On E-Orlicz Theory

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***Abstract*—In this paper, based on concepts of E-convex sets, E-convex functions and E-continuous, we establish the E-Orlicz theory which is a generalization to the Orlicz theory by relaxing the concepts of N-function, Young function, strong Young function and Orlicz function. In this theory, we introduce the definitions of E-Orlicz spaces, weak E-Orlicz spaces, E-Orlicz-Sobolev spaces, weak E-Orlicz-Sobolev spaces, E-Orlicz-Morrey spaces and weak E-Orlicz-Morrey spaces, E-Orlicz-Lorentz spaces and weak E-Orlicz-Lorentz spaces. However, we consider their implicit properties based on the effect of the operator E.**

***Index Terms*—E-N-function, E-Young function, E-strong Young function, E-Orlicz function, E-Orlicz spaces, E-Orlicz-Sobolev space, E-Orlicz-Morrey Space , E-Orlicz-Lorentz Spaces.**

I. INTRODUCTION

BIRNBAUM and Orlicz introduced the Orlicz spaces in 1931 as a generalization of the classical Lebesgue spaces, where the function u^p is replaced by a more general convex function Φ [2]. The concept of E-convex sets and E-convex functions were introduced by Youness to generalize the classical concepts of convex sets and convex functions to extend the studying of the optimality for non-linear programming problems in 1999 [3]. Chen defined the semi-E-convex functions and studied its basic properties in 2002 [3]. The concepts of pseudo E-convex functions and E-quasiconvex functions and strictly E-quasiconvex functions were introduced by Syau and Lee in 2004 [6]. The concept of Semi strongly E-convex functions was introduced by Youness and Tarek Emam in 2005 [8]. Sheiba Grace and Thangavelu considered the algebraic properties of E-convex sets in 2009 [4]. E-differentiable convex functions was defined by Meghed, Gomma, Youness and El-Banna to transform a non-differentiable function to a differentiable function in 2013 [5]. Semi-E-convex function was introduced by Ayache and Khaled in 2015 [1].

The purpose behind this paper is to define the E-N-functions, E-Young functions, E-strong Young functions and E-Orlicz functions using the concepts of E-convex sets, E-convex functions and E-continuous functions to generalize and extend the studying of the classical Orlicz theory via defining a new class of Orlicz spaces equipped by the luxemburg norms and generated by non-Young functions but E-Young functions with a map E, like E-Orlicz spaces, weak E-Orlicz spaces, E-Orlicz-Sobolev spaces, weak E-Orlicz-Sobolev spaces, E-Orlicz-Morrey space, weak E-Orlicz-Morrey space, E-Orlicz-Lorentz spaces and weak E-Orlicz-Lorentz spaces.

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Contents of the paper. For our study, we present the definitions of E-N-function, E-Young function, E-strong Young function and E-Orlicz function in section II. We consider the elementary properties of E-N-functions, E-Young functions, E-strong Young functions and E-Orlicz functions and their relationships in section III and IV respectively. In section V, we state the definitions of E-Orlicz space, weak E-Orlicz space, E-Orlicz-Sobolev space, weak E-Orlicz-Sobolev space, E-Orlicz-Morrey space, weak E-Orlicz-Morrey space, E-Orlicz-Lorentz space and weak E-Orlicz-Lorentz space. In addition, we study the implicit properties of these new spaces.

II. PRELIMINARIES

The setting for this paper is n-dimensional Euclidean space $R^n, n \geq 1$. Let Ω be a nonempty subset of R^n and (Ω, Σ, μ) be a measure space. A set Ω is said to be E-convex iff there is a map $E: R^n \longrightarrow R^n$ such that $\lambda E(x) + (1-\lambda)E(y) \in \Omega$, for each $x, y \in \Omega, 0 \leq \lambda \leq 1$. A function $f: R^n \longrightarrow R$ is said to be E-convex on a set Ω iff there is a map $E: R^n \longrightarrow R^n$ such that Ω is an E-convex set and

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \leq \lambda f\big(E(x)\big) + (1-\lambda) f\big(E(y)\big),$$

for each $x, y \in M$ and $0 \leq \lambda \leq 1$. And f is called E-concave on a set Ω if

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \geq \lambda f\big(E(x)\big) + (1-\lambda) f\big(E(y)\big),$$

for each $x, y \in \Omega$ and $0 \leq \lambda \leq 1$ (see [6]). A function $f: \Omega \longrightarrow R^m$ is said to be E-continuous at $a \in \Omega$ iff there is a map $E: R^n \longrightarrow R^n$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ implies

$$\big\| f\big(E(x)\big) - f\big(E(a)\big) \big\| < \varepsilon$$

whenever

$$\|x - a\| < \delta$$

and f is said to be E-continuous on Ω iff f is E-continuous at every $x \in \Omega$.

Definition 1. A function $\Phi: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ is called an E-N-function if there exists a map $E: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty)$ such that for μ-a.e. $t \in \Omega$, $[0, \infty)$ is an E-convex and Φ is an E-even, E-continuous, E-convex of u on $[0, \infty)$, $\Phi\big(E(t, u)\big) > 0$ for any $u \in (0, \infty)$,

$$\lim_{u \to 0^+} \frac{\Phi\big(E(t,u)\big)}{u} = 0 \,, \lim_{u \to \infty} \frac{\Phi\big(E(t,u)\big)}{u} = \infty$$

and for each $u \in [0, \infty)$, $\Phi\big(E(t, u)\big)$ is an μ-measurable function of t on Ω.

Remark 2. Every N-function is an E-N-function if the map E is taken as the identity map. But not every E-N-function is an N-function.

Examples 3. We cite examples of E-N-function which is not N-function

i. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = tu^2$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined as $E(t, u) = (|t|, u)$. Then Φ is an E-N-function but it is not an N-function because, for μ-a.e. $t \in \mathbb{R}$, $\Phi(t, u)$ is concave of u for $t \in (-\infty, 0)$.

ii. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = (1 - t)u^2 + t \exp(u)$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (t, \ln u^2)$. Then, Φ is an E-N-function but it is not an N-function since, for μ-a.e. $t \in \mathbb{R}, \Phi(t, u)$ is not even.

Definition 4. A function $\Phi: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ is called an E-Young function if there exists a map $E: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty]$ such that for μ-a.e.$t \in \Omega, [0, \infty)$ is an E-convex and Φ is an E-convex of u on $[0, \infty)$,

$$\Phi\big(E(t, 0)\big) = \lim_{u \to 0^+} \Phi\big(E(t, u)\big) = 0,$$

$$\lim_{u \to \infty} \Phi\big(E(t, u)\big) = \infty$$

and for each $u \in [0, \infty), \Phi\big(E(t, u)\big)$ is an μ-measurable function of t on Ω.

Remark 4. Every Young function is an E-Young function if the map E is taken as the identity map. But not every E-Young function is a Young function.

Examples 5. We cite examples of E-Young function which is not Young function

i. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = e^{t+u} - 1$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (u, u)$. Then, Φ is an E-Young function but it is not a Young function because for μ-a.e. $t \in \mathbb{R}, \Phi(t, 0) = e^t - 1 \neq 0$.

ii. Let $\Phi: \mathbb{C} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by

$$\Phi(t, u) = \begin{cases} t\ln(u), & u > 1 \\ 0, & 0 \leq u \leq 1 \end{cases}$$

and let $E: \mathbb{C} \times [0, \infty) \longrightarrow \mathbb{C} \times [0, \infty)$ be defined by $E(t, u) = (-|t|, u)$. So, Φ is an E-Young function but it is not a Young function because, for μ-a.e. $t \in \mathbb{C}$, $\Phi(t, u)$ is not convex because for $t \in (0, \infty)$, $\frac{\partial^2 \Phi}{\partial u^2} = -\frac{t}{u^2} < 0$.

Definition 6. A function $\Phi: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ is called an E-strong Young function if there exists a map $E: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty)$ such that for μ-a.e. $t \in \Omega, [0, \infty)$ is an E-convex and Φ is an E-convex E-continuous of u on $[0, \infty), \Phi\big(E(t, 0)\big) = 0 \Leftrightarrow u = 0$,

$$\lim_{u \to \infty} \Phi\big(E(t, u)\big) = \infty$$

and for each $u \in [0, \infty), \Phi\big(E(t, u)\big)$ is an μ-measurable function of t on Ω.

Remark 7. Every strong Young function is an E-strong Young function if the map E is taken as the identity map. But not every E-strong Young function is a strong Young function.

Example 8. We cite examples of E-strong Young function which is not strong Young function

i. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = e^{u^t} - 1$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (|t|, u)$. Then Φ is an E-strong Young function but it is not a strong Young function, where $\Phi(t, u) = e^{u^t} - 1$ is not convex because for $t \in (-\infty, 0), u^t$ is not convex.

ii. Let $\Phi: [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = \cosh(te^u) - 1$ and let $E: [0, \infty) \times [0, \infty) \longrightarrow [0, \infty) \times [0, \infty)$ be defined by $E(t, u) = (u, 0)$. Then Φ is an E-strong Young function but it is not a strong Young function since for μ-a.e. $t \in [0, \infty), \Phi(t, 0) = \cosh(t) - 1 \neq 0$.

Definition 9. A function $\Phi: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ is called an E-Orlicz function if there exists a map $E: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty)$ such that for μ-a.e. $t \in \Omega$, $[0, \infty)$ is an E-convex and Φ is an E-convex of u on $[0, \infty)$, $\Phi\big(E(t, 0)\big) = 0, \Phi\big(E(t, u)\big) > 0$ for any $u \in (0, \infty)$,

$$\lim_{u \to \infty} \Phi\big(E(t, u)\big) = \infty,$$

Φ is left E-continuous at

$$U_\Phi = \sup\{u > 0: \Phi\big(E(t, u)\big) < +\infty\}$$

and for each $u \in [0, \infty), \Phi\big(E(t, u)\big)$ is an μ-measurable function of t on Ω.

Remark 10. Every Orlicz function is an E-Orlicz function if the map E is taken as the identity map. But not every E-Orlicz function is an Orlicz function.

Examples 11. We cite examples of E-Orlicz function which is not Orlicz function

i. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = -t + u$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (0, u^p)$, $p \geq 1$. Then Φ is an E-Orlicz function but it is not an Orlicz function because for μ-a.e. $t \in \mathbb{R}, \Phi(t, 0) = -\mathrm{t} \neq 0$.

ii. Let $\Phi: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ be defined by $\Phi(t, u) = t + u^{\frac{p}{(1-t)}}$, $p \geq 1$ and let $E: \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R} \times [0, \infty)$ be defined by $E(t, u) = (0, u)$. Then Φ is an E-Orlicz function but it is not an Orlicz function because for μ-a.e.$t \in \mathbb{R}, \Phi(t, 0) = t \neq 0$.

III. ELEMENTARY PROPERTIES

A. *Properties of E-N-Functions*

Theorem 12. Let $\Phi_1, \Phi_2: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ be E-N-functions with respect to $E: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty)$. Then $\Phi_1 + \Phi_2$ and $c\Phi_1, c \geq 0$ are E-N-functions with respect to E.

Theorem 13. Let $\Phi: \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ be a linear E-N-function with respect to $E_1, E_2: \Omega \times [0, \infty) \longrightarrow \Omega \times [0, \infty)$. Then Φ is an E-N-function with respect to $E_1 + E_2$ and $cE_1, c \geq 0$.

Theorem 14. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a linear E-N-function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-N-function with respect to $E_1 \circ E_2$ and $E_2 \circ E_1$.

Theorem 15. Let $\Phi_i: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ for $i = 1, \cdots, n$ be E-N -functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then $\Phi = \max_i \Phi_i$ is an E-N-function with respect to E.

Theorem 16. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-N-function with respect to $E_i: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty), i = 1, \cdots, n$. Then Φ is an E-N-function with respect to $E_M = \max_i E_i$ and $E_m = \min_i E_i$.

Theorem 17. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-N-functions defined on a compact set $\Omega \times [0,\infty)$ with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$. Then Φ is an E-N-function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-N-functions with respect to a map E such that $\Phi_n \longrightarrow \Phi$ uniformly on compact set $\Omega \times [0,\infty)$ and Φ is continuous on $\Omega \times [0,\infty)$. Then $\Phi_n(E) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\left(E(t,u)\right)$$

is even continuous convex of u on $[0,\infty)$, $\Phi\big(E(t,u)\big) > 0$ for any $u \in (0,\infty)$,

$$\lim_{u\to 0} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to 0} \frac{\Phi_n\big(E(t,u)\big)}{u} = 0,$$

$$\lim_{u\to \infty} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to \infty} \frac{\Phi_n\big(E(t,u)\big)}{u} = \infty$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 18. Let Φ be a continuous E-N-function defined on a compact set $\Omega \times [0,\infty)$ with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}}, E_n: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$, such that $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-N-function with respect to E.

Proof. Suppose that Φ is a continuous E-N-function with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}}$ such that $E_n \longrightarrow E$ uniformaly on a compact set $\Omega \times [0,\infty)$. Then $\Phi(E_n) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi\big(E_n(t,u)\big)$$

is even continuous convex of u on $[0,\infty), \Phi\big(E(t,u)\big) > 0$ for $u \in (0,\infty)$,

$$\lim_{u\to 0} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to 0} \frac{\Phi\big(E_n(t,u)\big)}{u} = 0,$$

$$\lim_{u\to \infty} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to \infty} \frac{\Phi\big(E_n(t,u)\big)}{u} = \infty$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 19. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-N-functions defined on a compact set $\Omega \times [0,\infty)$ with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}, E_n: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$, such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ and $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-N-function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-N-functions with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}$ such that $\Phi_n \longrightarrow \Phi$ uniformly and $E_n \longrightarrow E$ uniformly on a compact set $\Omega \times [0,\infty)$ and Φ and E are continuous on $\Omega \times [0,\infty)$. So $\Phi_n(E_n) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$, that

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\big(E_n(t,u)\big)$$

is even continuous convex of u on $[0,\infty)$, $\Phi\big(E(t,u)\big) > 0, u \in (0,\infty)$,

$$\lim_{u\to 0} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to 0} \frac{\Phi_n\big(E_n(t,u)\big)}{u} = 0,$$

$$\lim_{u\to \infty} \frac{\Phi\big(E(t,u)\big)}{u} = \lim_{n\to\infty}\lim_{u\to \infty} \frac{\Phi_n\big(E_n(t,u)\big)}{u} = \infty$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

B. Properties of E-Young Functions

Theorem 20. Let $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then $\Phi_1 + \Phi_2$ and $c\Phi_1, c \geq 0$ are E-Young functions with respect to E.

Theorem 21. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a linear E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Young functions with respect to $E_1 + E_2$ and $cE_1, c \geq 0$.

Theorem 22. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a linear E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Young functions with respect to $E_1 \circ E_2$ and $E_2 \circ E_1$.

Theorem 23. Let $\Phi_i: \Omega \times [0,\infty) \longrightarrow \mathbb{R}, i = 1, \dots, n$ be E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then $\Phi = \max_i \Phi_i$ is an E-Young function with respect to E.

Theorem 24. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-Young function with respect to $E_i: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty), i = 1, \cdots, n$. Then Φ is an E-Young function with respect to $E_M = \max_i E_i$ and $E_m = \min_i E_i$.

Theorem 25. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-Young functions defined on a compact set $\Omega \times [0,\infty)$ with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$. Then Φ is an E-Young function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-Young functions with respect to a map E such that $\Phi_n \longrightarrow \Phi$ uniformly on a compact set $\Omega \times [0,\infty)$ and Φ is continuous on $\Omega \times [0,\infty)$. Then $\Phi_n(E) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\left(E(t,u)\right)$$

is convex of u on $[0,\infty)$,

$$\Phi\big(E(t,0)\big)=\lim_{u\to 0^+}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to 0^+}\Phi_n\big(E(t,u)\big)=0,$$
$$\lim_{u\to\infty}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to\infty}\Phi_n\left(E(t,u)\right)=\infty$$
and for each $u\in[0,\infty),\Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 26. Let $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be a continuous E-Young function defined on a compact set $\Omega\times[0,\infty)$ with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}},E_n:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$, such that $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a map $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then Φ is an E-Young function with respect to E.

Proof. Suppose that Φ is a continuous E-Young function with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}}$ such that $E_n\longrightarrow E$ uniformaly on a compact set $\Omega\times[0,\infty)$. Then $\Phi(E_n)\longrightarrow\Phi(E)$ uniformaly on $\Omega\times[0,\infty)$ and for μ-a.e. $t\in\Omega$,
$$\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\Phi\big(E_n(t,u)\big)$$
is convex of u on $[0,\infty)$,
$$\Phi\big(E(t,0)\big)=\lim_{u\to 0^+}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to 0^+}\Phi\big(E_n(t,u)\big)=0,$$
$$\lim_{u\to\infty}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to\infty}\Phi\left(E_n(t,u)\right)=\infty$$
and for each $u\in[0,\infty),\Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 27. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-Young functions defined on a compact set $\Omega\times[0,\infty)$ with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}},E_n:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$, such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ and $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous map $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then Φ is an E-Young function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-Young functions with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}$ such that $\Phi_n\longrightarrow\Phi$ and $E_n\longrightarrow E$ uniformly on a compact set $\Omega\times[0,\infty)$ and Φ and E are continuous on $\Omega\times[0,\infty)$. Then $\Phi_n(E_n)\longrightarrow\Phi(E)$ uniformaly on $\Omega\times[0,\infty)$ and for μ-a.e.$t\in\Omega$,
$$\Phi\big(E(t,u)\big)=\lim\nolimits_{n\to\infty}\Phi_n\big(E_n(t,u)\big)$$
is convex of u on $[0,\infty)$,
$$\Phi\big(E(t,0)\big)=\lim_{u\to 0^+}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to 0^+}\Phi_n\big(E_n(t,u)\big)=0,$$
$$\lim_{u\to\infty}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to\infty}\Phi_n\big(E_n(t,u)\big)=\infty$$
and for each $u\in[0,\infty),\Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

C. Properties of E-Strong Young Functions

Theorem 28. Let $\Phi_1,\Phi_2:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be E-strong Young functions with respect to $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then $\Phi_1+\Phi_2$ and $c\Phi_1,c\geq 0$ are E-strong Young functions with respect to E.

Theorem 29. Let $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be a linear E-strong Young function with respect to $E_1,E_2:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then Φ is an E-strong Young function with respect to E_1+E_2 and $cE_1,c\geq 0$.

Theorem 30. Let $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be a linear E-strong Young function with respect to $E_1,E_2:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then Φ is an E-strong Young function with respect to $E_1\circ E_2$ and $E_2\circ E_1$.

Theorem 31. Let $\Phi_i:\Omega\times[0,\infty)\longrightarrow\mathbb{R},i=1,\cdots,n$ be E-strong Young functions with respect to $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then $\Phi=\max_i\Phi_i$ is an E-strong Young function with respect to E.

Theorem 32. Let $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be an E-strong Young function with respect to $E_i:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty),i=1,\cdots,n$. Then Φ is an E-strong Young function with respect to $E_M=\max_i E_i$ and $E_m=\min_i E_i$.

Theorem 33. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-strong Young functions defined on a compact set $\Omega\times[0,\infty)$ with respect to $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$. Then Φ is an E-strong Young function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-strong Young functions with respect to a map E such that $\Phi_n\longrightarrow\Phi$ uniformly on a compact set $\Omega\times[0,\infty)$ and Φ is continuous on $\Omega\times[0,\infty)$. Then $\Phi_n(E)\longrightarrow\Phi(E)$ uniformaly on $\Omega\times[0,\infty)$ and for μ-a.e. $t\in\Omega$,
$$\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\Phi_n\left(E(t,u)\right)$$
is convex, continuous of u on $[0,\infty)$,
$$\Phi\big(E(t,0)\big)=\lim_{n\to\infty}\Phi_n\left(E(t,0)\right)=0\Leftrightarrow u=0,$$
$$\lim_{u\to\infty}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to\infty}\Phi_n\left(E(t,u)\right)=\infty$$
and for each $u\in[0,\infty),\Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 34. Let $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ be a continuous E-strong Young function defined on a compact set $\Omega\times[0,\infty)$ with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}},E_n:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$ such that $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a map $E:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$. Then Φ is an E-strong Young function with respect to E.

Proof. Suppose that Φ is a continuous E-strong Young function with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}}$ such that $E_n\longrightarrow E$ uniformaly on a compact set $\Omega\times[0,\infty)$ and E is continuous on $\Omega\times[0,\infty)$. Then $\Phi(E_n)\longrightarrow\Phi(E)$ uniformaly on $\Omega\times[0,\infty)$ and for μ-a.e. $t\in\Omega$,
$$\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\Phi\big(E_n(t,u)\big)$$
is convex continuous of u on $[0,\infty)$,
$$\Phi\big(E(t,0)\big)=\lim_{n\to\infty}\Phi\big(E_n(t,0)\big)=0\Leftrightarrow u=0,$$
$$\lim_{u\to\infty}\Phi\big(E(t,u)\big)=\lim_{n\to\infty}\lim_{u\to\infty}\Phi\left(E_n(t,u)\right)=\infty$$
and for each $u\in[0,\infty),\Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 35. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-strong Young functions defined on a compact set $\Omega\times[0,\infty)$ with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}},E_n:\Omega\times[0,\infty)\longrightarrow\Omega\times[0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi:\Omega\times[0,\infty)\longrightarrow\mathbb{R}$ and $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous map $E:\Omega\times$

$[0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-strong Young function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuou E-strong Young functions with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}$ such that $\Phi_n \longrightarrow \Phi$ and $E_n \longrightarrow E$ uniformly on a compact set $\Omega \times [0,\infty)$ and Φ and E are continuous on $\Omega \times [0,\infty)$. So, $\Phi_n(E_n) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\big(E_n(t,u)\big)$$

is convex continuous of u on $[0,\infty)$,

$$\Phi\big(E(t,0)\big) = \lim_{n\to\infty} \Phi_n\big(E_n(t,0)\big) = 0 \Leftrightarrow u = 0,$$

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \lim_{n\to\infty}\lim_{u\to\infty} \Phi_n\left(E_n(t,u)\right) = \infty$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

D. Properties of E-Orlicz Functions

Theorem 36. Let $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be E-Orlicz functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then $\Phi_1 + \Phi_2$ and $c\Phi_1, c \geq 0$ are E-Orlicz functions with respect to E.

Theorem 37. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a linear E-Orlicz function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Orlicz function with respect to $E_1 \circ E_2$ and $E_2 \circ E_1$.

Theorem 38. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a linear E-Orlicz function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Orlicz function with respect to $E_1 + E_2$ and $cE_1, c \geq 0$.

Theorem 39. Let $\Phi_i: \Omega \times [0,\infty) \longrightarrow \mathbb{R}, i = 1,\cdots,n$ be E-Orlicz functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then $\Phi = \max_i \Phi_i$ is an E-Orlicz function with respect to E.

Theorem 40. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-Orlicz function with respect to $E_i: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty), i = 1,\cdots,n$. Then Φ is an E-Orlicz function with respect to $E_M = \max_i E_i$ and $E_m = \min_i E_i$.

Theorem 41. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-Orlicz functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$. Then Φ is an E-Orlicz function withrespect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-Orlicz functions with respect to a map E such that $\Phi_n \longrightarrow \Phi$ uniformly on a compact set $\Omega \times [0,\infty)$ and Φ is continuous on $\Omega \times [0,\infty)$. Then $\Phi_n(E) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\left(E(t,u)\right)$$

is convex of u on $[0,\infty)$,

$$\Phi\big(E(t,0)\big) = \lim_{n\to\infty} \Phi_n\left(E(t,0)\right) = 0,$$

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \lim_{n\to\infty}\lim_{u\to\infty} \Phi_n\left(E(t,u)\right) = \infty,$$

$0 < \Phi\big(E(t,u)\big) < \infty$ for any $u \in (0,\infty)$, $\Phi\big(E(t,u)\big)$ is left continuous at

$$U_\Phi = \sup\{u > 0: \Phi\big(E(t,u)\big) < +\infty\}.$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 42. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be a continuous E-Orlicz function defined on a compact set $\Omega \times [0,\infty)$ with respect to a sequence of maps $(E_n)_{n\in\mathbb{N}}, E_n: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that $(E_n)_{n\in\mathbb{N}}$ converges uniformaly to a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Orlicz function with respect to E.

Proof. Suppose that Φ is a continuous E-Orlicz function with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}$ such that $E_n \longrightarrow E$ uniformaly on a compact set $\Omega \times [0,\infty)$ and E is continuous on $\Omega \times [0,\infty)$. Then $\Phi(E_n) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi\big(E_n(t,u)\big)$$

is convex of u on $[0,\infty)$,

$$\Phi\big(E(t,0)\big) = \lim_{n\to\infty} \Phi\big(E_n(t,0)\big) = 0,$$

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \lim_{n\to\infty}\lim_{u\to\infty} \Phi\big(E_n(t,u)\big) = \infty,$$

$0 < \Phi\big(E(t,u)\big) < \infty$ for any $u \in (0,\infty)$ and $\Phi\big(E(t,u)\big)$ is left continuous at

$$U_\Phi = \sup\{u > 0: \Phi\big(E(t,u)\big) < +\infty\}.$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

Theorem 43. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of continuous E-Orlicz functions defined on a compact set $\Omega \times [0,\infty)$ with respect to a sequence of continuous maps $(E_n)_{n\in\mathbb{N}}, E_n: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that $(\Phi_n)_{n\in\mathbb{N}}$ converges uniformaly to a continuous function $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ and $(E_n)_{n\in\mathbb{N}}$ con verges uniformaly to a continuous map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then Φ is an E-Orlicz function with respect to E.

Proof. Assume that $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of continuous E-Orlicz functions with respect to a sequence of continuous maps $(\Phi_n)_{n\in\mathbb{N}}$ such that $\Phi_n \longrightarrow \Phi$ and $E_n \longrightarrow E$ uniformly on a compact set $\Omega \times [0,\infty)$ and Φ and E are continuous on $\Omega \times [0,\infty)$. Then $\Phi_n(E_n) \longrightarrow \Phi(E)$ uniformaly on $\Omega \times [0,\infty)$ and for μ-a.e. $t \in \Omega$,

$$\Phi\big(E(t,u)\big) = \lim_{n\to\infty} \Phi_n\big(E_n(t,u)\big)$$

is convex of u on $[0,\infty)$,

$$\Phi\big(E(t,0)\big) = \lim_{n\to\infty} \Phi_n\big(E_n(t,0)\big) = 0,$$

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \lim_{n\to\infty}\lim_{u\to\infty} \Phi_n\big(E_n(t,u)\big) = \infty,$$

$0 < \Phi\big(E(t,u)\big) < \infty$ for any $u \in (0,\infty)$, $\Phi(E)$ is left continuous at

$$U_\Phi = \sup\{u > 0: \Phi\big(E(t,u)\big) < +\infty\}.$$

and for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω.

IV. RELATIONSHIPS BETWEEN E-CONVEX FUNCTIONS

In this section, we generalize the theorems in [9] to consider the relationships between E-N-functions, E-Young functions, E-strong Young functions and E-Orlicz functions.

Theorem 44. If Φ is an E-N-function, then Φ is an E-strong Young function.

Proof. Assume $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an E-N-function with a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. So, for μ-a.e. $t \in \Omega$, $\Phi\big(E(t,u)\big)$ is convex continuous of u on $[0,\infty)$ satisfying

$$\forall \varepsilon > 0, \exists \delta > 0, 0 < u < \delta \Longrightarrow \left|\frac{\Phi\big(E(t,u)\big)}{u}\right| < \varepsilon$$

because

$$\lim_{u\to 0^+} \frac{\Phi\big(E(t,u)\big)}{u} = 0.$$

Letting $\delta < 1$, we get

$$0 \leq \left|\Phi\big(E(t,u)\big)\right| < \left|\frac{\Phi\big(E(t,u)\big)}{\delta}\right| < \left|\frac{\Phi\big(E(t,u)\big)}{u}\right| < \varepsilon.$$

By the squeeze theorem for functions, we get $\Phi\big(E(t,0)\big) = 0 \Leftrightarrow u = 0$ because Φ is continuous at $u = 0$ and $\Phi\big(E(t,u)\big) > 0$ for any $u \in (0,\infty)$. Moreover,

$$\forall M \in \mathbb{R}, \exists u_M > 0, u > u_M \Longrightarrow \frac{\Phi\big(E(t,u)\big)}{u} > M$$

because

$$\lim_{u\to\infty} \frac{\Phi\big(E(t,u)\big)}{u} = \infty.$$

Taking $u_M > 1$, we have that

$$\Phi\big(E(t,u)\big) > Mu > Mu_M > M.$$

That is,

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \infty.$$

Furthermore, for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω which completes the proof.

Remark 45. The converse of theorem 44 is not correct. That is, an E-strong Young function may not be an E-N-function. For example, Let the function $\Phi: \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ be defined as $\Phi(t,u) = e^{u^t} - 1$ with the map $E: \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R} \times [0,\infty)$ defined by $E(t,u) = (1,u)$. Then Φ is an E-strong Young function but it is not an E-N-function because for μ-a.e. $t \in \mathbb{R}$,

$$\lim_{u\to 0} \frac{e^{u}-1}{u} = 1 \neq 0.$$

Theorem 46. If Φ is an E-strong Young function, then Φ is an E-Orlicz function.

Proof. Suppose that $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an E-strong Young function with a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then for μ-a.e. $t \in \Omega, \Phi\big(E(t,u)\big)$ is convex continuous of u on $[0,\infty)$ satisfying $\Phi\big(E(t,0)\big) = 0, \Phi\big(E(t,u)\big) > 0$ for any $u \in (0,\infty)$ because $\Phi\big(E(t,0)\big) = 0 \Leftrightarrow u = 0$ and

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \infty$$

and $\Phi\big(E(t,u)\big)$ is left continuous at $U_\Phi = +\infty$ because

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \infty.$$

Moreover, for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω. Hence, Φ is an E-Orlicz function.

Remark 47. The converse of theorem 46 is not correct. That is, not every E-strong Young function is an E-Orlicz function. For instance, let the function $\Phi: \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R}$ be defined as

$$\Phi(t,u) = \begin{cases} u - |t|, 0 \leq u < 1 \\ u + |t| - 2, 1 \leq u \end{cases}$$

with a map $E: \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{R} \times [0,\infty)$ defined by $E(t,u) = (u,u)$. Then Φ is an E-Orlicz function but it is not an E-strong Young function because, for μ-a.e. $t \in \Omega, \Phi\big(E(t,1)\big) = 0$.

Theorem 48. If Φ is an E-Orlicz function, then Φ is an E-Young function.

Proof. Assume that $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an E-Orlicz function with a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. Then, for μ-a.e. $t \in \Omega$, $\Phi\big(E(t,u)\big)$ is convex of u on $[0,\infty)$ satisfying $\Phi\big(E(t,0)\big) = 0$, $0 < \Phi\big(E(t,u)\big), u \in (0,\infty)$,

$$\lim_{u\to\infty} \Phi\big(E(t,u)\big) = \infty,$$

and $\Phi\big(E(t,u)\big)$ is left continuous at U_Φ. We only need to show that

$$\lim_{u\to 0^+} \Phi\big(E(t,u)\big) = 0.$$

In other words, we need to prove that

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, 0 < u < \delta_\varepsilon \Longrightarrow 0 \leq \Phi\big(E(t,u)\big) < \varepsilon.$$

For arbitrary $\varepsilon > 0$, consider

$$a_\Phi = inf\{u > 0 : \Phi\big(E(t,u)\big) > 0\}.$$

If $a_\Phi > 0$, then $\Phi\big(E(t,u)\big) = 0$ for all $u \in (0, a_\Phi)$. Taking $\delta_\varepsilon = a_\Phi > 0$, then $\Phi\big(E(t,u)\big) = 0 < \varepsilon$ for all $0 < u < \delta_\varepsilon$. That is,

$$\lim_{u\to 0^+} \Phi\big(E(t,u)\big) = 0$$

If $a_\Phi = 0$, then $\Phi\big(E(t,u)\big) > 0$ for all $u > 0$ and there exists $0 < u_0 < \infty$ such that $0 < \Phi\big(E(t,u_0)\big) < \infty$. That is, for all $\varepsilon > 0, \exists u_\varepsilon \in (0,\infty)$ such that $0 < \Phi\big(E(t,u_\varepsilon)\big) < \infty$. If $\Phi\big(E(t,u_0)\big) < \varepsilon$, then $\Phi\big(E(t,u_\varepsilon)\big) < \infty$ for $u_\varepsilon = u_0$ and if $\Phi\big(E(t,u_0)\big) \geq \varepsilon$, then for $u_\varepsilon = \alpha u_0$, where $0 \leq \alpha = \frac{\varepsilon}{2\Phi(E(t,u_0))} < 1$, that

$$\Phi\big(E(t,u_\varepsilon)\big) = \Phi\big(E(t,\alpha u_0)\big) \leq \alpha \Phi\big(E(t,u_0)\big) \leq \frac{\varepsilon}{2} < \varepsilon$$

because Φ is E-convex of u on $[0,\infty)$. Taking $\delta_\varepsilon = u_\varepsilon > 0$, we get, for $0 < u < \delta_\varepsilon$,

$$0 \leq \Phi\big(E(t,u)\big) \leq \Phi\big(E(t,\delta_\varepsilon)\big) = \Phi\big(E(t,u_\varepsilon)\big) < \varepsilon,$$

because $\Phi\big(E(t,u)\big)$ is increasing of u on $[0,\infty)$. Furthermore, for each $u \in [0,\infty), \Phi\big(E(t,u)\big)$ is an μ-measurable function of t on Ω. Hence, Φ is an E-Young function.

Remark 49. The converse of theorem 48 is not correct. That is, not every E-Young function is an E-Orlicz function. For example, let the function $\Phi: [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be defined as

$$\Phi(t,u) = \begin{cases} -\log\left(u + |t|^{\frac{1}{p}} + 1\right), 0 \leq u < 1 \\ \quad +\infty, \qquad\qquad 1 \leq u \end{cases}$$

with a map $E: [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty)$ defined by $E(t,u) = (u^p, u), p \geq 1$. Then Φ is an E-Young function but it is not an E-Orlicz function because $\Phi\big(E(t,u)\big)$ is not left continuous at $U_\Phi = 1$, where

$$\lim_{u\to 1} \Phi\big(E(t,u)\big) = -\log(3) \neq +\infty = \Phi\big(E(t,1)\big).$$

Corollary 50. E-N-function $\Rightarrow$ E-strong Young function $\Rightarrow$ E-Orlicz function $\Rightarrow$ E-Young function.

Corollary 51. E-N-function $\nLeftarrow$ E-strong Young function $\nLeftarrow$ E-Orlicz function $\nLeftarrow$ E-Young function.

V. MAIN RESULTS

In this section, we are going to study a class of Orlicz spaces equiped by E-luxemburg norms and generated by E-Young functions and then we establish their inclusion properties.

Lemma 52. Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \le E_2(t,x)$. Then, for μ-a.e. $t \in \Omega$, $\Phi(E_1(t,x)) \le \Phi(E_2(t,x))$.

Lemma 53. Let $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega, \Phi_1(t,x) \le \Phi_2(t,x)$. So, for μ-a.e. $t \in \Omega$, $\Phi_1(E(t,x)) \le \Phi_2(E(t,x))$.

A. *E-Orlicz Spaces and Weak E-Orlicz Spaces*

Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-Young function with respect to a map $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. The E-Orlicz space generated by Φ is defined by

$$EL_{\Phi(E)}(\Omega,\Sigma,\mu) = \{f \in BS_\Omega : \|f\|_{\Phi(E)} < \infty\},$$

$$\|f\|_{\Phi(E)} = \inf\left\{\lambda > 0: \int_\Omega \Phi\left(E\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu \le 1\right\}$$

and the weak E-Orlicz space generated by Φ is

$$EL_{\Phi(E),weak}(\Omega,\Sigma,\mu) = \{f \in BS_\Omega : \|f\|_{\Phi(E),weak} < \infty\},$$

$$\|f\|_{\Phi(E),weak} = \inf\left\{\lambda > 0: \sup_u \Phi(E(t,u))\, m(\Omega, f/\lambda, u) \le 1\right\},$$

where BS_Ω is the set of all μ-measurable functions f from Ω to BS such that $(BS, \|\cdot\|_{BS})$ is a Banach space and

$$m(\Omega, f, u) = \mu\{t \in \Omega: \| f(t) \|_{BS} > u\}.$$

Example 54. We have seen from example 8-i that $\Phi(t,u) = e^{t+u} - 1$ is an E-Young function with respect to the map $E(t,u) = (u,u)$. Then the E-Orlicz space and the weak E-Orlicz space generated by $\Phi(E(t,u)) = e^{2u} - 1$ are equipped with the norm

$$\|f\|_{\Phi(E)} = \inf\left\{\lambda > 0: \int_\Omega \left(exp\left(\frac{2 \| f(t) \|_{BS}}{\lambda}\right) - 1\right) d\mu \le 1\right\},$$

for all $f \in EL_{\Phi(E)}(\Omega,\Sigma,\mu)$ and

$$\|f\|_{\Phi(E),weak} = \inf\left\{\lambda > 0: \sup_u (e^{2u} - 1)\, m(\Omega, f/\lambda, u) \le 1\right\}$$

for all $f \in EL_{\Phi(E),weak}(\Omega,\Sigma,\mu)$.

If $\Phi_p(E(t,u)) = u^p, p \ge 1$, we get

$$EL_p(\Omega,\Sigma,\mu) = EL_{\Phi_p(E)}(\Omega,\Sigma,\mu) = \{f \in X_\Omega : \|f\|_p < \infty\},$$

$$\|f\|_p = \inf\left\{\lambda > 0: \int_\Omega \left|\frac{f}{\lambda}\right|^p d\mu \le 1\right\} = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

for all $f \in EL_p(\Omega,\Sigma,\mu)$ and

$$EL_{p,weak}(\Omega,\Sigma,\mu) = EL_{\Phi_p,weak}(\Omega,\Sigma,\mu) = \{f \in X_\Omega : \|f\|_{p,weak} < \infty\},$$

$$\|f\|_{p,weak} = \inf\left\{\lambda > 0: \sup_u u^p\, m(\Omega, f/\lambda, u) \le 1\right\}.$$

Example 55. Let $\Phi: \mathbb{C} \times [0,\infty) \longrightarrow \mathbb{R}$ be defined as

$$\Phi(t,u) = \begin{cases} t\ln(u), & u > 1 \\ 0, & 0 \le u \le 1 \end{cases}$$

with respect to $E: \mathbb{C} \times [0,\infty) \longrightarrow \mathbb{C} \times [0,\infty)$ such that

$$E(t,u) = \begin{cases} (1, e^{u^p}), & 1 \le p, \\ (1,0), & 1 < u, p = +\infty, \\ (0,0), & 0 \le u \le 1, p = +\infty. \end{cases}$$

Then, for μ-a.e. $t \in \mathbb{C}$, that

$$\Phi(E(t,u)) = \begin{cases} u^p, & 1 \le p, \\ +\infty, & 1 < u, p = +\infty, \\ 0, & 0 \le u \le 1, p = +\infty \end{cases}$$

is an E-Young function and the obtained spaces are $EL_p(\Omega,\Sigma,\mu)$ and $EL_{p,weak}(\Omega,\Sigma,\mu)$ for $1 \le p \le \infty$.

Theorem 56. If $\Phi: \Omega \times [0,\infty) \to \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \le E_2(t,x)$. Then

$$EL_{\Phi(E_2)}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1)}(\Omega,\Sigma,\mu)$$

and

$$EL_{\Phi(E_2),weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),weak}(\Omega,\Sigma,\mu).$$

Theorem 57. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \to \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1(E(t,x)) \le \Phi_2(E(t,x))$. Then

$$EL_{\Phi_2(E)}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E)}(\Omega,\Sigma,\mu)$$

and

$$EL_{\Phi_2(E),weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E),weak}(\Omega,\Sigma,\mu).$$

Theorem 58. If $\Phi: \Omega \times [0,\infty) \to \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \le E_2(t,x)$. Then

$$EL_{\Phi(E_2)}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),weak}(\Omega,\Sigma,\mu)$$

and if $\Omega \times [0,\infty)$ is compact, then

$$EL_{\Phi(E_2),weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1)}(\Omega,\Sigma,\mu).$$

Proof. Let $f \in EL_{\Phi(E_2)}(\Omega,\Sigma,\mu)$ and let Φ be an increasing E-Young function. Then, by Lemma 52, we have

$$\begin{aligned} \Phi(E_1(t,u)) m(\Omega, f/\lambda, u) &\le \Phi(E_2(t,u)) m(\Omega, f/\lambda, u) \\ &\le \int_{\left\{t \in \Omega: \frac{\|f(t)\|_{BS}}{\lambda} > u\right\}} \Phi\left(E_2\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu \\ &\le \int_\Omega \Phi\left(E_2\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu \le 1. \end{aligned}$$

Since u is arbitrary, we have

$$\sup_u \Phi(E_1(t,u))\, m(\Omega, f/\lambda, u) \le 1$$

and $f \in EL_{\Phi(E_1),weak}(\Omega,\Sigma,\mu)$ with

$$\|f\|_{\Phi(E_1),weak} \le \|f\|_{\Phi(E_2)}.$$

Let $f \in EL_{\Phi(E_2),weak}(\Omega,\Sigma,\mu)$ and assume that $\Omega \times [0,\infty)$ is compact. Then

$$\int_{\Omega} \Phi\left(E_1\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu$$
$$= \sup_u \Phi\big(E_1(t, \mathrm{u})\big)\, m(\Omega, f/\lambda, u)$$
$$\leq \sup_u \Phi\big(E_2(t, \mathrm{u})\big)\, m(\Omega, f/\lambda, u) \leq 1.$$

That is, $f \in EL_{\Phi(E_1),}(\Omega, \Sigma, \mu)$ with $\|f\|_{\Phi(E_1)} \leq \|f\|_{\Phi(E_2),weak}$.

Theorem 59. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1\big(E(t,x)\big) \leq \Phi_2\big(E(t,x)\big)$. Then

$$EL_{\Phi_2(E)}(\Omega, \Sigma, \mu) \subseteq EL_{\Phi_1(E),weak}(\Omega, \Sigma, \mu)$$

and if $\Omega \times [0,\infty)$ is compact, then

$$EL_{\Phi_2(E),weak}(\Omega, \Sigma, \mu) \subseteq EL_{\Phi_1(E)}(\Omega, \Sigma, \mu).$$

Proof. Let $f \in EL_{\Phi_2(E)}(\Omega, \Sigma, \mu)$. Then

$$\Phi_1\big(E(t,u)\big) m(\Omega, f/\lambda, u) \leq \Phi_2\big(E(t,u)\big) m(\Omega, f/\lambda, u)$$
$$\leq \int_{\left\{t\in\Omega: \frac{\|f(t)\|_{BS}}{\lambda} > u\right\}} \Phi_2\left(E\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu$$
$$\leq \int_{\Omega} \Phi_2\left(E\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu \leq 1.$$

Since u is arbitrary, we have

$$\sup_u \Phi_1\big(E(t,u)\big)\, m(\Omega, f/\lambda, u) \leq 1$$

and $f \in EL_{\Phi(E_1),weak}(\Omega, \Sigma, \mu)$ with

$$\|f\|_{\Phi(E_1),weak} \leq \|f\|_{\Phi(E_2)}.$$

Let $f \in EL_{\Phi_2(E),weak}(\Omega, \Sigma, \mu)$ and assume that $\Omega \times [0,\infty)$ is compact. Then

$$\int_{\Omega} \Phi_1\left(E\left(t, \frac{\| f(t) \|_{BS}}{\lambda}\right)\right) d\mu$$
$$= \sup_u \Phi_1\big(E(t, \mathrm{u})\big)\, m(\Omega, f/\lambda, u)$$
$$\leq \sup_u \Phi_2\big(E(t, \mathrm{u})\big)\, m(\Omega, f/\lambda, u) \leq 1.$$

That is, $f \in EL_{\Phi(E_1)}(\Omega, \Sigma, \mu)$ with $\|f\|_{\Phi_1(E)} \leq \|f\|_{\Phi_2(E),weak}$.

B. E-Orlicz-Sobolev Space and Weak E-Orlicz-Sobolev Space

Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-Young function with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$. The E-Orlicz-Sobolev space $EW^kL_{\Phi(E)}(\Omega, \Sigma, \mu)$ generated by $\Phi(E)$ is

$$EW^kL_{\Phi(E)}(\Omega, \Sigma, \mu) = \{f \in EL_{\Phi(E)}(\Omega, \Sigma, \mu): D^\alpha f \in EL_{\Phi(E)}(\Omega, \Sigma, \mu), \forall\, |\alpha| \leq k\},$$

$$\|f\|_{k,\Phi(E)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{\Phi(E)}$$

for all $f \in EW^kL_{\Phi(E)}(\Omega, \Sigma, \mu)$ and the weak E-Orlicz-Sobolev space is

$$EW^kL_{\Phi(E),weak}(\Omega, \Sigma, \mu) = \{f \in EL_{\Phi(E),weak}(\Omega, \Sigma, \mu): D^\alpha f \in EL_{\Phi(E),weak}(\Omega, \Sigma, \mu), \forall\, |\alpha| \leq k\},$$

$$\|f\|_{k,\Phi(E),weak} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{\Phi(E),weak}$$

for all $f \in EW^kL_{\Phi(E),weak}(\Omega, \Sigma, \mu)$.

If $\Phi_p\big(E(t,u)\big) = u^p, p \geq 1$, we get the E-Sobolev space

$$EW^kL_{\Phi_p(E)}(\Omega, \Sigma, \mu) = EW^{k,p}(\Omega, \Sigma, \mu) = \{f \in EL_p(\Omega, \Sigma, \mu): D^\alpha f \in EL_p(\Omega, \Sigma, \mu), \forall\, |\alpha| \leq k\}$$

equipped with the norm

$$\|f\|_{k,p} = \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|_p\right)^{\frac{1}{p}}$$

for all $f \in EL_p(\Omega, \Sigma, \mu)$ and the weak E-Sobolev space

$$EW^kL_{\Phi_p(E),weak}(\Omega, \Sigma, \mu) = EW^{k,p,weak}(\Omega, \Sigma, \mu) = \{f \in EL_{p,weak}(\Omega, \Sigma, \mu): D^\alpha f \in EL_{p,weak}(\Omega, \Sigma, \mu), \forall\, |\alpha| \leq k\},$$

$$\|f\|_{k,p,weak} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{p,weak}$$

for all $f \in EW^kL_{p,weak}(\Omega, \Sigma, \mu)$.

Theorem 60. If $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \leq E_2(t,x)$. Then

$$EW^kL_{\Phi(E_2)}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi(E_1)}(\Omega, \Sigma, \mu)$$

and

$$EW^kL_{\Phi(E_2),weak}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi(E_1),weak}(\Omega, \Sigma, \mu).$$

Theorem 61. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1\big(E(t,x)\big) \leq \Phi_2\big(E(t,x)\big)$. Then

$$EW^kL_{\Phi_2(E)}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi_1(E)}(\Omega, \Sigma, \mu)$$

and

$$EW^kL_{\Phi_2(E),weak}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi_1(E),weak}(\Omega, \Sigma, \mu).$$

Theorem 62. If $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e.$t \in \Omega$, $E_1(t,x) \leq E_2(t,x)$. Then

$$EW^kL_{\Phi(E_2)}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi(E_1),weak}(\Omega, \Sigma, \mu)$$

and if $\Omega \times [0,\infty)$ is a compact set, then

$$EW^kL_{\Phi(E_2),weak}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi(E_1)}(\Omega, \Sigma, \mu).$$

Theorem 63. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1\big(E(t,x)\big) \leq \Phi_2\big(E(t,x)\big)$. Then

$$EW^kL_{\Phi_2(E)}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi_1(E),weak}(\Omega, \Sigma, \mu)$$

and if $\Omega \times [0,\infty)$ is a compact set, then

$$EW^kL_{\Phi_2(E),weak}(\Omega, \Sigma, \mu) \subseteq EW^kL_{\Phi_1(E)}(\Omega, \Sigma, \mu).$$

C. E-Orlicz-Morrey Space and Weak E-Orlicz-Morrey Space

Let $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-convex function with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ and let $\phi: (0,\infty) \longrightarrow (0,\infty)$ be a function such that $\phi(r)$ is almost decreasing and $\phi(r)r^n$ is almost increasing and let B denote the ball $B(a,r) = \{t \in \Omega: |t-a| < r\}$. The E-Orlicz-Morrey space is

$$EL_{\Phi(E),\phi}(\Omega, \Sigma, \mu) = \{f \in X_\Omega: \|f\|_{\Phi(E),\phi} < \infty\},$$

$$\|f\|_{\Phi(E),\phi} = \sup_B \inf\left\{ \lambda > 0: \frac{1}{|B|\phi(r)} \int_B \Phi\left(E\left(t, \frac{\|f(t)\|_{BS}}{\lambda}\right)\right) d\mu \quad \le 1 \right\},$$

and the weak E-Orlicz-Morrey space is

$$EL_{\Phi(E),\phi,weak}(\Omega,\Sigma,\mu) = \{f \in X_\Omega: \|f\|_{\Phi(E),\phi,weak} < \infty\},$$

$$\|f\|_{\Phi(E),\phi,weak} = \sup_B \inf\left\{\lambda > 0: \sup_u \frac{\Phi(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le 1\right\}.$$

If $\Phi_p(E(t,u)) = u^p, p \ge 1$, then

$$EL_{\Phi_p(E),\phi}(\Omega,\Sigma,\mu) = EL_{p,\phi}(\Omega,\Sigma,\mu) = \{f \in X_\Omega: \|f\|_{p,\phi} < \infty\},$$

$$\|f\|_{p,\phi} = \sup_B \left(\frac{1}{|B|\phi(r)} \int_B \|f(t)\|_{BS}^p d\mu\right)^{\frac{1}{p}},$$

$$EL_{\Phi_p(E),\phi,weak}(\Omega,\Sigma,\mu) = EL_{p,\phi,weak}(\Omega,\Sigma,\mu) = \{f \in X_\Omega: \|f\|_{p,\phi,weak} < \infty\},$$

$$\|f\|_{p,\phi,weak} = \sup_B \sup_u \frac{u^p m(B,f,u)}{|B|\phi(r)}.$$

If $\phi(r) = r^{-n}$, we get

$$EL_{\Phi(E),\phi}(\Omega,\Sigma,\mu) = EL_{\Phi(E)}(\Omega,\Sigma,\mu),$$

$$EL_{\Phi(E),\phi,weak}(\Omega,\Sigma,\mu) = EL_{\Phi(E),weak}(\Omega,\Sigma,\mu).$$

If $\Phi_p(E(t,u)) = u^p, p \ge 1$ and $\phi(r) = r^{\lambda-n}$, we get the Morrey space

$$EL_{\Phi_p(E),\phi}(\Omega,\Sigma,\mu) = EL_{p,\lambda}(\Omega,\Sigma,\mu) = \{f \in X_\Omega: \|f\|_{p,\lambda} < \infty\},$$

$$\|f\|_{p,\lambda} = \sup_B \left(\frac{1}{r^\lambda} \int_B \|f(t)\|_{BS}^p d\mu\right)^{\frac{1}{p}}$$

and the weak Morrey space is

$$EL_{\Phi_p(E),\phi,weak}(\Omega,\Sigma,\mu) = EL_{p,\lambda,weak}(\Omega,\Sigma,\mu) = \{f \in X_\Omega: \|f\|_{p,\lambda,weak} < \infty\},$$

$$\|f\|_{p,\lambda,weak} = \sup_B \sup_u \frac{u^p m(B,f,u)}{r^\lambda}.$$

Theorem 64. If $\Phi: \Omega \times [0,\infty) \to \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \le E_2(t,x)$. Then

$$EL_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu)$$

and

$$EL_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu).$$

Theorem 65. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \to \mathbb{R}$ are E-Young function with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1(E(t,x)) \le \Phi_2(E(t,x))$. Then

$$EL_{\Phi_2(E),\phi}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E),\phi}(\Omega,\Sigma,\mu)$$

and

$$EL_{\Phi_2(E),\phi,weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E),\phi,weak}(\Omega,\Sigma,\mu).$$

Theorem 66. If $\Phi: \Omega \times [0,\infty) \to \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \le E_2(t,x)$. Then

$$EL_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu)$$

and if $\Omega \times [0,\infty)$ is a compact set, then

$$EL_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu).$$

Proof. Let $f \in EL_{\Phi(E_2),\phi}(\Omega,\Sigma,\mu)$ and let Φ be an increasing E-Young function. By Lemma 52, we have

$$\frac{\Phi(E_1(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \frac{\Phi(E_2(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \frac{1}{|B|\phi(r)} \int_B \Phi\left(E_2\left(t, \frac{\|f(t)\|_{BS}}{\lambda}\right)\right) d\mu \le 1.$$

Since u is arbitrary, then

$$\sup_{u>0} \frac{\Phi(E_1(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le 1$$

and $f \in EL_{\Phi(E_1),\phi,weak}(\Omega,\Sigma,\mu)$ with

$$\|f\|_{\Phi(E_1),\phi,weak} \le \|f\|_{\Phi(E_2),\phi}.$$

Let $f \in EL_{\Phi(E_2),\phi,weak}(\Omega,\Sigma,\mu)$ and $\Omega \times [0,\infty)$ be a compact set. Then

$$\frac{1}{|B|\phi(r)} \int_B \Phi\left(E_1\left(t, \frac{\|f(t)\|_{BS}}{\lambda}\right)\right) d\mu = \sup_{u>0} \frac{\Phi(E_1(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \sup_{u>0} \frac{\Phi(E_2(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le 1.$$

So, $f \in EL_{\Phi(E_1),\phi}(\Omega,\Sigma,\mu)$ with

$$\|f\|_{\Phi(E_1)} \le \|f\|_{\Phi(E_2),weak}.$$

Theorem 67. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1(E(t,x)) \le \Phi_2(E(t,x))$. Then

$$EL_{\Phi_2(E),\phi}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E),\phi,weak}(\Omega,\Sigma,\mu)$$

and if $\Omega \times [0,\infty)$ is compact set, then

$$EL_{\Phi_2(E),\phi,weak}(\Omega,\Sigma,\mu) \subseteq EL_{\Phi_1(E),\phi}(\Omega,\Sigma,\mu).$$

Proof. Let $f \in EL_{\Phi_2(E),\phi}(\Omega,\Sigma,\mu)$. Then

$$\frac{\Phi_1(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \frac{\Phi_2(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \frac{1}{|B|\phi(r)} \int_B \Phi_2\left(E\left(t, \frac{\|f(t)\|_{BS}}{\lambda}\right)\right) d\mu \le 1.$$

Since u is arbitrary, we have

$$\sup_{u>0} \frac{\Phi_1(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le 1$$

and $f \in EL_{\Phi_1(E),\phi,weak}(\Omega,\Sigma,\mu)$ with

$$\|f\|_{\Phi_1(E),\phi,weak} \le \|f\|_{\Phi_2(E),\phi}.$$

Let $f \in EL_{\Phi_2(E),\phi,weak}(\Omega,\Sigma,\mu)$ and $\Omega \times [0,\infty)$ be a compact set. Then

$$\frac{1}{|B|\phi(r)} \int_B \Phi_1\left(E\left(t, \frac{\|f(t)\|_{BS}}{\lambda}\right)\right) d\mu = \sup_{u>0} \frac{\Phi_1(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le \sup_{u>0} \frac{\Phi_2(E(t,u))m(B,f/\lambda,u)}{|B|\phi(r)} \le 1.$$

So, $f \in EL_{\Phi_1(E),\phi}(\Omega,\Sigma,\mu)$ with

$$\|f\|_{\Phi_1(E)} \le \|f\|_{\Phi_2(E),weak}.$$

D. E-Orlicz-Lorentz Spaces

Let $\Phi: [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be an E-convex function with respect to $E: (0,\infty) \times [0,\infty) \longrightarrow (0,\infty) \times [0,\infty)$ and let ω: $[0,\infty) \longrightarrow [0,\infty)$ be a weight function and $W(t) = \int_0^t \omega(s)ds$. The E-Orlicz-Lorentz space is:

$$\Lambda_{\omega,\Phi(E)} = \{f \in X_{(0,\infty)}: \|f\|_{\omega,\Phi(E)} < \infty\},$$

$$\|f\|_{\omega,\Phi(E)} = \inf\left\{\lambda > 0: \int_0^\infty \Phi\big(E(t, f^*(t)/\lambda)\big)W(t)d\mu \leq 1\right\},$$

and the weak E-Orlicz-Lorentz space is

$$\Lambda_{\omega,\Phi(E),weak} = \{f \in X_{(0,\infty)}: \|f\|_{\omega,\Phi(E),weak} < \infty\},$$

$$\|f\|_{\omega,\Phi(E),weak} = \inf\{\lambda > 0: \Phi\big(E(t, f^*(t)/\lambda)\big)W(t) \leq 1\},$$

$$f^*(t) = \sup\{u: \mu(|f| \geq u) \geq t\}$$

for all $f \in \Lambda_{\omega,\Phi(E)}$.

If $\omega(t) = 1$ for $t \in (0,\infty)$, then

$$\Lambda_{\omega,\Phi(E)}(\Omega,\Sigma,\mu) = EL_{\Phi(E)}(\Omega,\Sigma,\mu),$$

$$\Lambda_{\omega,\Phi(E),weak}(\Omega,\Sigma,\mu) = EL_{\Phi(E),weak}(\Omega,\Sigma,\mu).$$

If $\Phi\big(E(t,u)\big) = u^p$ for $1 \leq p < \infty$, we get the Lorentz space

$$\Lambda_{\omega,\Phi(E)}(\Omega,\Sigma,\mu) = EL_{\omega,p}(\Omega,\Sigma,\mu),$$

and the weak Lorentz space

$$\Lambda_{\omega,\Phi(E),weak}(\Omega,\Sigma,\mu) = EL_{\omega,p,weak}(\Omega,\Sigma,\mu).$$

And if $\omega(t) = 1, t \in (0,\infty)$, and $\Phi\big(E(t,u)\big) = u^p, 1 \leq p < \infty$, then

$$\Lambda_{\omega,\Phi(E)}(\Omega,\Sigma,\mu) = EL_p(\Omega,\Sigma,\mu),$$

$$\Lambda_{\omega,\Phi(E),weak}(\Omega,\Sigma,\mu) = EL_{p,weak}(\Omega,\Sigma,\mu).$$

Theorem 68. If $\Phi: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is an increasing E-Young function with respect to $E_1, E_2: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $E_1(t,x) \leq E_2(t,x)$. Then

$$\Lambda_{\omega,\Phi(E_2)}(\Omega,\Sigma,\mu) \subseteq \Lambda_{\omega,\Phi(E_1)}(\Omega,\Sigma,\mu)$$

and

$$\Lambda_{\omega,\Phi(E_2),weak}(\Omega,\Sigma,\mu) \subseteq \Lambda_{\omega,\Phi(E_1),weak}(\Omega,\Sigma,\mu).$$

Theorem 69. If $\Phi_1, \Phi_2: \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ are E-Young functions with respect to $E: \Omega \times [0,\infty) \longrightarrow \Omega \times [0,\infty)$ such that, for μ-a.e. $t \in \Omega$, $\Phi_1\big(E(t,x)\big) \leq \Phi_2\big(E(t,x)\big)$. Then

$$\Lambda_{\omega,\Phi_2(E)}(\Omega,\Sigma,\mu) \subseteq \Lambda_{\omega,\Phi_1(E)}(\Omega,\Sigma,\mu)$$

and

$$\Lambda_{\omega,\Phi_2(E),weak}(\Omega,\Sigma,\mu) \subseteq \Lambda_{\omega,\Phi_1(E),weak}(\Omega,\Sigma,\mu).$$

VI. CONCLUSION

We have shown that the non N-functions, non Young functions, non strong Young functions and non Orlicz functions can be transferred using the E-convex theory to E-N-functions, E-Young functions, E-strong Young functions and E-Orlicz functions respectively. We also have shown that the Orlicz spaces can be generated by non-Young functions but E-Young functions with an appropriate map E to extend and generalize studying the classical Orlicz theory. Moreover, we have considered the inclusion properties of E-Orlicz spaces based on effects of the map E.

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