n-1}$.

If $M_1=\ldots=M_{n-i}=M$ and $M_{n-i+1}=\ldots=M_n=B$ in (1.3), and allow i is any real number and notice that $c^{(\tau)}(B,\cdot)=1$, then the general chord-integral of index i, $C_i^{(\tau)}(M)$, of $M\in\mathcal{S}_o^n$ is given by

$$C_i^{(\tau)}(M)=\frac{1}{n}\int_{S^{n-1}}c^{(\tau)}(M,u)^{n-i}dS(u).\tag{1.6}$$

Obviously, $C_i^{(\tau)}(B)=\omega_n$, and when $i=n$ in (1.6), we have

$$C_n^{(\tau)}(M)=\frac{1}{n}\int_{S^{n-1}}dS(u)=\omega_n.\tag{1.7}$$

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Juan Zhang is with the Department of Mathematics, China Three Gorges University, Yichang, 443002, China, e-mail: juanzhang892x@163.com.

*Weidong Wang is corresponding author with the Department of Mathematics and Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, 443002, China, e-mail: wangwd722@163.com.

Besides, Feng and Wang (see [8]) established the following result:

Theorem 1.A *If $M \in \mathcal{S}_o^n$ and $\tau \in (-1, 1)$, then*

$$C^{(\tau)}(M) \leq V(M),$$

with equality if and only if M is centered at the origin.

For $M \in \mathcal{S}_o^n$, the intersection body, IM, of M is a star body symmetric with respect to origin whose radial function on S^{n-1} is given by (see [19]):

$$\rho(IM, u) = v_{n-1}(M \cap u^{\perp}),$$

for all $u \in S^{n-1}$. Here v_{n-1} is $(n-1)$-dimensional volume and $M \cap u^{\perp}$ denotes the intersection of K with the subspace $u^{\perp}$ that passes through the origin and orthogonal to u. Regarding the investigations of intersection bodies, we refer to [21], [28], [29], [31].

Based on the properties of intersection bodies, Schuster [24] introduced the notion of radial Blaschke-Minkowski homomorphisms as follows:

A map Ψ: $\mathcal{S}_o^n \to \mathcal{S}_o^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Ψ is continuous.

(b) For all $M, N \in \mathcal{S}_o^n$,

$$\Psi(M \check{+} N) = \Psi M \tilde{+} \Psi N. \tag{1.8}$$

(c) For all $M \in \mathcal{S}_o^n$ and every $\vartheta \in SO(n)$, $\Psi(\vartheta M) = \vartheta \Psi M$.

Here, $SO(n)$ is the group of rotations in n dimensions. $\Psi M \tilde{+} \Psi N$ denotes the radial Minkowski addition of ΨM and ΨN (see (1.1)), $M \check{+} N$ denotes the radial Blaschke addition of star bodies M and N (see (1.2)).

The radial Blaschke-Minkowski homomorphism is a special real valued valuation in Brunn-Minkowski theory, many results of radial Blaschke-Minkowski homomorphisms can be seen in these articles, see [6], [9], [30], [33]. A systematic study was initiated by Blaschke in the 1930s and continued by Hadwiger, culminating in his famous classification of continuous and rigid motion invariant valuations on convex bodies. The book (see [16]) is excellent source for the classical theory of valuations. For some of the more recent results, see [1], [2], [3], [4], [12], [13], [14], [15], [22], [23], [25], [26], [27], [32].

In this paper, associated with the radial Blaschke-Minkowski homomorphisms, we continuously research the general chord-integrals of index i. First, we establish the following Brunn-Minkowski type inequality.

Theorem 1.1 *Let $M, N \in \mathcal{S}_o^n$, $\tau \in (-1, 1)$, $i, j \in \mathbf{R}$ and $i \neq j$. If $i \leq n-1 \leq j \leq n$, then*

$$\left(\frac{C_i^{(\tau)}(\Psi(M \check{+} N))}{C_j^{(\tau)}(\Psi(M \check{+} N))}\right)^{\frac{1}{j-i}} \leq \left(\frac{C_i^{(\tau)}(\Psi M)}{C_j^{(\tau)}(\Psi M)}\right)^{\frac{1}{j-i}} + \left(\frac{C_i^{(\tau)}(\Psi N)}{C_j^{(\tau)}(\Psi N)}\right)^{\frac{1}{j-i}}; \tag{1.9}$$

if $n-1 \leq i \leq n \leq j$, then inequality (1.9) is reversed. Equality holds in (1.9) if and only if ΨM and ΨN have similar general chord. For $i = n-1$ and $j = n$, (1.9) is identical.

Let $j = n$ in (1.9), and combining with (1.7), we have the following fact:

Corollary 1.1 *If M, $N \in \mathcal{S}_o^n$, i is any real and $\tau \in (-1, 1)$, then for $i < n-1$,*

$$C_i^{(\tau)}(\Psi(M \check{+} N))^{\frac{1}{n-i}} \leq C_i^{(\tau)}(\Psi M)^{\frac{1}{n-i}} + C_i^{(\tau)}(\Psi N)^{\frac{1}{n-i}}, \tag{1.10}$$

with equality if and only if ΨM and ΨN have similar general chord; for $n-1 < i < n$, inequality (1.10) is reversed. For $i = n-1$, (1.10) is identical.

Because of the intersection body is a special example of the radial Blaschke-Minkowski homomorphisms, from Corollary 1.1 we obtain the following result:

Corollary 1.2 *Let M, $N \in \mathcal{S}_o^n$, i is any real and $\tau \in (-1, 1)$, then for $i < n-1$,*

$$C_i^{(\tau)}(I(M \check{+} N))^{\frac{1}{n-i}} \leq C_i^{(\tau)}(IM)^{\frac{1}{n-i}} + C_i^{(\tau)}(IN)^{\frac{1}{n-i}};$$

this inequality is reversed for $n-1 < i < n$. With equality if and only if IM and IN have similar general chord. For $i = n-1$, this inequality is an identity.

Since IM is origin-symmetric, by the equality condition of Theorem 1.A, then $C^{(\tau)}(IM) = V(IM)$. Let $i = 0$ in Corollary 1.2, we have that

Corollary 1.3 *If M, $N \in \mathcal{S}_o^n$, then*

$$V(I(M \check{+} N))^{\frac{1}{n}} \leq V(IM)^{\frac{1}{n}} + V(IN)^{\frac{1}{n}},$$

with equality if and only if IM and IN are homothetic.

Then, we obtain another form of the Brunn-Minkowski type inequality for general chord-integrals of index i.

Theorem 1.2 *If $M \in \mathcal{S}_o^n$ and N is a ball in $\mathbf{R}^n$, $\tau \in (-1, 1)$, then for all $i = 0, \ldots, n-1$,*

$$\frac{C_i^{(\tau)}(\Psi(M \check{+} N))}{C_{i+1}^{(\tau)}(\Psi(M \check{+} N))} \leq \frac{C_i^{(\tau)}(\Psi M)}{C_{i+1}^{(\tau)}(\Psi M)} + \frac{C_i^{(\tau)}(\Psi N)}{C_{i+1}^{(\tau)}(\Psi N)}. \tag{1.11}$$

Finally, as an application of Corollary 1.1 and its equality condition, we give an analogous version of the volume differences inequality, which is related to the radial Blaschke-Minkowski homomorphisms for the general chord-integrals of index i.

Theorem 1.3 *Let M, N, D, $D' \in \mathcal{S}_o^n$, $\tau \in (-1, 1)$, $i \in \mathbf{R}$ and $\Psi D \subset \Psi M$, $\Psi D' \subset \Psi N$, ΨM and ΨN have similar general chord. If $i < n-1$, then*

$$\begin{aligned}(C_i^{(\tau)}(\Psi(M \check{+} N)) - C_i^{(\tau)}(\Psi(D \check{+} D')))^{\frac{1}{n-i}} \\ \geq (C_i^{(\tau)}(\Psi M) - C_i^{(\tau)}(\Psi D))^{\frac{1}{n-i}} \\ + (C_i^{(\tau)}(\Psi N) - C_i^{(\tau)}(\Psi D'))^{\frac{1}{n-i}};\end{aligned} \tag{1.12}$$

if $n-1 < i < n$, inequality (1.12) is reversed. Equality holds in (1.12) if and only if ΨD and $\Psi D'$ have similar general chord and there exists a constant λ, such that $(C_i^{(\tau)}(\Psi M), C_i^{(\tau)}(\Psi D)) = \lambda(C_i^{(\tau)}(\Psi N), C_i^{(\tau)}(\Psi D'))$. For $i = n-1$, (1.12) is identical.

II. Results and Proofs

In this section, we will give the proofs of Theorems 1.1-1.3. First, in order to prove Theorem 1.1, the following lemmas are required.

Lemma 2.1 (The Beckenbach-Dresher inequality [5], [7]) *Let function $f, g \geq 0$, $\mathbf{E}$ is a bounded measurable subset in $\mathbf{R}^n$ and ϕ is a distribution function. If $p \geq 1 \geq r \geq 0$, then*

$$\left(\frac{\int_{\mathbf{E}}(f+g)^p d\phi}{\int_{\mathbf{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbf{E}} f^p d\phi}{\int_{\mathbf{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbf{E}} g^p d\phi}{\int_{\mathbf{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{2.1}$$

with equality if and only if the functions f and g are positively proportional. For $p = 1$ and $r = 0$, (2.1) is an identity.

Lemma 2.2 (The inverse Beckenbach-Dresher inequality [17]) *Let function $f, g \geq 0$, $\mathbf{E}$ is a bounded measurable subset in $\mathbf{R}^n$ and ϕ is a distribution function. If $1 \geq p \geq 0 \geq r$, then*

$$\left(\frac{\int_{\mathbf{E}}(f+g)^p d\phi}{\int_{\mathbf{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \geq \left(\frac{\int_{\mathbf{E}} f^p d\phi}{\int_{\mathbf{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbf{E}} g^p d\phi}{\int_{\mathbf{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{2.2}$$

with equality if and only if the functions f and g are positively proportional. For $p = 1$ and $r = 0$, (2.2) is an identity.

Proof of Theorem 1.1 Since $i \leq n-1 \leq j \leq n$ and $i \neq j$, let $p = n-i$, $r = n-j$, then $0 \leq r \leq 1 \leq p$ and $p \neq r$. For $M, N \in \mathcal{S}_o^n$, any $u \in S^{n-1}$ and $\tau \in (-1,1)$, combining with (1.6) and (1.8), we get

$$\begin{aligned} & C_{n-p}^{(\tau)}(\Psi(M \check{+} N)) \\ &= \frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(\Psi M \tilde{+} \Psi N, u)^p dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} (c^{(\tau)}(\Psi M, u) + c^{(\tau)}(\Psi N, u))^p dS(u). \end{aligned} \tag{2.3}$$

Similarly to the proof of (2.3),

$$\begin{aligned} & C_{n-r}^{(\tau)}(\Psi(M \check{+} N)) \\ &= \frac{1}{n}\int_{S^{n-1}} (c^{(\tau)}(\Psi M, u) + c^{(\tau)}(\Psi N, u))^r dS(u). \end{aligned} \tag{2.4}$$

It follows from Lemma 2.1, (2.3) and (2.4), we know that

$$\begin{aligned} & \left(\frac{C_{n-p}^{(\tau)}(\Psi(M \check{+} N))}{C_{n-r}^{(\tau)}(\Psi(M \check{+} N))}\right)^{\frac{1}{p-r}} \\ &= \left(\frac{\int_{S^{n-1}}(c^{(\tau)}(\Psi M, u) + c^{(\tau)}(\Psi N, u))^p dS(u)}{\int_{S^{n-1}}(c^{(\tau)}(\Psi M, u) + c^{(\tau)}(\Psi N, u))^r dS(u)}\right)^{\frac{1}{p-r}} \\ &\leq \left(\frac{\int_{S^{n-1}} c^{(\tau)}(\Psi M, u)^p dS(u)}{\int_{S^{n-1}} c^{(\tau)}(\Psi M, u)^r dS(u)}\right)^{\frac{1}{p-r}} \\ &\quad + \left(\frac{\int_{S^{n-1}} c^{(\tau)}(\Psi N, u)^p dS(u)}{\int_{S^{n-1}} c^{(\tau)}(\Psi N, u)^r dS(u)}\right)^{\frac{1}{p-r}} \\ &= \left(\frac{C_{n-p}^{(\tau)}(\Psi M)}{C_{n-r}^{(\tau)}(\Psi M)}\right)^{\frac{1}{p-r}} + \left(\frac{C_{n-p}^{(\tau)}(\Psi N)}{C_{n-r}^{(\tau)}(\Psi N)}\right)^{\frac{1}{p-r}}. \end{aligned} \tag{2.5}$$

Let $i = n-p$ and $j = n-r$ in (2.5), then inequality (1.9) is given.

Similar to the above method, for $n-1 \leq i \leq n \leq j$, the inverse of (1.9) follows from Lemma 2.2, (2.3) and (2.4).

The equality condition of inequality (2.1) imply that equality holds in (1.9) if and only if $c^{(\tau)}(\Psi M, u)$ and $c^{(\tau)}(\Psi N, u)$ are positively proportional, i.e., ΨM and ΨN have similar general chord.

For $i = n-1$ and $j = n$, by (1.7), we get (1.9) is identical.

Lemma 2.3 ([8]) *If $M \in \mathcal{S}_o^n$, $\tau \in (-1,1)$, $j = 0, 1, \ldots, n$ and $\mu > 0$, then*

$$C_j^{(\tau)}(M_\mu) = \sum_{i=0}^{n-j} \begin{pmatrix} n-j \\ i \end{pmatrix} c_{j+i}^{(\tau)}(M)\mu^i,$$

where $M_\mu = M \tilde{+} \mu B$.

Lemma 2.4 ([8]) *If $M_1, \ldots, M_n \in \mathcal{S}_o^n$, $\tau \in (-1,1)$ and $1 < m \leq n$, then*

$$\begin{aligned} & C^{(\tau)}(M_1, \ldots, M_n)^m \\ &\leq \prod_{i=1}^{m} C^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-i+1}), \end{aligned}$$

with equality if and only if $M_{n-m+1}, \ldots, M_n$ are all of similar general chord.

Proof of Theorem 1.2 For $M \in \mathcal{S}_o^n$, $\tau \in (-1,1)$ and $i = 0, 1, \ldots, n-2$, let

$$f_i(\mu) = C_i^{(\tau)}(\Psi M \tilde{+} \mu B), \quad \mu > 0. \tag{2.6}$$

From Lemma 2.3, it follows that

$$\begin{aligned} & f_i(\mu + \varepsilon) \\ &= C_i^{(\tau)}(\Psi M \tilde{+} \mu B \tilde{+} \varepsilon B) \\ &= C_i^{(\tau)}(\Psi M \tilde{+} \mu B) + \varepsilon(n-i) C_{i+1}^{(\tau)}(\Psi M \tilde{+} \mu B) + o(\varepsilon^2) \\ &= f_i(\mu) + \varepsilon(n-i) f_{i+1}(\mu) + o(\varepsilon^2). \end{aligned}$$

Thus,

$$f_i'(\mu) = (n-i) f_{i+1}(\mu).$$

By Lemma 2.4, for $i = 0, 1, \ldots, n-2$, we have

$$C_{i+1}^{(\tau)}(\Psi M \tilde{+} \mu B)^2 \leq C_i^{(\tau)}(\Psi M \tilde{+} \mu B) C_{i+2}^{(\tau)}(\Psi M \tilde{+} \mu B),$$

i.e.,

$$f_{i+1}(\mu)^2 \leq f_i(\mu) f_{i+2}(\mu).$$

Now, we define

$$F_i(\mu) = \frac{f_i(\mu)}{f_{i+1}(\mu)}, \quad i = 0, 1, \ldots, n-2. \tag{2.7}$$

This implies

$$\begin{aligned} & F_i'(\mu) \\ &= \frac{f_i'(\mu) f_{i+1}(\mu) - f_i(\mu) f_{i+1}'(\mu)}{f_{i+1}(\mu)^2} \\ &= \frac{f_{i+1}(\mu)^2 + (n-i-1)(f_{i+1}(\mu)^2 - f_i(\mu) f_{i+2}(\mu))}{f_{i+1}(\mu)^2} \\ &\leq 1. \end{aligned}$$

Thus for $\lambda > 0$, we obtain

$$\int_0^\lambda F_i'(\mu)d\mu \leq \int_0^\lambda 1 d\mu,$$

i.e.,

$$F_i(\lambda) \leq F_i(0) + \lambda. \tag{2.8}$$

From (2.6), (2.7) and (2.8), for $i = 0, 1, \ldots, n-2$, we have

$$\frac{C_i^{(\tau)}(\Psi M \tilde{+} \lambda B)}{C_{i+1}^{(\tau)}(\Psi M \tilde{+} \lambda B)} \leq \frac{C_i^{(\tau)}(\Psi M)}{C_{i+1}^{(\tau)}(\Psi M)} + \lambda. \tag{2.9}$$

But for the standard unit ball B, we know $\Psi B = B$ (see page 21 of [24]). Hence, if N is a ball in $\mathbf{R}^n$, then ΨN is also a ball. From this, let $\Psi N = \lambda B$, and combine with (1.4) and (1.6), we obtain

$$\begin{aligned}\frac{C_i^{(\tau)}(\Psi N)}{C_{i+1}^{(\tau)}(\Psi N)} &= \frac{C_i^{(\tau)}(\lambda B)}{C_{i+1}^{(\tau)}(\lambda B)} \\ &= \frac{\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(\lambda B, u)^{n-i} dS(u)}{\frac{1}{n}\int_{S^{n-1}} c^{(\tau)}(\lambda B, u)^{n-i-1} dS(u)} \\ &= \frac{\lambda^{n-i}}{\lambda^{n-i-1}} \\ &= \lambda.\end{aligned}$$

This together with (1.8) and (2.9), we get

$$\frac{C_i^{(\tau)}(\Psi(M \check{+} N))}{C_{i+1}^{(\tau)}(\Psi(M \check{+} N))} \leq \frac{C_i^{(\tau)}(\Psi M)}{C_{i+1}^{(\tau)}(\Psi M)} + \frac{C_i^{(\tau)}(\Psi N)}{C_{i+1}^{(\tau)}(\Psi N)}.$$

When $i = n-1$, inequality (1.11) always holds as an equality. This is just inequality (1.11).

According to Theorem 1.2, we may obtain a related inequality for intersection bodies.

Corollary 2.1 *If $M \in \mathcal{S}_o^n$ and N is a ball in $\mathbf{R}^n$, $\tau \in (-1, 1)$, then for all $i = 0, \ldots, n-1$,*

$$\frac{C_i^{(\tau)}(I(M \check{+} N))}{C_{i+1}^{(\tau)}(I(M \check{+} N))} \leq \frac{C_i^{(\tau)}(IM)}{C_{i+1}^{(\tau)}(IM)} + \frac{C_i^{(\tau)}(IN)}{C_{i+1}^{(\tau)}(IN)}.$$

Lemma 2.5 (Bellman's inequality [5]) *Let $\mathbf{a} = \{a_1, \ldots, a_n\}$ and $\mathbf{b} = \{b_1, \ldots, b_n\}$ be two series of positive real numbers. If $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^p - \sum_{i=2}^n b_i^p > 0$, then for $p > 1$,*

$$\begin{aligned}&\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \\ &\leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{\frac{1}{p}}.\end{aligned}$$

This inequality is reversed for $p < 0$ or $0 < p < 1$, with equality if and only if $\mathbf{a} = c\mathbf{b}$, where c is a constant.

Proof of Theorem 1.3 For M, N, D, $D' \in \mathcal{S}_o^n$ and $\tau \in (-1, 1)$, if $i < n-1$, using inequality (1.10), we get

$$\begin{aligned}&C_i^{(\tau)}(\Psi(D \check{+} D'))^{\frac{1}{n-i}} \\ &\leq C_i^{(\tau)}(\Psi D)^{\frac{1}{n-i}} + C_i^{(\tau)}(\Psi D')^{\frac{1}{n-i}},\end{aligned} \tag{2.10}$$

with equality if and only if ΨD and $\Psi D'$ have similar general chord. Since ΨM and ΨN have similar general chord, thus according to the equality condition of inequality (1.10), we have

$$\begin{aligned}&C_i^{(\tau)}(\Psi(M \check{+} N))^{\frac{1}{n-i}} \\ &= C_i^{(\tau)}(\Psi M)^{\frac{1}{n-i}} + C_i^{(\tau)}(\Psi N)^{\frac{1}{n-i}}.\end{aligned} \tag{2.11}$$

But $\Psi D \subset \Psi M$, $\Psi D' \subset \Psi N$, these and formula (1.6) give

$$\begin{aligned}&C_i^{(\tau)}(\Psi M) > C_i^{(\tau)}(\Psi D), \quad C_i^{(\tau)}(\Psi N) > C_i^{(\tau)}(\Psi D'), \\ &C_i^{(\tau)}(\Psi(M \check{+} N)) \\ &= C_i^{(\tau)}(\Psi M \tilde{+} \Psi N) \\ &> C_i^{(\tau)}(\Psi D \tilde{+} \Psi D') \\ &= C_i^{(\tau)}(\Psi(D \check{+} D')),\end{aligned}$$

from this, notice that $n - i > 1$ ($i < n-1$), and according to (2.10), (2.11) and Lemma 2.5, we obtain

$$\begin{aligned}&(C_i^{(\tau)}(\Psi(M \check{+} N)) - C_i^{(\tau)}(\Psi(D \check{+} D')))^{\frac{1}{n-i}} \\ &\geq \left[\left(C_i^{(\tau)}(\Psi M)^{\frac{1}{n-i}} + C_i^{(\tau)}(\Psi N)^{\frac{1}{n-i}}\right)^{n-i}\right. \\ &\quad \left. - \left(C_i^{(\tau)}(\Psi D)^{\frac{1}{n-i}} + C_i^{(\tau)}(\Psi D')^{\frac{1}{n-i}}\right)^{n-i}\right]^{\frac{1}{n-i}} \\ &\geq \left(C_i^{(\tau)}(\Psi M) - C_i^{(\tau)}(\Psi D)\right)^{\frac{1}{n-i}} \\ &\quad + \left(C_i^{(\tau)}(\Psi N) - C_i^{(\tau)}(\Psi D')\right)^{\frac{1}{n-i}}.\end{aligned}$$

This yields inequality (1.12).

Along the same line, for $n - 1 < i < n$, the inverse of (1.12) follows from Corollary 1.1, (1.6) and Lemma 2.5.

By the equality conditions of inequality (1.10) and Lemma 2.5, we see that equality holds in (1.12) if and only if ΨD and $\Psi D'$ have similar general chord and there exists constant λ such that $(C_i^{(\tau)}(\Psi M), C_i^{(\tau)}(\Psi D)) = \lambda(C_i^{(\tau)}(\Psi N), C_i^{(\tau)}(\Psi D'))$. For $i = n-1$, (1.12) is identical.

Associated with the intersection bodies, we have the following inequalities.

Corollary 2.2 *Let $M, N, D, D' \in \mathcal{S}_o^n$, $\tau \in (-1, 1)$, $i \in \mathbf{R}$ and $D \subset M$, $D' \subset N$, IM and IN have similar general chord. If $i < n-1$, then*

$$\begin{aligned}&(C_i^{(\tau)}(I(M \check{+} N)) - C_i^{(\tau)}(I(D \check{+} D')))^{\frac{1}{n-i}} \\ &\geq (C_i^{(\tau)}(IM) - C_i^{(\tau)}(ID))^{\frac{1}{n-i}} + (C_i^{(\tau)}(IN) - C_i^{(\tau)}(ID'))^{\frac{1}{n-i}};\end{aligned}$$

if $n - 1 < i < n$, this inequality is reversed. With equality if and only if ID and ID' have similar general chord and $(C_i^{(\tau)}(IM), C_i^{(\tau)}(ID)) = \lambda(C_i^{(\tau)}(IN), C_i^{(\tau)}(ID'))$, where λ is a constant. For $i = n-1$, this inequality is identical.

Note that IM is origin-symmetric, by the equality condition of Theorem 1.A, then the case $i = 0$ of Corollary 2.2 yields that

Corollary 2.3 *Let M, N, D and $D' \in \mathcal{S}_o^n$, if $D \subset M$, $D' \subset N$ and M is a homothetic copy of N, then*

$$\begin{aligned}&(V(I(M \check{+} N)) - V(I(D \check{+} D')))^{\frac{1}{n}} \\ &\geq (V(IM) - V(ID))^{\frac{1}{n}} + (V(IN) - V(ID'))^{\frac{1}{n}},\end{aligned}$$

with equality if and only if D and D' are homothetic and $(V(IM), V(ID)) = \lambda(V(IN), V(ID'))$, where λ is a constant.

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