Estimation of Error of Approximation in $Lip(\rho(t), r)$ Class by $(N_q^p.C_1)$ Transform

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***Abstract*—In this paper, error of approximation of a function in $Lip(\rho(t), r)$ class using $(N_q^p.C_1)$ transform of Fourier series is estimated. Some important corollaries have also been derived from our main theorem.**

***Index Terms*—Best Approximation, Cesàro means, Lipschitz class, Norlud means, $(N_q^p.C_1)$ transform, Fourier series.**

I. Introduction

STUDY of divergent series is the foundation of summability theory. The field of summability theory has become an active area of research due to its vast applications in mathematics such as in Fourier analysis, Approximation theory, Probability theory and Fixed point theory.
The study of best approximation of periodic functions in Lipschitz space using different summability methods has become an active and broad area of research. Several investigators like [1], [5], [7], [8], [11], [12], [13], [17], [18] have been studied best approximation of a function $h(x)$ in Lipschitz classes by a trigonometric polynomial using different single and product summability methods. Working in this direction, Kushwaha and Dhakal [15] investigated the degree of approximation of functions in $Lip(\alpha, r)$ class by $(N_{p,q}.C_1)$ product summability method of Fourier series. But the approximation of signals or functions in $Lip(\rho(t), r)$ class through $(N_q^p.C_1)$ product transform of Fourier series have not been studied so far. This motivated us to work in this direction. In fact, in the proposed work, a quite new theorem on degree of approximation of a function $h(x) \in Lip(\rho(t), r)$ class using trigonometric polynomial by $(N_q^p.C_1)$ product transform of Fourier series has been proved which generalizes many known results in this direction.

Let $L_{2\pi}$ be the space of 2π- periodic functions and $h \in L_{2\pi}$ is periodic integrable function in the sense of Lebesgue. Then the Fourier series of $h(x)$ is given by

$$h \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos jx + b_j \sin jx) \qquad (1)$$

with $(n+1)^{th}$ partial sum $s_n(h; x)$ which is also called trigonometric polynomial of n^{th} order (or degree) of Fourier series. Here a_j and b_j are Fourier coefficients.
L^r-norm of a function is defined by

$$\|h\|_r = \left(\int_0^{2\pi} |h(x)|^r dx\right)^{\frac{1}{r}}, 1 \leq r < \infty \qquad (2)$$

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A function (or signal) h is approximated by trigonometric polynomial $s_n(x)$ of order n and the degree of approximation $E_n(h)$ is defined by (Zygmund [10])

$$E_n(h) = \min_{t_n} \|s_n - h\|_r \qquad (3)$$

where $s_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n}(a_j \cos jx + b_j \sin jx)$ is trigonometric polynomial of degree n of the first $(n+1)$ terms of the series (1). This method of approximation is called trigonometric Fourier approximation (TFA).
A function $h(x) \in Lip\alpha$ class, if

$$|h(x+t) - h(x)| = O(|t^\alpha|) \text{ for } 0 < \alpha \leq 1 \qquad (4)$$

and $h(x) \in Lip(\alpha, r)$ for $0 \leq x \leq 2\pi, r \geq 1$, if

$$\left(\int_0^{2\pi} |h(x+t) - h(x)|^r dx\right)^{\frac{1}{r}} = O(|t|^\alpha), 0 < \alpha \leq 1,$$

(definition 5.38 of Mc Fadden [6])

A function $h(x) \in Lip(\rho(t), r)$, if

$$\left(\int_0^{2\pi} |h(x+t) - h(x)|^r dx\right)^{\frac{1}{r}} = O(\rho(t)), r \geq 1 \qquad (5)$$

where $\rho(t)$ is a positive inceasing function. $Lip(\rho(t), r)$ class is the generalization of $Lip(\alpha, r)$ and $Lip\alpha$ classes. It can be easily seen that

$$Lip(\rho(t), r) \xrightarrow{\rho(t) = t^\alpha} Lip(\alpha, r) \xrightarrow{r \to \infty} Lip\alpha$$

Let $\sum_{j=0}^{\infty} u_j$ be an infinite series such that its n^{th} partial sum is given by $s_n = \sum_{k=0}^{n} u_k$. If

$$(C, 1) = C_1 = \sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \to s \quad \text{as} \quad n \to \infty \quad (6)$$

then the infinite series $\sum_{j=0}^{\infty} u_j$ is said to be summable by Cesàro method $(C, 1)$ to a definite number's (Hardy [4]).

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of non-zero real constants defined as

$$\begin{gathered} P_n = p_0 + p_1 + \text{...........} + p_n = \sum_{\nu=0}^{n} p_\nu \neq 0 \ \forall n \geq 0, \\ P_{-1} = p_{-1} = 0, \ P_n \to \infty, \text{ as } n \to \infty \end{gathered} \qquad (7)$$

$$\begin{gathered} Q_n = \ q_0 + q_1 + \text{..........} + q_n = \sum_{\nu=0}^{n} q_\nu \neq 0 \ \forall n \geq 0, \\ Q_{-1} = q_{-1} = 0, \ Q_n \to \infty, \text{as } n \to \infty \end{gathered} \qquad (8)$$

$$R_n = p_0 q_n + p_1 q_{n-1} + ... + p_n q_0 = \sum_{\nu=0}^{n} p_\nu q_{n-\nu} \to \infty, \text{ as } n \to \infty \tag{9}$$

The convolution of $\{p_n\}$ and $\{q_n\}$ is denoted by $(p^* q)$ and defined as

$$R_n = (p^* q)_n = \sum_{k=0}^{n} p_{n-k} q_k, \forall n \geq 0. \tag{10}$$

And we write

$$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k \tag{11}$$

If $R_n \neq 0, \forall n$, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.
If $t_n^{p,q} \to s$ as $n \to \infty$, then the infinite series $\sum_{j=0}^{\infty} u_j$ or sequence $\{s_n\}$ is summable to a definite number s by generalized Nörlund method (Borwein [2]) and is denoted by

$$t_n^{p,q} \to s \left(N_q^p\right), \quad as \quad n \to \infty$$

The necessary and sufficient conditions for (N, p, q) method to be regular are

$$\sum_{k=0}^{n} |p_{n-k} q_k| = O\left(|R_n|\right) \tag{12}$$

and $p_{n-k} = o\left(|R_n|\right)$, as $n \to \infty$ for every fixed $k \geq 0$, for which $q_k \neq 0$.

The product of (N_q^p) transform with C_1 transform defines $(N_q^p.C_1)$ transform and is given by

$$t_n^{p,q,c} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \sigma_{n-k} \tag{13}$$

If $t_n^{p,q,c} \to s$ as $n \to \infty$, then the infinite series $\sum_{j=0}^{\infty} u_j$ or the sequence $\{s_n\}$ is said to be summable to the sum s by $\left(N_p^q.C_1\right)$ transform and is denoted by

$$t_n^{p,q,c} \to s \left(N_p^q.C_1\right), \quad as \quad n \to \infty$$

Since $s_n \to s \Rightarrow C_1 = \sigma_n \to s$ as $n \to \infty \Rightarrow C_1$ is regular.
The regularity of $\left(N_q^p\right)$ and C_1 transforms implies the regularity of $\left(N_q^p.C_1\right)$ transform.

A. Particular Cases of $\left(N_q^p.C_1\right)$

Some important particular cases of $\left(N_q^p.C_1\right)$ transform are given below:
(i) If $q_n = 1, \forall n$, then we get $(N, p_n)(C, 1)$ transform. A special case, if we take $p_n = \binom{n+\delta-1}{\delta-1}, \delta > 0$, then $(N, p_n)(C, 1)$ transform further reduces to $(C, \delta)(C, 1)$.
(ii) If $p_n = 1, \forall n$, then we get $\left(\overline{N}, q_n\right)(C, 1)$ transform.

B. Example

[15] Let us consider the infinite series

$$\sum_{j=1}^{\infty} u_j = 1 + 4\sum_{j=1}^{\infty} j(-1)^j$$

The $n^t h$ partial sum of above series is given by

$$s_n = 1 + 4\sum_{j=1}^{n} j(-1)^j = (2j+1)(-1)^j$$

Applying $(C, 1)$ summability, it gives

$$\sigma_n = \frac{1}{n+1} \sum_{j=0}^{n} (2j+1)(-1)^j = (-1)^j 0.99999$$

This shows that the above infinite series is not summable by Cesàro method but $(-1)^j$ is summable by (N_q^p) method. Therefore, the series is $(N_q^p.C_1)$ summable. Hence the product summability transform $(N_q^p.C_1)$ is more stronger than indivisual summbility transforms $(C, 1)$ and (N_q^p) .

C. Remark 1

The product summability transforms are more powerful than single summability transforms as they give better approximation for wider class of functions. Some more examples of product summability transforms can be seen in [16].
We use the following notations throughout this paper:

$$\phi(t) = h(x+t) + h(x-t) - 2h(x)$$

$$G_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} \frac{p_k q_{n-k}}{n-k+1} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}$$

II. MAIN THEOREM

In this paper, we estimate the error between input signal $h(x)$ and output signal $t_n^{p,q,c}(h; x)$ by establishing the following result:

A. Theorem

Let $\left(N_q^p\right)$ be a regular generalized Nörlund transform. Let $\{p_n\}$ and $\{q_n\}$ be two non-negative, monotonic, non-increasing sequences of real constant such that

$$\sum_{k=0}^{n} \frac{p_k q_{n-k}}{n-k+1} = O\left(\frac{R_n}{n+1}\right), \ \forall n \geq 0. \tag{14}$$

Suppose $\rho(t)$ be a modulus of continuity such that

$$\int_0^\mu \frac{\rho(t)}{t} dt = O\left(\rho\left(\mu\right)\right), \text{ where } 0 < t < \mu. \tag{15}$$

If a function $h : [-\pi, \pi] \to R$ be a 2π- periodic, Lebesgue integrable, belongs to $Lip\left(\rho(t), r\right) (r \geq 1)$- class, then the degree of approximation of $h(x)$ by $\left(N_q^p.C_1\right)$ transform of Fourier series (1) is given by

$$\|t_n^{p,q,c}\left(h : x\right) - h(x)\|_{L^r_{(\alpha)}} = O\left(\frac{1}{(n+1)} \int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2} dt\right) \tag{16}$$

III. LEMMAS

In this section, we discuss the following lemmas which are required for the proof of our theorem:

A. Lemma 1

$|G_n(t)| = O\left(\frac{1}{t}\right)$, for $0 \le t \le \frac{\pi}{n+1}$

Proof: For $0 \le t \le \frac{\pi}{n+1}$, $\sin nt \le n \sin t$,, $\sin\frac{t}{2} \ge \frac{t}{\pi}$, we have

$$\begin{aligned}
|G_n(t)| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right| \\
&\le \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1}\right. \\
&\left.\frac{(n-k+1)\sin\frac{t}{2}\sin(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right| \\
&\le \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1}\frac{(n-k+1)}{\frac{t}{\pi}}\right| \\
&\le \frac{1}{2\pi R_n}\sum_{k=0}^{n}p_{n-k}q_k \\
&= O\left(\frac{1}{t}\right)
\end{aligned}$$

■

B. Lemma 2

$$|G_n(t)| = O\left(\frac{1}{(n+1)t^2}\right), \frac{\pi}{n+1} < t \le \pi$$

Proof: For $\frac{\pi}{n+1} < t \le \pi$, $\sin^2(n-k+1)\frac{t}{2} \le 1$, and $\sin\frac{t}{2} \ge \frac{t}{\pi}$, we have

$$\begin{aligned}
G_n(t) &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\right| \\
&\le \frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1}\frac{\pi^2}{t^2} \text{ as } \sin^2(n-k+1)\frac{t}{2} \le 1 \\
&= \frac{\pi}{2R_n t^2}\sum_{k=0}^{n}\frac{p_{n-k}q_k}{n-k+1} \\
&= \frac{\pi}{2R_n t^2}O\left(\frac{R_n}{n+1}\right) \text{ by the hypothesis of the theorem} \\
&= O\left(\frac{1}{(n+1)t^2}\right)
\end{aligned}$$

■

C. Lemma 3

Let $h(x) \in Lip\,(\xi(t), r)$, $r \ge 1$, then

$$\left[\int_0^{2\pi}|\phi(x,t)|^r dx\right]^{\frac{1}{r}} = O(\rho(t))$$

Proof: Clearly,

$$\begin{aligned}
|\phi(x,t)| &= |h(x+t) + h(x-t) - 2h(x)| \\
&\le |h(x+t) - h(x)| + |h(x-t) - h(x)|
\end{aligned}$$

Then using Minkowski's inequality, we have

$$\begin{aligned}
&\left[\int_0^{2\pi}|\phi(x,t)|^r dx\right]^{\frac{1}{r}} \\
&\le \left[\int_0^{2\pi}\{|h(x+t)-h(x)| + |h(x-t)-h(x)|\}^r dx\right]^{\frac{1}{r}} \\
&\le \left[\int_0^{2\pi}\{|h(x+t)-h(x)|\}^r dx\right]^{\frac{1}{r}} + \\
&\quad \left[\int_0^{2\pi}\{|h(x-t)-h(x)|\}^r dx\right]^{\frac{1}{r}} \\
&= O(\rho(t)) + O(\rho(t)) \text{ using (5)} \\
&= O(\rho(t))
\end{aligned}$$

■

D. Lemma 4

Let $h(x,t) \in L_p([a,b]\times[c,d])$, $p \ge 1$. Then

$$\left\{\int_a^b\left|\int_c^d h(x,t)dt\right|^p dx\right\}^{\frac{1}{p}} \le \int_c^d\left(\int_a^b|h(x,t)|^p dx\right)^{\frac{1}{p}}dt$$

The above inequality is also known as generalized Minkowski's inequality [[10], p.19].

IV. Proof of the Theorem

Using integral representation of $s_n(h;x)$ of the Fourier series (1) and definition of $t_n^{p,q,c}(h:x)$ given in (13), we have

$$\begin{aligned}
&t_n^{p,q,c}(h;x) - h(x) \\
&= \frac{1}{R_n}\sum_{k=0}^{n}p_k q_{n-k}\{\sigma_{n-k}(x) - h(x)\} \\
&= \int_0^{\pi}\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\phi(x,t)dt \\
&\Rightarrow t_n^{p,q,c}(h;x) - h(x) = \int_0^{\pi}\phi(x,t)G_n(t)dt
\end{aligned}$$

which on applying Lemma 4, we have

$$\begin{aligned}
&\|t_n^{p,q,c}(h;x) - h(x)\|_{L^r_{(\alpha)}} \\
&= \left[\int_0^{2\pi}\left|t_n^{p,q,c}(h:x) - h(x)\right|^r dx\right]^{\frac{1}{r}} \\
&= \left[\int_0^{2\pi}\left|\int_0^{\pi}\phi(x,t)G_n(t)dt\right|^r dx\right]^{\frac{1}{r}} \\
&\le \int_0^{\pi}\left(\int_0^{2\pi}|\phi(x,t)|^r dx\right)^{\frac{1}{r}}|G_n(t)|dt \\
&\le \int_0^{\pi}O(\rho(t))|G_n(t)|\,dt \text{ using Lemma 3} \\
&\le O\left(\int_0^{\frac{\pi}{(n+1)}}(\rho(t))|G_n(t)|dt\right) + \\
&O\left(\int_{\frac{\pi}{(n+1)}}^{\pi}(\rho(t))|G_n(t)|dt\right) \\
&= I_1 + I_2 \text{ (say)}
\end{aligned} \tag{17}$$

in view of Lemma 1 and equation (15), we get

$$\begin{aligned}
I_1 &= O\left(\int_0^{\frac{\pi}{(n+1)}}(\rho(t))|G_n(t)|dt\right) \\
&= O\left(\int_0^{\frac{\pi}{(n+1)}}\frac{\rho(t)}{t}dt\right) \\
&= O\left(\rho\left(\frac{1}{n+1}\right)\right)
\end{aligned} \tag{18}$$

A. Remark 2

$$\rho\left(\frac{\pi}{n+1}\right) \leq \pi\rho\left(\frac{1}{n+1}\right), \text{ for} \frac{\pi}{n+1} \geq \frac{1}{n+1}$$

In order to evaluate I_2, we consider

$$I_2 = O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} (\rho(t))\,|G_n(t)|dt\right)$$

Using Lemma 2, we have

$$I_2 = O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{(n+1)t^2}dt\right)$$

$$= O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2}dt\right) \quad (19)$$

since

$$\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2}dt \geq \frac{\pi}{(n+1)}\rho\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{1}{t^2}dt$$

$$= \frac{\pi}{(n+1)}\rho\left(\frac{1}{(n+1)}\right)\left(\frac{-1}{t}\right)_{\frac{\pi}{(n+1)}}^{\pi}$$

$$= \frac{\pi}{(n+1)}\rho\left(\frac{1}{(n+1)}\right)\left(\frac{n+1}{\pi}\right)$$

$$\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2}dt \geq \xi\left(\frac{1}{(n+1)}\right) \quad (20)$$

inserting equation (20) in equation (19), we get

$$I_2 = \rho\left(\frac{1}{(n+1)}\right) = O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2}dt\right) \quad (21)$$

now combining the equations (17), (18) and (21), we get required result

$$\|t_n^{p,q,c}(h:x) - h(x)\|_{L^r_{(\alpha)}} = O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{\rho(t)}{t^2}dt\right) \quad (22)$$

This completes the proof of the theorem.

V. APPLICATIONS

Approximation theory has many applications such as approximation of functions (signals) using summability methods through trigonometric Fourier approximations are used to improve the quality of digital filters. These filters are used in signal, speech and image processing.
Following Corollaries can be derived from our main theorem:

A. Corollary 1

If $\frac{\rho(t)}{t}$ is non-increasing in $\left(\frac{\pi}{(n+1)}, \pi\right)$ in main theorem, then degree of approximation of $h(x) \in Lip\,(\rho(t), r)$ class by $\left(N_p^q.C_1\right)$ transform of Fourier series is given by

$$t_n^{p,q,c}(h:x) - h(x) = O\left(\rho\left(\frac{1}{(n+1)}\right)\log(n+1)\right)$$

Proof: Since it is given that $\frac{\rho(t)}{t}$ is non-increasing in $\left(\frac{\pi}{(n+1)}, \pi\right)$, so using this condition and second mean value theorem for integrals in (22), we get

$$t_n^{p,q,c}(h:x) - h(x) = O\Bigg(\frac{1}{(n+1)}(n+1).$$

$$\rho\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{1}{t}dt\Bigg)$$

$$t_n^{p,q,c}(h:x) - h(x) = O\left(\rho\left(\frac{1}{(n+1)}\right)\log(n+1)\right) \quad (23)$$

as we know that

$$\rho\left(\frac{\pi}{(n+1)}\right) \leq \pi\rho\left(\frac{1}{(n+1)}\right), \text{ for } \frac{\pi}{(n+1)} \geq \frac{1}{(n+1)}$$

This completes the proof of the corollary 1. ■

B. Corollary 2

If we put $\rho(t) = t^{\alpha}$, then $Lip\,(\rho(t), r)$ class reduces to $Lip\,(\alpha, r)$ class and the degree of approximation by $\left(N_q^p.C_1\right)$ transform is given by

$$t_n^{p,q,c}(h;x) - h(x) = \begin{cases} O(n+1)^{\alpha},\ 0<\alpha<1 \\ O\left(\frac{\log(n+1)}{n+1}\right),\ \alpha=1 \end{cases}$$

Proof: Putting $\rho(t) = t^{\alpha}$ in main theorem, we get

$$t_n^{p,q,c}(h;x) - h(x) = O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{t^{\alpha}}{t^2}dt\right)$$

$$= O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} t^{\alpha-2}dt\right)$$

$$= O\left(\frac{1}{(n+1)}\left(\frac{t^{\alpha-1}}{(\alpha-1)}\right)_{\frac{\pi}{(n+1)}}^{\pi}\right)$$

$$= O\left(\frac{1}{(n+1)}\left(\frac{1}{n+1}\right)^{\alpha-1}\right)$$

$$= O(n+1)^{-\alpha} \text{ where } 0 < \alpha < 1. \quad (24)$$

If $\alpha = 1$, then

$$t_n^{p,q,c}(h:x) - h(x) = O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{t}{t^2}dt\right)$$

$$= O\left(\frac{1}{(n+1)}\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{1}{t}dt\right)$$

$$= O\left(\frac{1}{(n+1)}(\log t)_{\frac{\pi}{(n+1)}}^{\pi}\right)$$

$$= O\left(\frac{\log(n+1)}{(n+1)}\right) \quad (25)$$

From equations (24) and (25), we have

$$t_n^{p,q,c}(h;x) - h(x) = \begin{cases} O(n+1)^{\alpha},\ 0<\alpha<1 \\ O\left(\frac{\log(n+1)}{n+1}\right),\ \alpha=1 \end{cases}$$

This completes the proof of the corollary 2 and result of Kushwaha and Dhakal [15] become a particular case of of our result. ■

VI. CONCLUSION

Many important results on best approximation of functions belonging to various Lipschitz classes using different summability means have been reviewed. The result which we obtain in this paper is more general than the previous existing results. The result is very useful and can be extended in future by the researchers working in this direction.

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