# Strengthened Change Point Detection Model for Weak Mean Difference Data

Qi Zhou[1], Shaoqian Huang[2]

*Abstract*—**The lifetime difference of components in adjacent parallel structures decreases as the number of components belonging to each level of parallel structures increases. To restore the system structure, we must differentiate the components that belong to different levels of parallel structures. Hence, detecting the small lifetime difference in components is extremely important. A strengthened change point detection model (SCPDM) for weak mean difference data (WMDD) is established. The concept of WMDD usually means that the effect of a large variance renders the mean difference nonsignificant in two subsignals of a signal sample. Traditional change point detection models become insensitive and ineffective for WMDD. For WMDD that can be collected repeatedly, we perform two enhanced operations that double the mean difference by using the variance information and subsequently analyze the asymptotic properties of the enhanced data. Then, we propose SCPDM based on the asymptotic results. Finally, we compare SCPDM with two other main change point detection models and verify that SCPDM is superior to other models by simulation analysis.**

*Index Terms*—**single change point detection, weak mean difference, asymptotic analysis, enhanced operations, simulation**

## I. Introduction

THE change point is the location at which a certain variable in a model suddenly changes [1]. The change point often represents a qualitative change for the object of focus. Historically, Page [2], [3] first proposed the study of change points in the field of sample testing. To detect the change points in a signal sample, the following several steps are usually followed. First, an associated cost function [4] is selected to measure the homogeneity of each subsignal. Second, according to whether the number of change points is fixed, a discrete optimization problem is solved to estimate the location of the change point. In different change point detection models, selecting a suitable cost function for a signal sample is the most important step [5]. The first type of change point detection model detects the mean change point. Many classical change point detection models have been proposed for various kinds of signals, and these models can be divided into the following three types. First, for piecewise independent and identically distributed (i.i.d.) signals, the mean shift model was first established for normal random signal samples with a piecewise constant mean and constant variance [3], [6], [7], [8], [9]. Second, certain signals may have mean shifts along with shifts of their variances. For example, mean shift and scale shift models were established for normal random signal samples with piecewise constant means and variances [10], [11]. Finally, the rate shift model has been proposed and studied for Poisson distribution signals with piecewise constant rate parameters [12], [13]. The second type of change point detection model is suitable for signal samples with a linear dependence between the variables and changes happening at certain unknown instances, which are called structural changes [14], [15], [16]. For this situation, several well-known models were established, such as the autoregressive model [1], [17] and multiple regression models[16], [18]. Other commonly used change point detection models include kernel change point detection[19], [20], [21], [22] and the Mahalanobis-type metric [23]. Kernel change point detection can be performed on the high-dimensional mapping of the original signal that is implicitly defined by a kernel function. Certain machine learning techniques may be involved in this kind of method, such as a support vector machine or clustering [24], [25]. In addition, for certain clustering methods, the Mahalanobis-type metric is usually used to replace the cost function in the mean shift model [26], [27], [28].

In addition to the models mentioned above, several classical models based on algorithms have been proposed for inferring change points, and these models mainly include the following four types [10]. The first type is based on the likelihood ratio. Csorgo and Horváth[29] established a change point detection model under the assumption of a multivariate Gaussian distribution. This model is mainly used to analyze the change points for time series data. The second type is based on the Bayes method. A number of researchers have studied this type of model. Kander and Zacks [30] studied the exponential family to establish a change point detection model, while Gardner [31] established a model based on the normal distribution. Later, the model was extended to the large sample distribution theory, multivariate normal distribution and general linear regression field [9], [32], [33]. The third type is based on the maximum likelihood. This kind of model employs the mature large sample theory. For example, Fotopoulos et al. [34], [35] established exact computable expressions, bounds and approximations for certain analysis results. The last type is based on samples. This kind of model focused on the nonparametric method, which has a distribution-free advantage.

Among the research sub-directions of change point detection, signal samples with a mean shift have always been a research hot spot. Hawkins et al. [36] used the sample variance without degrees of freedom to detect the change point. In [37], the maximum likelihood estimation method was utilized to analyze the change point to verify the type of population distribution. Later, prior knowledge was involved in establishing a change point detection model in [38]. In [39], the change point location was determined by analyzing the local information near a point, which involved complex

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Qi Zhou is with the Department of Mathematics, Beijing Jiaotong University, Beijing, 100044 China; e-mail: (17121641@bjtu.edu.cn).
Shaoqian Huang is with the Department of Mathematics, Beijing Forestry University, Beijing, 100083 China; e-mail: (huangshq@bjfu.edu.cn).
Corresponding author: Qi Zhou (phone: 8618811190038).

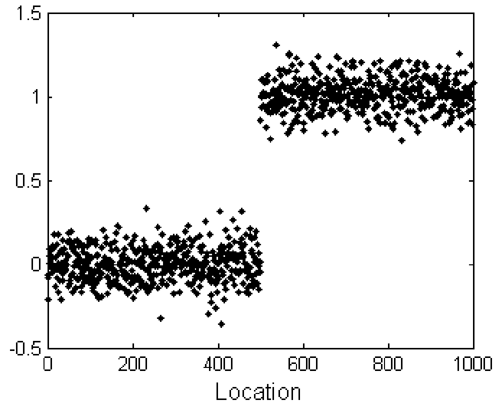
distribution information that usually substituted for certain approximate results. In recent years, some new methods have also been proposed regarding change point detection. As indicated in [40], an optimal algorithm was introduced to determine the location of a change point. In [41], an adapted algorithm was established by the polynomial maximization method. In [42], partition models were set up to test the existence of a mean shift and estimate the location of the change point. In addition, lasso methods were established and improved by many authors [43], [44], [45].

Weak mean difference data (WMDD) are representative of a type of data in which the information of the mean difference is reduced by a large variance. For example, as shown in Fig. (1), when the mean difference significantly exceeds the standard deviation, the location of the change point is easily detected. If the standard deviation is too large to cover the information of the mean difference, then the accuracy of change point detection may be decreased, and the location of the change point can hardly be detected by current models.

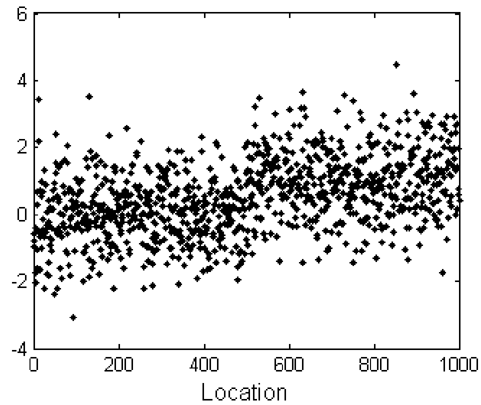
An important example of WMDD originated in the reliability field. Assume that the resistance values of all the components in Fig. (2) are equal. Jin et al. [46] pointed out that components that are tested in a laboratory environment differ in significant ways from those that have experienced operations in fielded systems, as the fielded environment will cause homogeneous components to suffer different degrees of damage. As shown below, the voltages of components belonging to adjacent parallel structures become closer as the number of components belonging to the same level of parallel structure increases, which means that the lifetime difference in components belonging to adjacent parallel structures becomes extremely small. To restore the system structure, we must differentiate the components that belong to different levels of parallel structures. As is known, components in the same level of parallel structures have homogeneous lifetime data. Hence, detecting the weak lifetime difference in components is extremely important to distinguish whether components belong to the same level of parallel structure and helps to establish a topology diagram of the system structure.

The difficulty of detecting change points for WMDD lies in capturing the small differences between subsignals. To solve this problem, we perform two enhanced operations to increase the mean difference between subsignals by utilizing the variance. In addition, we analyze the asymptotic properties of the enhanced data. Next, we propose a strengthened change point detection model (SCPDM) according to the asymptotic results. Finally, we compare the SCPDM with two current models and verify that the SCPDM has a higher efficiency in terms of change point detection for WMDD by simulation analysis.

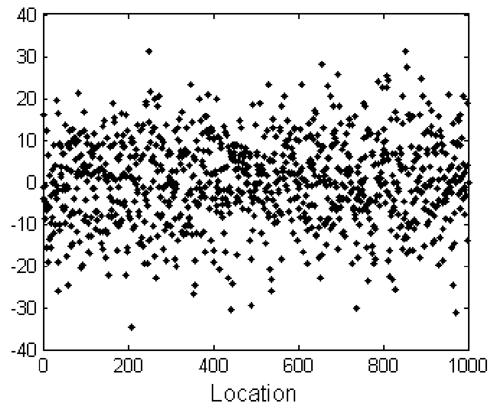
The paper is organized as follows. In II, the two enhanced operations and their asymptotic properties are provided. Based on asymptotic results, SCPDM is proposed. In Section III, using simulation analysis, we verify the correctness of theorem 2 and its remark. In addition, the SCPDM and the other two detection models are compared. The paper concludes with Section IV, which discusses the study findings and future implications.




(a)




(b)




(c)

Fig. 1. Diagram of the difficulty in change point detection that is influenced by various ratios between the variance and mean difference. The figures represent points from two normal distributions, $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$. (a) $\sigma = 0.1, |\mu_1 - \mu_2| = 1$. (b) $\sigma = 1, |\mu_1 - \mu_2| = 1$. (c) $\sigma = 10, |\mu_1 - \mu_2| = 1$.

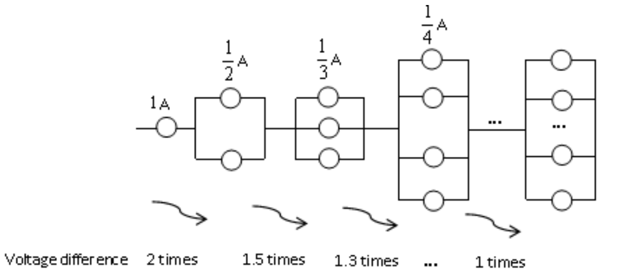



Fig. 2. Variation in the mean difference between components in the adjacent parallel structure. Because different components are subjected to different voltage intensities, the degree of deterioration over the lifetime varies by component; we can reflect the deterioration by the mean. When the resistance of the component is 1, then as the branch increases, the voltage experienced by the component is reduced.

## II. METHODOLOGY

### A. Two enhanced operations

Because the standard deviation is far larger than the mean difference, it is unwise and inefficient to perform change point detection for WMDD using traditional models. In fact, compared with the information of the mean difference, the variance is very remarkable and may supply more information. Therefore, we consider utilizing the variance that characterizes a type of disturbance information to analyze change points by performing the following two enhancements.

Now, we believe that $t^*(t^* \in \{2,3,...,T\})$ is the only abrupt location in sequence $y_1,...,y_T$. For $y_t(t \in \{2,3,...,T\})$, we first conduct an operation called enhanced-I at $t$ and obtain the enhanced-I sequence $y_1^{'},...,y_T^{'}$ in (1).

$$\begin{aligned} &y_1^{'} = y_1, y_i^{'} = min(y_{i-1}, y_i), i = 1,...,t-1, \\ &y_t^{'} = y_t, y_i^{'} = max(y_{i-1}, y_i), i = t+1,...,T. \end{aligned} \tag{1}$$

Then, we conduct a second operation, called enhanced-II at $t$, and obtain the enhanced-II sequence $y_1^{''},...,y_T^{''}$ in (2).

$$\begin{aligned} &y_1^{''} = y_1, y_i^{''} = max(y_{i-1}, y_i), i = 1,...,t-1, \\ &y_t^{''} = y_t, y_i^{''} = min(y_{i-1}, y_i), i = t+1,...,T. \end{aligned} \tag{2}$$

Intuitively, these kinds of enhancements utilize the variance information directly by taking a larger value or a smaller value between adjacent samples. Variance indicates the degree of fluctuation of the data, that is, in a set of data, the data is either larger or smaller according to the size of the variance centered on the mean. Therefore, when the first half of a set of data takes a smaller value (larger value) between adjacent samples, and the second half takes a larger value (smaller value), the mean difference of the strengthened data can well reflect the variance information.

To illustrate this, we generate normal random numbers with different means and variances. Then, under both enhanced operations, the mean difference is calculated separately, as shown in Table. (I). It can be seen that the mean difference has nothing to do with the mean, and the mean difference is close to the variance.

Now, we provide several symbolic explanations. For a signal sample $y_1,...,y_T$, we perform the above two operations at location $t(t \in \{2,3,...,T\})$, where $\mu_1^{'}(t,y^{'})$ indicates the sample mean of $y_1^{'},...,y_{t-1}^{'}$; $\mu_2^{'}(t,y^{'})$ indicates the sample mean of $y_t^{'},...,y_T^{'}$; $\mu_1^{''}(t,y^{''})$ indicates the sample mean of $y_1^{''},...,y_{t-1}^{''}$; and $\mu_2^{''}(t,y^{''})$ indicates the sample mean of $y_t^{''},...,y_T^{''}$. In addition, $\circ$ and $\bullet$ represent operations on the homogeneous signal sample and the signal sample containing a change point, respectively.

### B. Asymptotic property of the enhanced sequence

We consider establishing a certain asymptotic property of the enhanced sequence in the following theorem. First, we point out that there are sufficient data for a signal sample. In other words, we assume that there are sufficient data before and after the location we are examining in a signal sample. When we consider $y_t$, the assumption shows that there are sufficient data before $y_t$. From the perspective of mathematics, it corresponds to $t \to \infty$. Likewise, the existence of sufficient data after $y_t$ corresponds to $T-t \to \infty$. Furthermore, for a signal sample that has one change location at $t^*(t^* \in \{2,...,T\})$, $\mu_1$ indicates the population mean of the first subsignal, and $\mu_2$ indicates the population mean of the second subsignal. $\sigma^2$ indicates the constant population variance. Generally, WMDD indicates that $\frac{|\mu_2-\mu_1|}{\sigma} \leq 1$.

**Theorem 1**. Assume that $y_1,...,y_T$ are i.i.d., i.e., homogeneous with a constant variance $\sigma^2$. The above two operations are performed $n_c$ times independently at location $t(t \in \{2,...,T\})$. Then, we have the following asymptotic property in (3).

$$\begin{aligned} P(\lim_{n_c,t,T-t\to\infty} \frac{1}{n_c}\sum_{i=1}^{n_c}(|\mathring{\mu}_{1i}^{'}(t,y^{'}) \\ -\mathring{\mu}_{2i}^{'}(t,y^{'})| - |\mathring{\mu}_{1i}^{''}(t,y^{''}) - \mathring{\mu}_{2i}^{''}(t,y^{''})|) = 0) = 1 \end{aligned} \tag{3}$$

where $\mathring{\mu}_{1i}^{'}(t,y^{'})$, $\mathring{\mu}_{2i}^{'}(t,y^{'})$, $\mathring{\mu}_{1i}^{''}(t,y^{''})$ and $\mathring{\mu}_{2i}^{''}(t,y^{''})|)$ represent the mean in the $i$th ($i = 1,2,...,n_c$) test sample.

**Proof**. Because $y_1,...,y_T$ are independent and identically distributed, $min(y_1,y_2)$, $min(y_2,y_3)$,...,$min(y_{T-1},y_T)$ are identically distributed. We use $\mu_{min}$ for the mean of $min(y_i,y_j)$. Likewise, we use $\mu_{max}$ for the mean of $max(y_i,y_j)$. Therefore, $y_1^{'},y_2^{'},...,y_{t-1}^{'}$ are identically distributed according to the distribution of $min(y_i,y_j), \forall i \neq j, i,j \in \{1,2,...,T\}$. According to the law of large numbers, we have the following:

$$\mathring{\mu}_{1i}^{'}(t,y^{'}) \to \mu_{min} \qquad a.s. \qquad t\to\infty \tag{4}$$

Likewise, we have the following:

$$\mathring{\mu}_{2i}^{'}(t,y^{'}) \to \mu_{max} \qquad a.s. \qquad T-t\to\infty \tag{5}$$

As a result, we have the following in (6):

$$\begin{aligned} \mathring{\mu}_{1i}^{'}(t,y^{'}) - \mathring{\mu}_{2i}^{'}(t,y^{'}) \to \mu_{min} - \mu_{max} \\ a.s. \qquad t, T-t\to\infty \end{aligned} \tag{6}$$

On the other hand, in (7):

$$\begin{aligned} &\mathring{\mu}_{1i}^{''}(t,y^{''}) \to \mu_{max} \qquad a.s. \qquad t\to\infty \\ &\qquad \mathring{\mu}_{2i}^{''}(t,y^{''}) \to \mu_{min} \qquad a.s. \qquad T-t\to\infty \end{aligned} \tag{7}$$

Consequently, we have the following result in (8):

$$\mathring{\mu}_{1i}^{''}(t,y^{''}) - \mathring{\mu}_{2i}^{''}(t,y^{''}) \to \mu_{max} - \mu_{min} \qquad a.s. \qquad t, T-t \to \infty \tag{8}$$

In addition, $f(x,y) = |x| - |y|$ is a continuous function, and thus $a.s.$ (almost sure) convergence can be preserved under the transformation of this function; therefore, we have the following (9):

$$|\mathring{\mu}_{1i}^{'}(t,y^{'}) - \mathring{\mu}_{2i}^{'}(t,y^{'})| - |\mathring{\mu}_{1i}^{''}(t,y^{''}) - \mathring{\mu}_{2i}^{''}(t,y^{''})| \to |\mu_{min} - \mu_{max}| - |\mu_{max} - \mu_{min}| = 0 \qquad a.s. \qquad t, T-t \to \infty \tag{9}$$

Thus, because the process is independently performed $n_c$ times, $|\mathring{\mu}_{1i}^{'}(t,y^{'}) - \mathring{\mu}_{2i}^{'}(t,y^{'})| - |\mathring{\mu}_{1i}^{''}(t,y^{''}) - \mathring{\mu}_{2i}^{''}(t,y^{''})|(i = 1, ..., n_c)$ can be viewed as independently and identically distributed. According to the law of large numbers, we have the following (10):

$$\frac{1}{n_c}\sum_{i=1}^{n_c}(|\mathring{\mu}_{1i}^{'}(t,y^{'}) - \mathring{\mu}_{1i}^{'}(t,y^{'})| - |\mathring{\mu}_{1i}^{''}(t,y^{''}) - \mathring{\mu}_{1i}^{''}(t,y^{''})|) \to 0 \qquad a.s. \qquad t, T-t \to \infty, n_c \to \infty \tag{10}$$

**Remark 1**. Theorem 1 demonstrates that, when $y_1, ..., y_T$ are homogeneous and the above two operations are performed $n_c$ times independently at any location $t(t \in (\{2, ..., T\}))$, the value of $\frac{1}{n_c}\sum_{i=1}^{n_c}|\mathring{\mu}_{1i}^{''}(t,y^{'}) - \mathring{\mu}_{2i}^{''}(t,y^{'})| - |\mathring{\mu}_{1i}^{''}(t,y^{''} - \mathring{\mu}_{2i}^{''}(t,y^{''})|$ fluctuates at approximately 0.

Next, we will give the asymptotic property when there is only one change point among the signal samples. We will present the asymptotic results about the position of the change point.

**Theorem 2**. Assume that the independent $y_1, ..., y_T$ has only one change point $t^*$ with a constant variance $\sigma^2$. At $t^*$, the above two enhanced operations are independently performed $n_c$ times. Then, we have the following asymptotic property:

(1) if $\frac{|\mu_1 - \mu_2|}{\sigma} > 1$, then we have (11):

$$P(\lim_{n_c, t^*, T-t^* \to \infty}\frac{1}{n_c}\sum_{i=1}^{n_c}(||\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})|| - (|\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})|)) = 0) = 1 \tag{11}$$

The meanings of $\mathring{\mu}_{1i}^{'}(t^*,y^{'}), \mathring{\mu}_{2i}^{'}(t^*,y^{'}), \mathring{\mu}_{1i}^{''}(t^*,y^{''})$ and $\mathring{\mu}_{2i}^{''}(t^*,y^{''})$ are the same as in theorem 2.

(2) if $\frac{|\mu_1 - \mu_2|}{\sigma} \leq 1$, then we have (12):

$$P(\lim_{n_c, t^*, T-t^* \to \infty}\frac{1}{n_c}\sum_{i=1}^{n_c}(||\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})||) = 2|\mu_2 - \mu_1|) = 1 \tag{12}$$

**Proof**. (1) First, we prove the first part. The following formula is clear when $\mu_1 < \mu_2$, in (13)

$$|\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| \to (\mu_2 - \mu_1) \to a.s. \qquad t^*, T-t^* \to \infty \tag{13}$$

Because $\frac{|\mu_1 - \mu_2|}{\sigma} > 1$, an incorrect enhanced operation cannot reverse the direction of the mean difference; therefore, we have the following (14):

$$|\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})| \to (\mu_2 - \mu_1) \to a.s. \qquad t^*, T-t^* \to \infty \tag{14}$$

Therefore,

$$|\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})| - (|\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})|) \to 0 \qquad a.s. \qquad t^*, T-t^* \to \infty \tag{15}$$

Likewise, when $\mu_1 > \mu_2$, we have the following two relationships in (16) and (17):

$$|\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| \to (\mu_1 - \mu_2) \to a.s. \qquad t^*, T-t^* \to \infty \tag{16}$$

$$|\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})| - |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})| \to (\mu_1 - \mu_2) \to a.s. \qquad t^*, T-t^* \to \infty \tag{17}$$

Therefore,

$$|\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})| - |\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - (|\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})|) \to 0 \qquad a.s. \qquad t^*, T-t^* \to \infty \tag{18}$$

In summary, we have the following in (19):

$$||\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})|| - (|\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})|) \to 0 \qquad a.s. \qquad t^*, T-t^* \to \infty \tag{19}$$

Thus, because the process is independently performed $n_c$ times, $||\overset{\bullet}{\mu}_{1i}^{'}(t^*,y^{'}) - \overset{\bullet}{\mu}_{2i}^{'}(t^*,y^{'})| - |\overset{\bullet}{\mu}_{1i}^{''}(t^*,y^{''}) - \overset{\bullet}{\mu}_{2i}^{''}(t^*,y^{''})|| - (|\mathring{\mu}_{1i}^{'}(t^*,y^{'}) - \mathring{\mu}_{2i}^{'}(t^*,y^{'})| + |\mathring{\mu}_{1i}^{''}(t^*,y^{''}) - \mathring{\mu}_{2i}^{''}(t^*,y^{''})|)$ can be viewed as being independent and identically distributed. According to the law of large numbers, we have the following in (20):

$$\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y')-\dot{\mu}'_{2i}(t^*,y')|-|\dot{\mu}''_{1i}(t^*,y'')-\dot{\mu}''_{2i}(t^*,y'')||$$
$$-(|\mathring{\mu}'_{1i}(t^*,y')-\mathring{\mu}'_{2i}(t^*,y')|+|\mathring{\mu}''_{1i}(t^*,y'')-\mathring{\mu}''_{2i}(t^*,y'')|)) \to 0$$
$$a.s. \qquad t^*, T-t^*, n_c \to \infty \quad (20)$$

**Proof**. (2) The following formula in (21) is clear when $\mu_1 < \mu_2$,

$$|\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\mathring{\mu}'_{1i}(t^*,y') - \mathring{\mu}'_{2i}(t^*,y')|$$
$$\to (\mu_2-\mu_1) \to a.s. \qquad t^*, T-t^* \to \infty \quad (21)$$

Due to $\frac{|\mu_2-\mu_1|}{\sigma} \leq 1$, the enhanced operation can reverse the mean difference, and thus we have the following in (22):

$$|\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{2i}(t^*,y'')| - |\mathring{\mu}''_{1i}(t^*,y'') - \mathring{\mu}''_{2i}(t^*,y'')|$$
$$\to -(\mu_2-\mu_1) \to a.s. \qquad t^*, T-t^* \to \infty \quad (22)$$

Likewise, when $\mu_1 > \mu_2$, we have the following two relationships in (23)and (24):

$$|\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\mathring{\mu}'_{1i}(t^*,y') - \mathring{\mu}'_{2i}(t^*,y')|$$
$$\to -(\mu_1-\mu_2) \to a.s. \qquad t^*, T-t^* \to \infty$$
$$(23)$$

$$|\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{2i}(t^*,y'')| - |\mathring{\mu}''_{1i}(t^*,y'') - \mathring{\mu}''_{2i}(t^*,y'')|$$
$$\to (\mu_1-\mu_2) \to a.s. \qquad t^*, T-t^* \to \infty \quad (24)$$

Thus, in (25)

$$||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{2i}(t^*,y'')||$$
$$\to 2|\mu_2-\mu_1| \qquad a.s. \qquad t^*, T-t^* \to \infty \quad (25)$$

Because the process is independently performed $n_c$ times, $||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{2i}(t^*,y'')||(i = 1,...,n_c)$ can be viewed as being independently and identically distributed. Therefore, we have the following in (26):

$$\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y')-\dot{\mu}'_{1i}(t^*,y')|-|\dot{\mu}''_{1i}(t^*,y'')-\dot{\mu}''_{1i}(t^*,y'')||$$
$$\to 2|\mu_2-\mu_1| \qquad a.s. \qquad t, T-t, n_c \to \infty \quad (26)$$

**Remark 2.** For (13), (14), (21) and (22), we provide the four schematic diagrams, as shown in Fig. (5). to illustrate the establishment of the four formulas. The black lines in the figure represent original data at different mean levels, whereas the red lines indicate enhanced data at different mean levels.

**Remark 3.** For WMDD, theorem 2 demonstrates that when $y_1,...,Y_T$ has a change at $t^*(t^* \in 2,...,T)$ and if the above two operations are performed $n_c$ times independently at $t^*(t^* \in 2,...,T)$, then the value of $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{1i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')||$ will reach the value $2|\mu_2-\mu_1|$. Furthermore, by theorem 1, we know that at any non-change location, $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{1i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')||$ will become $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\mathring{\mu}'_{1i}(t^*,y') - \mathring{\mu}'_{1i}(t^*,y')| - |\mathring{\mu}''_{1i}(t^*,y'') - \mathring{\mu}''_{1i}(t^*,y'')||$. Because the data exemplify the weak mean difference, this value will still approximate zero. Therefore, if the above two operations are performed $n_c$ times independently at any non-change location, then the value of $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{1i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')||$ will be less than the value $2|\mu_2 - \mu_1|$. The latter finding is verified in Section 3 in detail.

### C. SCPDM

For WMDD, the different asymptotic properties in theorem 1 and theorem 2 are important information for judging whether there is a change point and the location of the change point. Consequently, we propose a model for detecting the change point in WMDD by sampling repeatedly in this section.

Assume that there is only one change point whose location is $t^*(1 < t^* \leq T)$ in $y_1,...,y_T$. To obtain a better estimate of $t^*$, we should establish a contrast function [5] to measure the goodness-of-fit of the signal sample. First, at $t(t = 2,...,T)$, the above two enhanced operations are performed $n_c$ times. The following contrast function is established in (23):

$$\hat{t}^*(y) = argmax\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y')$$
$$-\dot{\mu}'_{2i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')|| \quad (23)$$

According to theorem 2, the value of $||\underset{t}{\dot{\mu}'_1} - \underset{t}{\dot{\mu}'_2}| - |\underset{t}{\dot{\mu}''_1} - \underset{t}{\dot{\mu}''_2}||$ is expected to be large when $t^*$ is well-estimated.

During the establishment of the SCPDM, solving this discrete optimization problem becomes clear. We need only to calculate $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')||$ at (t=2,...,T) and to make $\hat{t}^*(y) = argmax\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}'_{1i}(t^*,y') - \dot{\mu}'_{2i}(t^*,y')| - |\dot{\mu}''_{1i}(t^*,y'') - \dot{\mu}''_{1i}(t^*,y'')||$. Details are elaborated in the next section, where we perform several simulation studies to estimate $t^*$ under a normal distribution with various kinds of parameters.

## III. VALIDATION OF THE METHODOLOGY

In this section, we will perform simulation analysis in two parts. In the first part, we verify the correctness of theorem 2 and remark 2. In the second part, we perform several simulation studies to estimate the potential location of change points under a normal distribution with different parameters and compare the SCPDM with two current models to verify that the SCPDM has a higher efficiency than those of the other models for WMDD.

### A. Verifying the correctness of the theorem

To verify the correctness of theorem 2, we generate random numbers based on the normal distribution, and the parameter settings are shown in the caption of each figure.

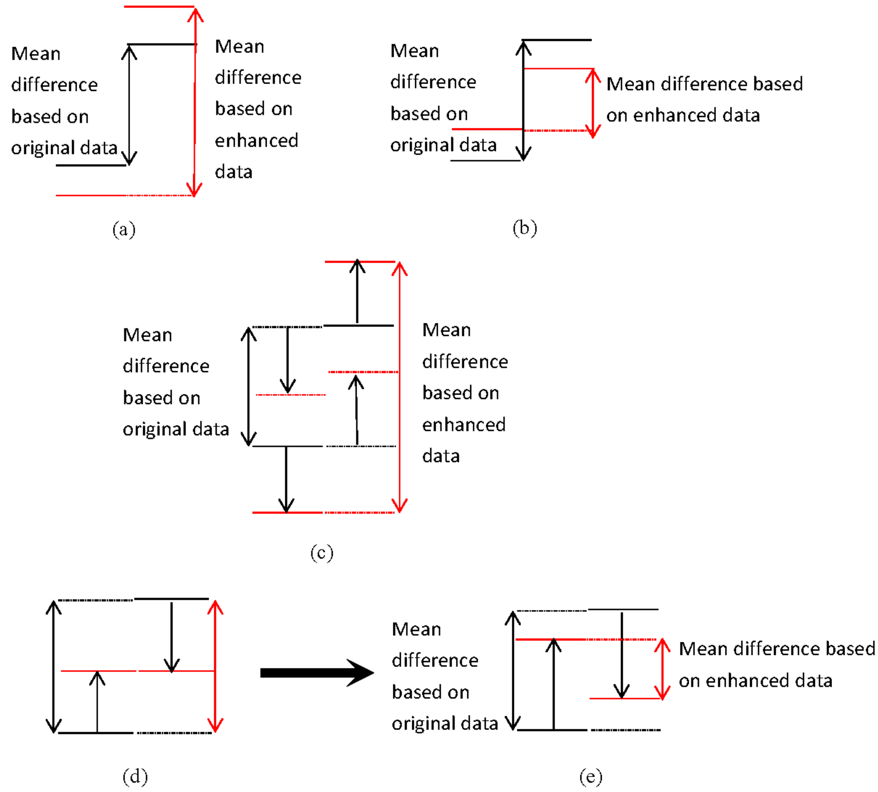



Fig. 3. The illustration of formula (13), (14), (21) and (22). (a) illustrates formula (13); (b) illustrates formula (14); (c) illustrates formula (21); The process from (d) to (e) illustrates formula (22)

For the public parameters, we set $T = 1000$, $t^* = 501$, and $n_c = 1000$. Both the number of data in the first half and the number in the second half are all 500, which can be considered large. Therefore, it can be seen as the case of $t^* \to \infty$ and $T - t^* \to \infty$. For $\frac{|\mu_1-\mu_2|}{\sigma} > 1$, the asymptotic result should be near 0. We carry out two sets of simulation analyses. As shown in Fig. (4), simulated results agree with theoretical results.

When $\frac{|\mu_1-\mu_2|}{\sigma} \leq 1$, as shown in Fig. (5), we carry out four sets of simulation analyses.

To verify the correctness of remark 2, when $\frac{|\mu_1-\mu_2|}{\sigma} \leq 1$, at $t(t = 400, ..., 600)$, $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}_{1i}'(t^*, y') - \dot{\mu}_{2i}'(t^*, y')| - |\dot{\mu}_{1i}''(t^*, y'') - \dot{\mu}_{2i}''(t^*, y'')||)$ is calculated, and the results are shown in Fig. (6).

It can be seen from Fig. (6) that when $t$ is near $t^* = 501$, $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\dot{\mu}_{1i}'(t, y') - \dot{\mu}_{2i}'(t, y')| - |\dot{\mu}_{1i}''(t, y'') - \dot{\mu}_{2i}''(t, y'')||)$ reaches a maximum value, which is approximately $2|\mu_2 - \mu_1|$. The simulation result is consistent with theorem 2.

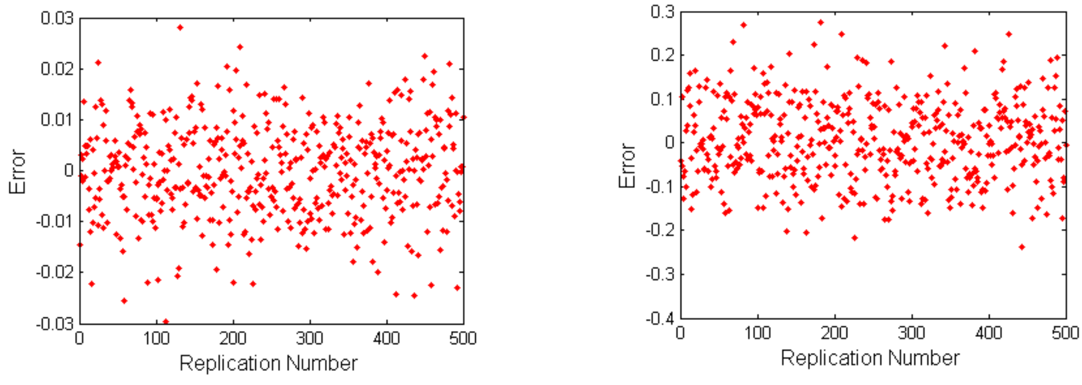
### B. Model comparison

When $\frac{|\mu_1-\mu_2|}{\sigma} \leq 1$, to verify that the SCPDM's estimation of $t^*$ is better than that of the other models, we present certain simulation results based on traditional models, including the least squares model and Bayes method [47]. We generate random samples $y_1, ..., y_T$ based on the normal distribution, and the parameter settings are shown in the corresponding figure; we set the public parameters, namely, $T = 1000$, $t^* = 500$, and $n_c = 1000$. For the three models with the same parameter settings, we repeat the same operation 1000 times, compute the estimation results of $t^*$ and regard the frequency of each $\hat{t}^*$ as the probability of being a real change point, which reflects the accuracy of each model. The results are shown in Fig. (7), Fig. (8) and Fig. (9).

When $\frac{|\mu_1-\mu_2|}{\sigma} > 1$, i.e., the parameters are set to be $\sigma = 0.5$ and $|\mu_1 - \mu_2| = 1$ for 1000 repeated tests, the least squares model results in $Pr(\hat{t}^* \in [499, 503]) = 0.942$ and $Pr(\hat{t}^* = 501) = 0.661$, and the Bayes model results in $Pr(\hat{t}^* \in [499, 503]) = 0.87$ and $Pr(\hat{t}^* = 501) = 0.61$. Both methods have a high accuracy for change point detection.

By setting different parameters, the detection accuracy of the change point interval, i.e., Pr($\hat{t}^* \in [499, 503]$), and change point location, i.e., Pr($\hat{t}^* = 5.1$), of the three models are compared in Table. (II), Table. (III), Table. (IV) and Table. (V).

When $\frac{|\mu_1-\mu_2|}{\sigma} \leq 1$, the type of data is WMDD, and we consider $Pr(\hat{t}^* \in [499, 503])$. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 0.1$, the accuracy of the SCPDM is 42% higher than that of the least squares model and 43% higher than that of the Bayes model. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 0.7$, the accuracy of the SCPDM is 54% higher than that of the least squares model and 60.7% higher than that of the Bayes model. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 1$, the accuracy of the SCPDM is 31% higher than the accuracy of the least squares model and 41.1% higher than the accuracy of the Bayes model. When the parameters are set to $\sigma = 10$ and $|\mu_1 - \mu_2| = 1$, the accuracy of the SCPDM is 31% higher than the accuracy of the least squares model and 41.1% higher than the accuracy of the Bayes model. When the parameters




(a) (b)

Fig. 4. The value of $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\overset{\bullet}{\mu}'_{1i}(t^*,y')-\overset{\bullet}{\mu}'_{2i}(t^*,y')|-|\overset{\bullet}{\mu}''_{1i}(t^*,y'')-\overset{\bullet}{\mu}''_{2i}(t^*,y'')||-(|\overset{\circ}{\mu}'_{1i}(t^*,y')-\overset{\circ}{\mu}'_{2i}(t^*,y')|+|\overset{\circ}{\mu}''_{1i}(t^*,y'')-\overset{\circ}{\mu}''_{2i}(t^*,y'')|))$ for 500 repetitions. (a) $\sigma=0.1$, $|\mu_1-\mu_2|=0.5$. (b) $\sigma=1$, $|\mu_1-\mu_2|=5$.

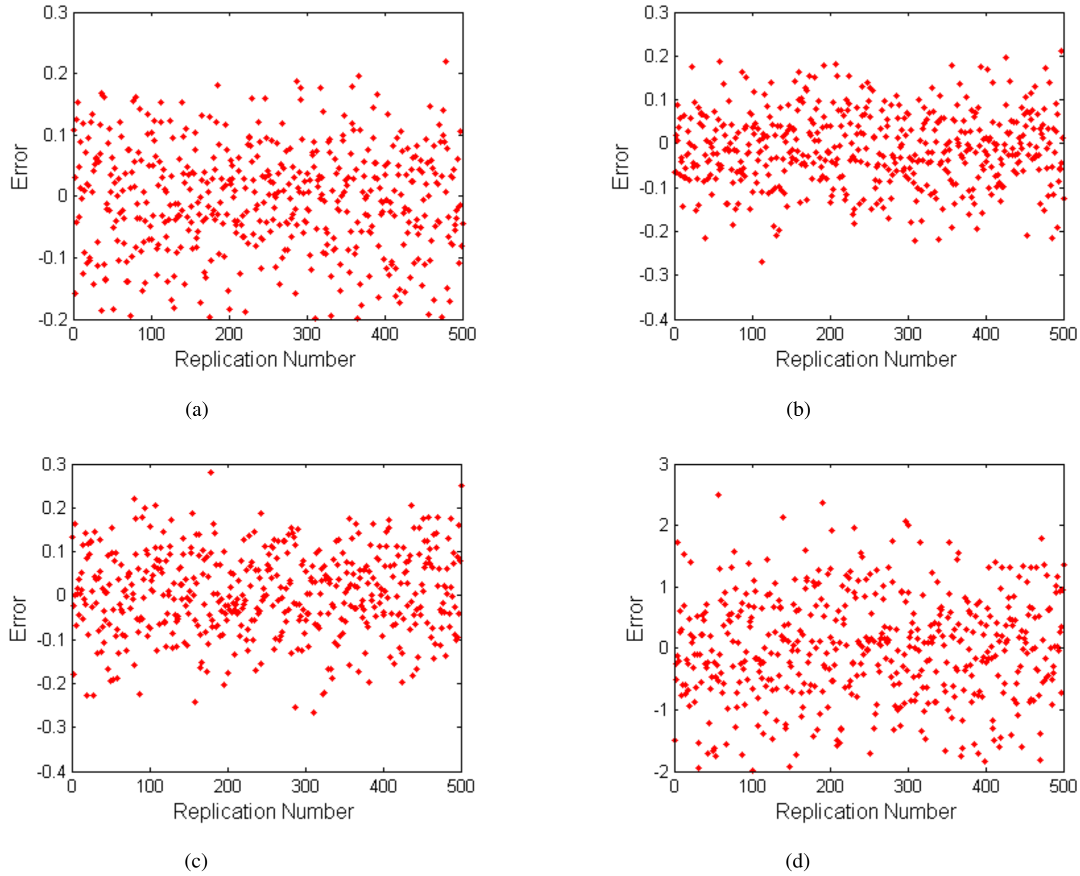



Fig. 5. The value of $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\overset{\bullet}{\mu}'_{1i}(t^*,y')-\overset{\bullet}{\mu}'_{2i}(t^*,y')|-|\overset{\bullet}{\mu}''_{1i}(t^*,y'')-\overset{\bullet}{\mu}''_{2i}(t^*,y'')||)-2|\mu_2-\mu_1|$ for 500 repetitions. (a) $\sigma=1$, $|\mu_1-\mu_2|=0.1$. (b) $\sigma=1$, $|\mu_1-\mu_2|=0.7$. (c) $\sigma=1$, $|\mu_1-\mu_2|=1$. (d) $\sigma=10$, $|\mu_1-\mu_2|=1$.

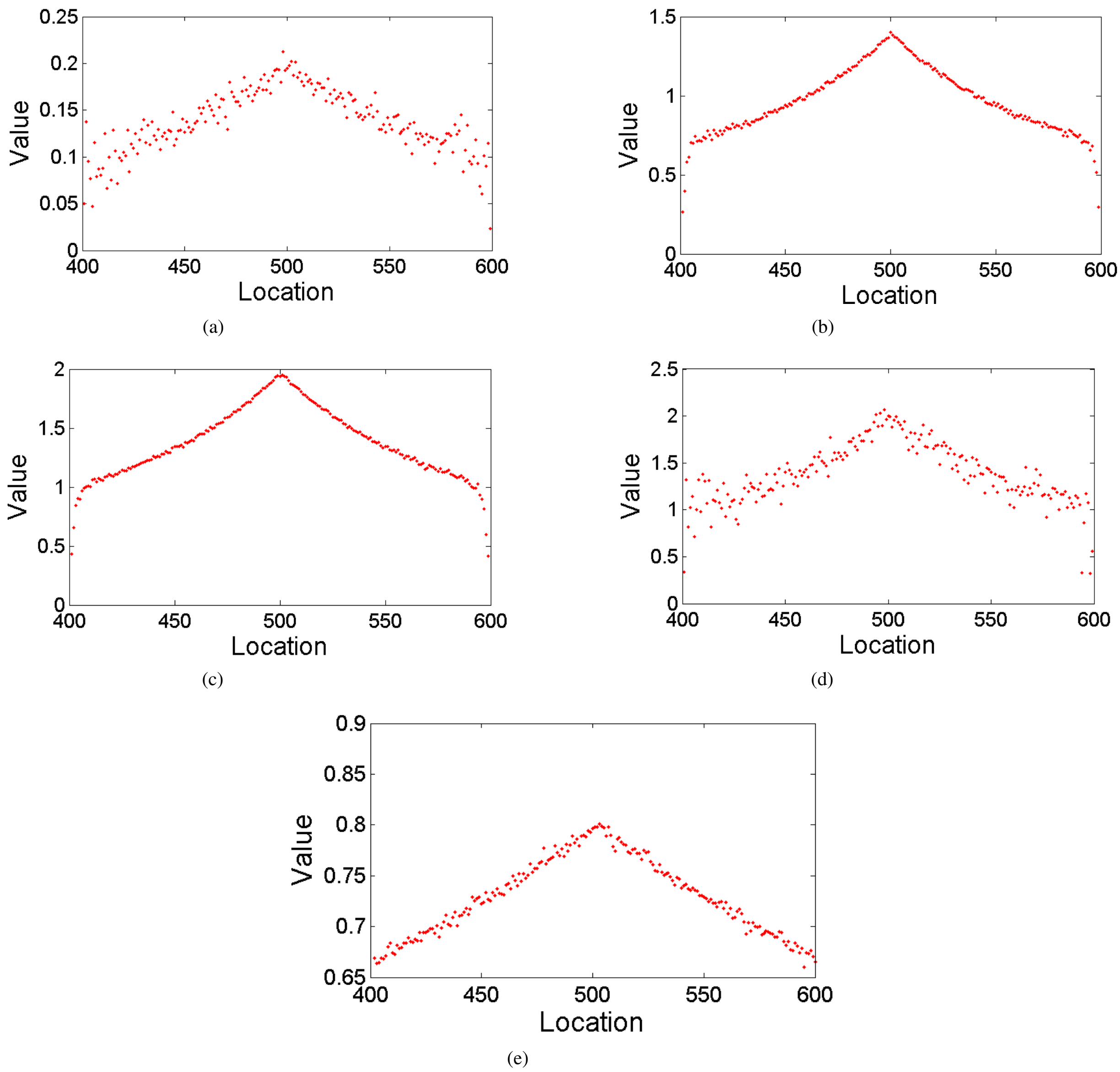



Fig. 6. The value of $\frac{1}{n_c}\sum_{i=1}^{n_c}(||\overset{\bullet}{\mu}{}'_{1i}(t,y')-\overset{\bullet}{\mu}{}'_{2i}(t,y')|-|\overset{\bullet}{\mu}{}''_{1i}(t,y'')-\overset{\bullet}{\mu}{}''_{2i}(t,y'')||)$ at location $t$. (a) $\sigma=1$, $|\mu_1-\mu_2|=0.1$. (b) $\sigma=1$, $|\mu_1-\mu_2|=0.7$. (c) $\sigma=1$, $|\mu_1-\mu_2|=1$. (d) $\sigma=10$, $|\mu_1-\mu_2|=1$. (e) $\sigma=1$, $|\mu_1-\mu_2|=0.4$.

are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 0.4$, the accuracy of the SCPDM is 62.6% higher than the accuracy of the least squares model and 70.1% higher than the accuracy of the Bayes model.

When $\frac{|\mu_1-\mu_2|}{\sigma} \leq 1$, we consider $Pr(\hat{t}^* = t^*)$. When the parameters are set to $\sigma = 1$ and $|\mu_1-\mu_2| = 0.1$, the accuracy of the SCPDM is 9.1% higher than that of the least squares model and 9.5% higher than that of the Bayes model. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 0.7$, the accuracy of the SCPDM is 27.7% higher than that of the least squares model and 32.1% higher than that of the Bayes model. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 1$, the accuracy of the SCPDM is 25% higher than the accuracy of the least squares model and 33.6% higher than the accuracy of the Bayes model. When the parameters are set to $\sigma = 10$ and $|\mu_1-\mu_2| = 1$, the accuracy of the SCPDM is 9.2% higher than the accuracy of the least squares model and 9.5% higher than the accuracy of the Bayes model. When the parameters are set to $\sigma = 1$ and $|\mu_1 - \mu_2| = 0.4$, the accuracy of the SCPDM is 23.2% higher than the accuracy of the least squares model and 28.8% higher than the accuracy of the Bayes model.

## IV. Conclusions and Future Study

This paper focuses on detecting the change points for weak mean difference data. We perform asymptotic analysis and establish a strengthened change point detection model. According to theorem 2 (2), the enhanced sequence uses significant variance information so that the weak mean difference increases from $|\mu_1 - \mu_2|$ to $2|\mu_1 - \mu_2|$, which makes the change point easier to detect and increases the accuracy of change point detection. In addition, for WMDD, the traditional methods can be improved by adding the sample capacity to the sequence. Meanwhile, for the same amount of data, the SCPDM greatly increases the efficiency of change point detection by repeatedly detecting sequences with the same data structure. Furthermore, repeated measurements are possible for the lifetime data of components at the

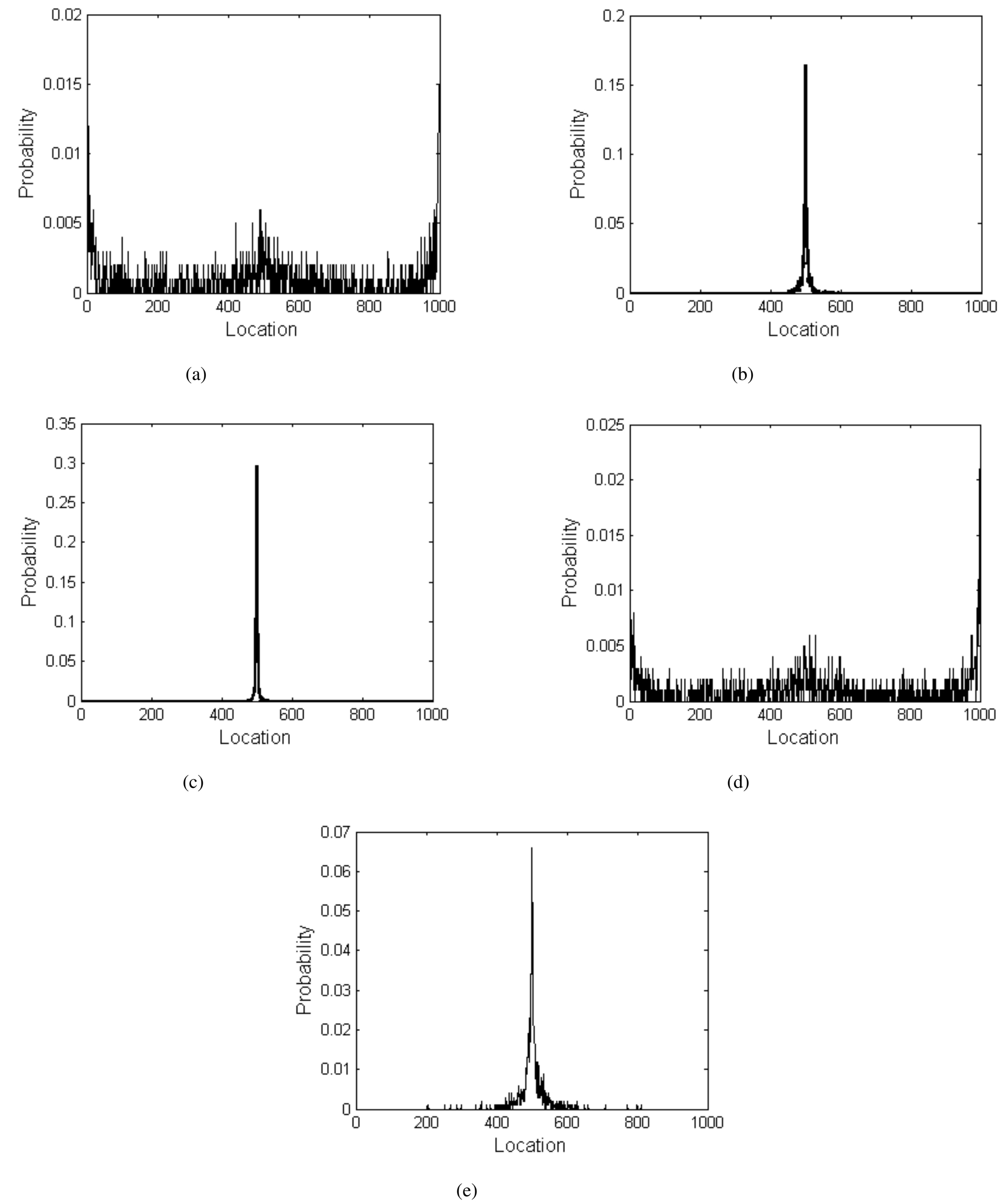



Fig. 7. Probability of each location becoming a change point in the least squares model. $(a)\sigma = 1, |\mu_1 - \mu_2| = 0.1.(b)\sigma = 1, |\mu_1 - \mu_2| = 0.7.(c)\sigma = 1, |\mu_1 - \mu_2| = 1.(d)\sigma = 10, |\mu_1 - \mu_2| = 1.(e)\sigma = 1, |\mu_1 - \mu_2| = 0.4.$

same location. Hence, compared with traditional methods, the SCPDM can effectively detect change points. Although the accuracy of change point detection has been improved, this paper also has several limitations. First, we only discuss that $y_1, ..., y_T$ are independent with a normal distribution and there exists only a single change point. Second, the reason why the relationship between $|\mu_1 - \mu_2|$ and $\sigma$ has an important influence on the accuracy of change point detection is not discussed in depth. We define the ratio boundary of WMDD based on only experience and simulations. In a future study, we will extend the SCPDM to other distribution types and multiple point detection. In addition, for theorem 2, we will reprove the theorem by introducing the relationship between $|\mu_1 - \mu_2|$ and $\sigma$.

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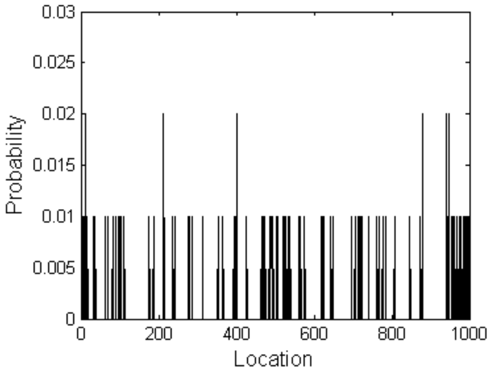
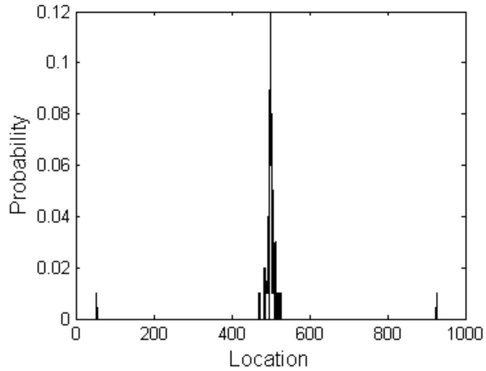
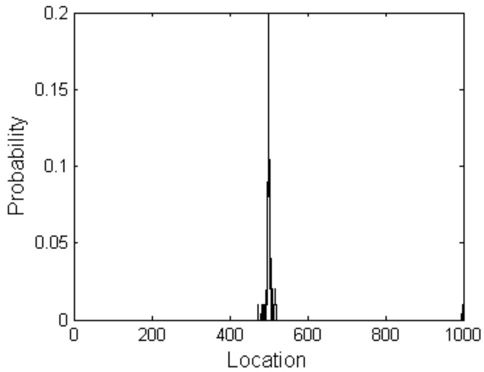


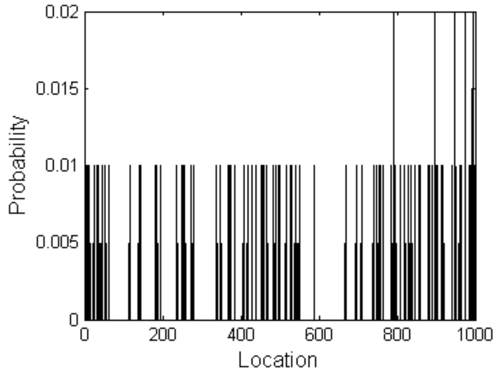

(a)



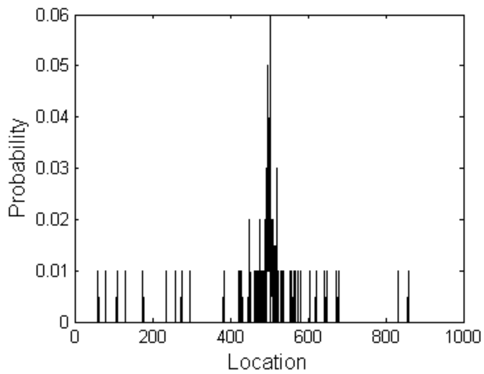

(b)




(c)




(d)

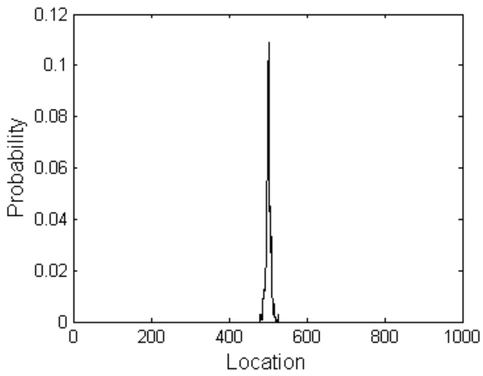



(e)

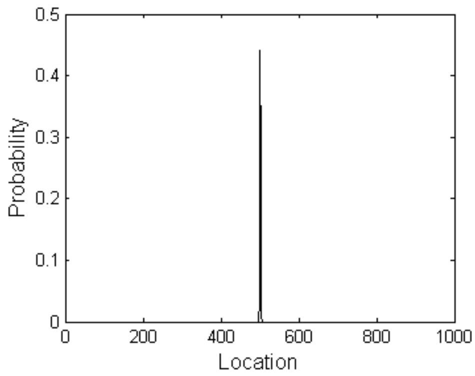
Fig. 8. Probability of each location becoming a change point in the Bayes model. $(a)\sigma = 1, |\mu_1 - \mu_2| = 0.1.(b)\sigma = 1, |\mu_1 - \mu_2| = 0.7.(c)\sigma = 1, |\mu_1 - \mu_2| = 1.(d)\sigma = 10, |\mu_1 - \mu_2| = 1.(e)\sigma = 1, |\mu_1 - \mu_2| = 0.4.$

TABLE I
THE MEAN DIFFERENCE UNDER BOTH ENHANCED OPERATIONS FOR NORMAL RANDOM NUMBERS WITH DIFFERENT MEANS AND VARIANCES

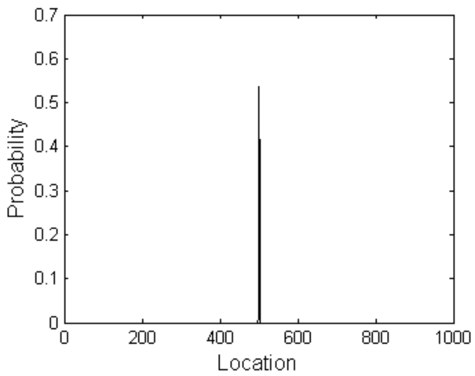
| | $\sigma^2 = 0.1$ | $\sigma^2 = 1$ |
|---|---|---|
| $\mu = 1$ | 0.1025 | 1.1244 |
| | 0.1130 | 1.0855 |
| $\mu = 5$ | 0.1170 | 1.0722 |
| | 0.1133 | 1.1860 |
| $\mu = 10$ | 0.1084 | 1.0814 |
| | 0.1180 | 1.1332 |



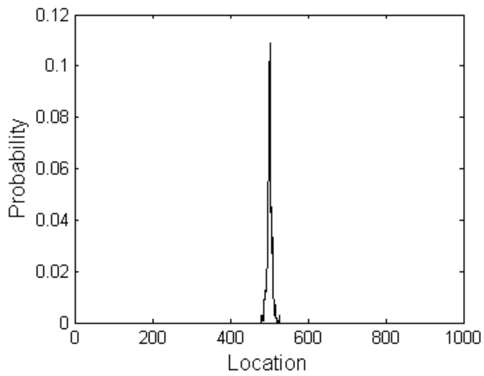

(a)



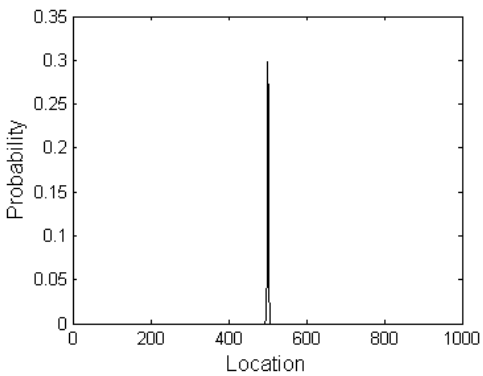

(b)




(c)




(d)




(e)

Fig. 9. Probability of each location becoming a change point in the SCPDM. $(A)\sigma = 1, |\mu_1 - \mu_2| = 0.1.(B)\sigma = 1, |\mu_1 - \mu_2| = 0.7.(C)\sigma = 1, |\mu_1 - \mu_2| = 1.(D)\sigma = 10, |\mu_1 - \mu_2| = 1.(E)\sigma = 1, |\mu_1 - \mu_2| = 0.4.$

TABLE II
ESTIMATE OF THE PROBABILITY OF A CHANGE POINT IN CERTAIN INTERVALS FOR THE LEAST SQUARES METHOD

| $\mu_1 - \mu_2$ | $\sigma$ | 484-488 | 489-493 | 494-498 | 499-503 | 504-508 | 509-513 | 514-518 |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 1 | 0.009 | 0.014 | 0.015 | 0.010 | 0.015 | 0.011 | 0.009 |
| 0.7 | 1 | 0.039 | 0.048 | 0.166 | 0.417 | 0.148 | 0.048 | 0.034 |
| 1 | 1 | 0.010 | 0.031 | 0.113 | 0.681 | 0.113 | 0.021 | 0.013 |
| 1 | 10 | 0.006 | 0.008 | 0.010 | 0.019 | 0.013 | 0.007 | 0.012 |
| 0.4 | 1 | 0.054 | 0.075 | 0.113 | 0.215 | 0.107 | 0.055 | 0.043 |

TABLE III
ESTIMATE OF THE PROBABILITY OF A CHANGE POINT IN CERTAIN INTERVALS FOR THE BAYES METHOD

| $\mu_1 - \mu_2$ | $\sigma$ | 484-488 | 489-493 | 494-498 | 499-503 | 504-508 | 509-513 | 514-518 |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 1 | 0.01 | 0.03 | 0.01 | 0 | 0.02 | 0 | 0 |
| 0.7 | 1 | 0.06 | 0.06 | 0.18 | 0.35 | 0.18 | 0.07 | 0.01 |
| 1 | 1 | 0.01 | 0.02 | 0.11 | 0.58 | 0.13 | 0.07 | 0.03 |
| 1 | 10 | 0.01 | 0.01 | 0.01 | 0.01 | 0 | 0 | 0.01 |
| 0.4 | 1 | 0.02 | 0.04 | 0.08 | 0.14 | 0.10 | 0.06 | 0.02 |

TABLE IV
ESTIMATE OF THE PROBABILITY OF A CHANGE POINT IN CERTAIN INTERVALS FOR THE SCPDM METHOD

| $\mu_1 - \mu_2$ | $\sigma$ | 484-488 | 489-493 | 494-498 | 499-503 | 504-508 | 509-513 | 514-518 |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 1 | 0.019 | 0.065 | 0.171 | 0.430 | 0.201 | 0.073 | 0.023 |
| 0.7 | 1 | 0 | 0 | 0.010 | 0.957 | 0.033 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0.001 | 0.991 | 0.008 | 0 | 0 |
| 1 | 10 | 0.019 | 0.065 | 0.171 | 0.430 | 0.201 | 0.073 | 0.023 |
| 0.4 | 1 | 0 | 0 | 0.040 | 0.841 | 0.118 | 0.001 | 0 |

TABLE V
ESTIMATE OF THE PROBABILITY OF $\hat{t}^* = t^*$ FOR THE THREE MODELS

| MD | SD | Least Squares Method | Bayes Method | SCPDM |
|---|---|---|---|---|
| 0.1 | 1 | 0.004 | 0 | 0.095 |
| 0.7 | 1 | 0.164 | 0.12 | 0.441 |
| 1 | 1 | 0.286 | 0.2 | 0.536 |
| 1 | 10 | 0.003 | 0 | 0.095 |
| 0.4 | 1 | 0.066 | 0.01 | 0.298 |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**Qi Zhou** was born in HeBei, China, in 1993. He received a bachelor's degree in statistics from North China University of Science and Technology. His main research interests are statistics, linear Bayesian estimator, change point analysis and simulation.

**Shaoqian Huang** was born in HeBei, China, in 1994. He received a bachelor's degree in statistics from North China University of Science and Technology. His main research interests are statistics, linear Bayesian estimator, change point analysis and simulation.