# On the Chirp Function, the Chirplet Transform and the Optimal Communication of Information

Jonathan Blackledge

Abstract—The purpose of this extended paper is to provide a review of the chirp function and the chirplet transform and to investigate the application of chirplet modulation for digital communications, in particular, the transmission of binary strings. The significance of the chirp function in the solution to a range of fundamental problems in physics is revisited to provide a background to the case and to present the context in which the chirp function plays a central role, the material presented being designed to show a variety of problems with solutions and applications that are characterized by a chirp function in a fundamental way.

A study is then provided whose aim is to investigate the uniqueness of the chirp function in regard to its use for convolutional coding and decoding, the latter case (i.e. decoding) being related to the autocorrelation of the chirp function which provides a unique solution to the deconvolution problem. Complementary material in regard to the uniqueness of a chirp is addressed through an investigation into the selfcharacterization of the chirp function upon Fourier transformation. This includes a short study on the eigenfunctions of the Fourier transform, leading to a uniqueness conjecture which is based on an application of the Bluestein decomposition of a Fourier transform. The conjecture states that the chirp function is the only phase-only function to have a self-characteristic Fourier transform, and, for a specific scaling constant, a conjugate eigenfunction. In the context of this conjecture, we consider the transmission of information through a channel characterized by additive noise and the detection of signals with very low Signal-to-Noise Ratios. It is shown that application of chirplet modulation can provide a simple and optimal solution to the problem of transmitting binary strings through noisy communication channels, a result which suggests that all digital communication systems should ideally by predicated on the application of chirplet modulation.

In the latter part of the paper, a method is proposed for securing the communication of information (in the form of a binary string) through chirplet modulation that is based on prime number factorization of the chirplet (angular) bandwidth. Coupled with a quantum computer for factorizing very large prime numbers using Shor's algorithm, the method has the potential for designing a communications protocol specifically for users with access to quantum computing when the factorization of very large prime numbers is required. In this respect, and, in the final part of the paper, we investigate the application of chirplet modulation for communicating through the 'Water-Hole'. This includes the introduction of a method for distinguishing between genuine 'intelligible' binary strings through the Kullback-Leibler divergence which is shown to be statistically significant for a number of natural languages. Finally, a conjecture is developed in regard to focusing on the emission of intelligent signals from multiple star systems in the search for extraterrestrial intelligence. Prototype MATLAB code is given in the Appendix so that interested readers can reproduce some of the results given and modify and/or develop further the algorithms provided. The paper concludes with a number of open questions and some ideas for further investigation.

*Index Terms*—Chirp Function, Fourier Transform, Bluestein Decomposition, Fourier Eigenfunctions, Phase-only Functions, Chirplet Transform, Chirplet Modulation, Convolutional Coding, Encryption/Decryption, Bandwidth Factorization Key Exchange, Intelligibility of Binary Strings, Extraterrestrial Communications.

## I. INTRODUCTION

THE chirp function and the chirplet transformation are well known, having been studied and implemented in a wide range of information and communication engineering applications. These applications have their origins in work dating back to the 1950's and 1960's, in particular, the invention and patenting of chirp pulse based communications, e.g. [1], [2].

For a unit amplitude, the linear frequency modulated chirp is defined by a function of time t given by  $\exp(i\alpha t^2)$  where  $\alpha$  is a real constant known as the 'chirp rate' (with units of Time<sup>-2</sup>) or 'chirp parameter' in the multi-dimensional case. A chirp of this type can of course be equally well be expressed in terms of its conjugate,  $\exp(-i\alpha t^2)$ . The function  $\exp(\pm i\alpha t^2)$  is characterized by a quadratic phase function  $\theta(t) = \alpha t^2$  giving a linear frequency modulation of  $2\alpha t$ . This is obtained by taking the derivative of the phase function which yields the 'instantaneous frequency', i.e.  $\theta'(t) \equiv d\theta(t)/dt = 2\alpha t$ . This is an example of a simple linear chirp function and there are a number of variations that can be considered such as the 'Quadratic Chirp' when  $\theta'(t) = \alpha t^2$  and the 'Exponential Chirp' when  $\theta'(t) = \alpha^t$ .

The linear frequency modulated chirp represents a reoccurring theme in many areas of physics and in communications engineering. It is used in telecommunication and radio communication as a spread-spectrum technique where the bandwidth of a transmitted signal is spread in the frequency domain (resulting in a signal with a wider bandwidth) and forms part of the wireless telecommunications IEEE standards [3]. In practice, chirp functions are of compact support. For the two-sided case,  $t \in [-T/2, T/2]$  where T is the 'length' or 'period' of the pulse (its compact support in time), and, for the single-sided case,  $t \in [0,T]$ . Such functions are referred to as 'Chirplets'. In general, a chirplet is any part of a chirp function that has been windowed in time, windows that may be discontinuous or otherwise. Such functions have applications in signal and image processing through implementation of the chirplet transform [4].

Manuscript received 20, September, 2019; final revised version submitted on 25, March, 2020.

Jonathan Blackledge is the Science Foundation Ireland's Stokes Professor. He is an Honorary Professor in the School of Electrical and Electronic Engineering at the Technological University Dublin, Ireland, Distinguished Professor in the Center for Advanced Studies, Warsaw University of Technology, Poland, Visiting Professor, University of Wales, UK, Professor Extraordinaire in the Department of Computer Science, University of Western Cape and Honorary Professor in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, South Africa. email: jonathan.blackledge@tudublin.ie, web: https://www.linkedin.com/in/jonathan-blackledge-7643a5150/. This paper is dedicated to the memory of Giordano Bruno (1548-1600).

The reason for the wide ranging applications of the chirp function and the chirplet transform relate to some of their unique characteristics and properties which are re-visited in this paper. However, it should be appreciated that the chirp function and chirplet transform are not just useful 'inventions' with unique properties. They are re-occurring themes in physics where they provide some of the most fundamental characteristics of general solutions to specific physical models defined in terms of various partial differential equations. This is an aspect of the chirp function that is explored in the early part of the paper. In this context, we re-consider the characteristics of a chirp showing that it has a unique conjugate Fourier eigenfunction and that no phase-only function other than the chirp appears to have a self-characteristic Fourier transform. Coupled with the autocorrelation characteristics of a chirp function, this property underlies the uniqueness of the chirp function.

After studying the properties of the chirp function, the paper reconsiders the principles of transmitting information in the form of a binary string or bit-stream through a channel characterized by additive noise when the Signal-to-Noise Ratio (SNR) is very low using a chirplet modulation algorithm. The accuracy of the associated demodulation algorithm is then quantified by examining the Bit Error Rate. We then explore a new communications protocol where, provided the sender and recipient of a chirplet modulated signal have accurate prior knowledge of the operational bandwidth, modulation and demodulation can be undertaken by factorizing the value of a semi-prime (derived from the bandwidth) into two prime numbers. Providing the semiprime is large enough, we briefly explore how this protocol yields the potential to exercise a uniquely robust form of communications security, particularly in regard to the application of quantum computing to factorize two prime numbers using Shor's algorithm. In this context, we examine how this approach might be used in the detection and interpretation of signals transmitted through the 'waterhole'. This is coupled with the implementation of a binary entropy-based test to differentiate between a random and non-random 'intelligible' strings from which a machine learning strategy can be formulated. Finally a conjecture is considered relating to the most likely sources from which such 'intelligible' strings might be detected.

## A. Structure of Paper

This extended paper is structured as follows: Section II provides the mathematical preliminaries which are used throughout the paper including issues on the notation that is fundamental to comprehension of the material presented. For generality, the paper considers the multi-dimensional case where the chirp function is given by  $\exp(\pm i\alpha r^2)$ ,  $\mathbf{r} \in \mathbb{R}^n$ ; where  $r \equiv |\mathbf{r}|$  and n = 1, 2 or 3.

Section III provides an overview of the chirp function as a central kernel in the solution to a range of physical problems, all of which are considered in some detail. The purpose of this is to emphasize to the reader (especially those readers who are not familiar with the chirp function) that the chirp function is not just another 'basis function' that can be used as a kernel for an integral transform (the chirplet transform) but a fundamental manifestation of the physical world. This is complemented in Section IV which provides some examples of the how the chirp function is used in information engineering, including its role in real and synthetic aperture radar and optical fiber communications, for example.

Section V provides a short study on the eigenfunctions and self-characteristic functions of the Fourier transform. This is a digression from the principal theme of the work but is provided to 'set the scene' for what is arguably the most important exposition to be considered in the paper. This argument is given in Section VI and compounded in the 'uniqueness conjecture' which states that the chirp function is the only phase-only function to have a conjugate eigenfunction.

Section VII examines the background as to why the chirp function yields solutions to the propagation of information through channels with additive noise that are optimal. Up to this point in the paper, the analysis presented is concerned with chirp functions that are continuous and of infinite extent and in Section VIII, attention focuses on some of the equivalent properties of a continuous chirp that is of finite extent (of compact support), i.e. a Chirplet. This material provides the essential background to the application of chirplets for the modulation of a binary string which is the subject of Section IX and considers both continuous and discrete time modulation. In the latter case, we explore the conditions for generating a Nyquist sampled chirplet.

On the basis of this material, Section X explores example chirplet modulation and demodulation algorithms which couple to the prototype MATLAB exemplar functions provided in Appendix A. Example results are presented to introduce the reader to some of the principal characteristics of chirplet modulation and the conditions associated with its applications. This includes a short study of the Bit Error Rate associated with changes in the Signal-to-Noise Ratio of a chirplet modulated binary string. In this study, the chirplet is assumed to be a Nyquist sampled pulse with a period T, demodulation requiring this value to be known and thereby representing a fundamental key in the chirplet modulation/demodulation process. A method of exchanging this key is considered in Section XI based on the prime number factorization of a semi-prime formed from knowledge of the bandwidth of the communications channel to be used. This allows Alice to communicate to Bob using chirp modulation; all that is required to do so (apart from knowledge of the algorithm and its parameters), is for Alice and Bob know the bandwidth of the channel.

The demodulation scheme considered in Section X ensures that a binary string is output. This leaves the problem of testing to see whether the binary string is an intelligible string (associated with a natural language, for example) or a string of random bits obtained by demodulating noise without having *a priori* information on the binary input before chirp modulation is applied. A solution to this problem is investigated in Section XII based on a test that is predicated on the binary entropy. In this paper, the Kullback-Leibler Divergence test is applied which is a measure of how one probability distribution is different from another reference probability distribution.

Section XIII provides an investigation into how the ideas developed in this paper might apply to the interpretation of

signals recorded in the 'waterhole', a bandwidth of 0.25 GHz, which is the quietest channel in the interstellar radio noise background. The waterhole has been theorized to be the optimal frequency band for communicating with extraterrestrial intelligence and has consequently been used by SETI (Search for Extraterrestrial Intelligence) for many years without any success to date. This section also includes a proposition as to why sources of intelligent signals might be more common from binary and multiple star system. Finally Section XIV provides a some conclusions to the work including a review of the results obtained, directions for future work and some open questions.

#### B. Original Contributions

To the best of the authors knowledge, the components of this paper that provide original contributions to the field are as follows:

- Theorem VI.1, which shows that the chirp function has a self-characteristic Fourier transform based on an application of the Bluestein decomposition for the Fourier transform and leads to Conjectures VI.1, VI.2 and VI.3.
- The prime number factorization of the bandwidth for exchanging the chirplet period introduced in Section XI.
- Implementation of the relative entropy test to distinguish between intelligible and random binary strings as discussed in Section XII.
- The conjecture associated with increasing the likelihood of detecting intelligible signals using chirplet demodulation from sources that are assumed to be habitable planets in a stable orbit around two of more stars.

### II. MATHEMATICAL PRELIMINARIES

The principal mathematical analysis presented in this paper relies on the properties of the Fourier transform coupled with the convolution and correlation integrals in n-dimensions. In this section, these properties are briefly stated together with the associated notation used through this work.

For a square integrable function  $f(\mathbf{r}) \in L^2(\mathbb{R}^n) : \mathbb{C} \to \mathbb{C}$ , we define the (*n*-dimensional) Fourier and inverse Fourier transforms in 'non-unitary' form as

$$F(\mathbf{k}) = \mathcal{F}_n[f(\mathbf{r})] \equiv \int_{-\infty}^{\infty} f(\mathbf{r}) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^n \mathbf{r}$$

and

$$f(\mathbf{r}) = \mathcal{F}_n^{-1}[F(\mathbf{k})] \equiv \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} F(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}) d^n \mathbf{k}$$

respectively. Here, **r** is the *n*-dimensional spatial vector where  $r \equiv |\mathbf{r}| = (r_1^2 + r_2^2 + ... + r_n^2)^{\frac{1}{2}}$ . Similarly, **k** is the spatial frequency vector where  $k \equiv |\mathbf{k}| = 2\pi/\lambda$  for wavelength  $\lambda$  and  $\mathbf{k} \cdot \mathbf{r} = k_1 r_1 + k_2 r_2 + ... + k_n r_n$ . Note that  $\lambda$ is also used to denote an eigenvalue which may be complex. These integral transforms define a Fourier transform pair which, in this paper, is implied using the notation

$$F(\mathbf{k}) \leftrightarrow f(\mathbf{r}) \text{ or } f(\mathbf{r}) \leftrightarrow F(\mathbf{k})$$

We define the convolution integral of two functions  $f(\mathbf{r})$ and  $g(\mathbf{r})$  as

$$s(\mathbf{r}) = g(\mathbf{r}) \otimes f(\mathbf{r}) \equiv \int_{-\infty}^{\infty} g(\mathbf{r} - \mathbf{s}) f(\mathbf{s}) d^n \mathbf{s}$$

and the correlation integral as

$$s(\mathbf{r}) = g^*(\mathbf{r}) \odot f(\mathbf{r}) \equiv \int_{-\infty}^{\infty} g^*(\mathbf{r} + \mathbf{s}) f(\mathbf{s}) d^n \mathbf{s}$$

for a complex function  $g(\mathbf{r})$  with the conjugate  $g^*(\mathbf{r})$ , where  $[s(\mathbf{r}), g(\mathbf{r}), f(\mathbf{r})] \in L^2(\mathbb{R}^n) : \mathbb{C} \to \mathbb{C}$ . for which the following fundamental theorems apply:

• The Convolution Theorem

$$g(\mathbf{r}) \otimes f(\mathbf{r}) \leftrightarrow G(\mathbf{k})F(\mathbf{k})$$

where  $G(\mathbf{k}) \leftrightarrow g(\mathbf{r})$  and  $F(\mathbf{k}) \leftrightarrow f(\mathbf{r})$ 

- The Correlation Theorem
  - $g^*(\mathbf{r}) \odot f(\mathbf{r}) \leftrightarrow G^*(\mathbf{k})F(\mathbf{k})$
- The Product Theorem

$$g(\mathbf{r})f(\mathbf{r}) \leftrightarrow \frac{1}{(2\pi)^n}G(\mathbf{k}) \otimes F(\mathbf{k})$$

The dimensions associated with the integral operators  $\otimes$ and  $\odot$  are inferred from the dimension of the functions used. However, from time to time, multiple operators are used to emphasize the dimensionality of the convolution operation by including a subscript, i.e. for a function  $f(\mathbf{r})$ ,  $\mathbf{r} \in \mathbb{R}^n$ ,  $\otimes$ denotes  $\otimes_{r_1} \otimes_{r_2} \dots \otimes_{r_n}$ . Thus, for three-dimensional Cartesian coordinates, when  $\mathbf{r} \in \mathbb{R}^3$  and where, using conventional notation,  $r_1 \equiv x$ ,  $r_2 \equiv y$  and  $r_3 \equiv z$ ,  $\otimes \equiv \otimes_x \otimes_y \otimes_z$ .

In order to utilize scale invariance, from time to time, the 'unitary form' of the Fourier transform pair is used (specifically in Section V) when the forward and inverse transforms are defined as

$$F(\boldsymbol{\nu}) = \mathcal{F}_n[f(\mathbf{r})] \equiv \int_{-\infty}^{\infty} f(\mathbf{r}) \exp(-2\pi i \boldsymbol{\nu} \cdot \mathbf{r}) d^n \mathbf{r}$$

and

$$f(\mathbf{r}) = \mathcal{F}_n^{-1}[F(\boldsymbol{\nu})] \equiv \int_{-\infty}^{\infty} F(\boldsymbol{\nu}) \exp(2\pi i \boldsymbol{\nu} \cdot \mathbf{r}) d^n \boldsymbol{\nu}$$

respectively. These definitions restore the symmetry between the forward and inverse unitary transforms on  $L^2(\mathbb{R}^n)$ . In this case, both the convolution and product theorems are symmetric, i.e.  $g(\mathbf{r}) \otimes f(\mathbf{r}) \leftrightarrow G(\boldsymbol{\nu})F(\boldsymbol{\nu})$  and  $g(\mathbf{r})f(\mathbf{r}) \leftrightarrow$  $G(\boldsymbol{\nu}) \otimes F(\boldsymbol{\nu})$ .

One particular Fourier transform pair that is important in the context of this paper is the *n*-dimensional Fourier transform of a chirp function which is given by (for  $\mathbf{r} \in \mathbb{R}^n$ )

$$\exp(\pm i\alpha r^2) \leftrightarrow \exp\left(\pm \frac{in\pi}{4}\right) \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \exp\left(\mp \frac{ik^2}{4\alpha}\right) \quad (1)$$

This result is easily derived, given that the well known Fourier transform pair (for  $\mathbf{r} \in \mathbb{R}^1$ ), e.g. [5]

$$\exp(-\alpha x^2)\leftrightarrow \sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{k^2}{4\alpha}\right)$$

can be generalized for when  $\mathbf{r} \in \mathbb{R}^n$  to

$$\exp(-\alpha r^2) \leftrightarrow \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \exp\left(-\frac{k^2}{4\alpha}\right)$$

and with  $\alpha := \pm i\alpha$ , Relationship (1) is obtained.

For a (*n*-dimensional) Dirac delta function  $\delta^n(\mathbf{r})$  with the sampling property

$$\int_{-\infty}^{\infty} \delta^n(\mathbf{r}) f(\mathbf{r}) d^n \mathbf{r} = f(\mathbf{0})$$

it is clear that

$$\int_{-\infty}^{\infty} \delta^n(\mathbf{r}) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^n \mathbf{r} = 1, \; \forall \mathbf{k}$$

We can therefore define the Dirac delta function in terms of the inverse Fourier transform as

$$\delta^{n}(\mathbf{r}) = \mathcal{F}_{n}^{-1}[1] \equiv \frac{1}{(2\pi)^{n}} \int_{-\infty}^{\infty} \exp(i\mathbf{k} \cdot \mathbf{r}) d^{n}\mathbf{k} \qquad (2)$$

giving the Fourier transform pair

$$1 \leftrightarrow \delta^n(\mathbf{r})$$

We note the following scaling and symmetry properties of the delta function, i.e. for a non-zero constant  $\alpha$ 

 $\delta^n(\alpha \mathbf{r}) = \frac{1}{\mid \alpha \mid} \delta^n(\mathbf{r})$ 

and

$$\delta^n(\mathbf{r}) = \delta^n(-\mathbf{r})$$

respectively.

We also note the sampling property of the delta function which, using the unitary form of the Fourier transform, for a real constant X (the sampling period), is given by [6]

$$\sum_{n=-\infty}^{\infty} \delta(x - nX) \leftrightarrow \frac{1}{X} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{X}\right)$$

from which it follows that (for X = 1)

$$\sum_{m=-\infty}^{\infty} \delta(x-n) = \int_{-\infty}^{\infty} d\nu \exp(2\pi i\nu x) \sum_{k=-\infty}^{\infty} \delta(\nu-k)$$

$$=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\exp(2\pi i\nu x)\delta(\nu-k)=\sum_{k=-\infty}^{\infty}\exp(2\pi ikx)$$
(3)

Finally, we define the p-norm as

$$\|f(\mathbf{r})\|_{p} \equiv \left(\int_{\mathbb{R}^{n}} |f(\mathbf{r})|^{p} d^{n}\mathbf{r}\right)^{\frac{1}{p}}, \ 1 \le p \le \infty$$

with the uniform norm being given by

$$||f(\mathbf{r})||_{\infty} = \sup\{|f(\mathbf{r})|, \mathbf{r} \in \mathbb{R}^n\}$$

#### III. ON THE PHYSICAL SIGNIFICANCE OF THE CHIRP

The importance and (apparent) uniqueness of the quadratic phase-only function  $\exp(i\alpha r^2)$ , as discussed in this section, is a re-occurring theme in the physical world. The emergence of this function in Fresnel optics is well known and comes from analyzing the propagation and scattering (typically under the weak-scattering approximation) of a wave-field in the intermediate or Fresnel zone when the geometry of the wave-fronts of an expanding wave-field are taken to be parabolic rather than spherical (in the near-field) or planar (in the far-field).

The use of chirps for  $\mathbf{r} \in \mathbb{R}^1$  in communications engineering has already been briefly discussed in the introduction. Another related application includes pulse compression which is designed to maximize the sensitivity and resolution of radar systems by modifying (i.e. chirping) emitted pulses to improve their autocorrelation properties [7]. Chirped mirrors incorporate 'chirped spaces' in a dielectric designed to reflect wavelengths of light and compensate for the dispersion of wavelengths that can be created by some optical elements [8]. In the propagation of light pulses through optical fibers, spectral broadening and pulse compression can be extended to significantly higher pulse energies by using large-mode-area photonic crystal fibers in combination with chirping input pulses [9].

In quantum mechanics, fundamental transient phenomena such as those that occur in the 'quantum shutter problem' [13] are characterized by a Fresnel integral, i.e. an integral whose kernel is determined by a quadratic phase function. The chirp function even plays a role in cosmology given that chirps are a principal signature for the detection of gravitational waves generated by compact binary systems [10] which yield a 'Chirp Mass' given by

$$\mathcal{M} = C \left[ \frac{1}{\nu^{\frac{8}{3}}} (\ln \nu)' \right]^{\frac{3}{5}}$$

where  $\nu$  is the frequency of the gravitational wave and  $C = 0.0272c^3/G$  is a constant composed from the speed of light *c* and Newton's gravitational constant *G*. The chirp mass determines the principal order orbital evolution of a coalescing binary system as a result of energy loss from the gravitational waves emitted whose frequency evolution is characterized by  $\nu$ . As the orbital period of an inspiral system reduces, the energy of the gravitational waves emitted increases as does their frequency, thereby yielding a gravitational wave chirp [11].

In this section, we revisit some of the concepts discussed above in order to give the reader a back-ground to the importance that the (multi-dimensional or otherwise) chirp function plays in certain physical systems. The reason for doing this, is to illustrate that the chirp function is much more than a good invention for applications in communications engineering alone, but rather, a fundamental property of the physical world. In this way, we aim to emphasize the reoccurring role that the chirp function plays before investigating its unique properties for communicating information through noisy environments which is considered in Section VII. In this context, we revisit three fundamental examples in physics:

• the propagation and scattering of a wave-field in the Fresnel zone;

- the propagation of a wave-field under the beam approximation;
- a solution to the quantum shutter problem.

#### A. Fresnel Zone Solutions

Consider the inhomogeneous Helmholtz equation for the three-dimensional scalar wave function  $u(\mathbf{r}, k)$ ,  $\mathbf{r} \in \mathbb{R}^3$  with wave-number k and a source function  $f(\mathbf{r})$  which is taken to be of compact support and given by

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = -f(\mathbf{r}), \ f(\mathbf{r}) \exists \forall \mathbf{r} \in \mathbb{R}^3$$

For homogenous boundary conditions, when  $u(\mathbf{r}) = 0$  and  $\nabla u(\mathbf{r}) = \mathbf{0}$  on the surface of  $f(\mathbf{r})$ , it is well known that the outgoing Green's function solution to this equation is given by [12]

$$u(\mathbf{r},k) = g(r,k) \otimes f(\mathbf{r}) \equiv \int_{\mathbf{s} \in \mathbb{R}^3} g(\mathbf{r} \mid \mathbf{s},k) f(\mathbf{s}) d^3 \mathbf{s}$$

where

$$g(r,k) = \frac{\exp(ikr)}{4\pi r}$$

and  $\otimes$  denotes the (three-dimensional) convolution integral as defined in Section II.

Noting that

$$\mathbf{r} \mid \mathbf{s} \equiv \mid \mathbf{r} - \mathbf{s} \mid = [(\mathbf{r} - \mathbf{s}) \cdot (\mathbf{r} - \mathbf{s})]^{\frac{1}{2}} = (r^2 - 2\mathbf{r} \cdot \mathbf{s} + s^2)^{\frac{1}{2}}$$
$$= r \left(1 - \frac{2\mathbf{s} \cdot \mathbf{r}}{r^2} + \frac{s^2}{r^2}\right)^{\frac{1}{2}}$$

where  $s \equiv |\mathbf{s}|$ , binomial expansion to second order yields the result

$$|\mathbf{r} - \mathbf{s}| \simeq r - \frac{\mathbf{s} \cdot \mathbf{r}}{r} + \frac{s^2}{2r} = r - \hat{\mathbf{n}} \cdot \mathbf{s} + \frac{s^2}{2r}, \ \hat{\mathbf{n}} = \frac{\mathbf{r}}{r}$$

under the condition  $s^2/r^2 << 1$ . Thus, if s/r << 1,  $|\mathbf{r} - \mathbf{s}| \simeq r - \hat{\mathbf{n}} \cdot \mathbf{s}$  and

$$u(\mathbf{r},k) = \frac{\exp(ikr)}{4\pi r} F(\mathbf{k})$$

where  $\mathbf{k} = k\hat{\mathbf{n}}$  and

$$F(\mathbf{k}) = \mathcal{F}_3[f(\mathbf{s})] \equiv \int_{\mathbf{s} \in \mathbb{R}^3} f(\mathbf{s}) \exp(-i\mathbf{k} \cdot \mathbf{s}) d^3 \mathbf{s} \qquad (4)$$

It is then clear that  $u(\mathbf{r}, k)$  is determined by the Fourier transform of the function  $f(\mathbf{r})$  under the condition  $s/r \ll 1$  - the so-called far-field solution in the 'Fraunhofer zone'.

A different 'Fresnel solution' is obtained when we impose the condition  $s^2/r^2 << 1$  because in this case

$$u(\mathbf{r},k) = \frac{\exp(ikr)}{4\pi r}F(\mathbf{r},k)$$

where

$$F(\mathbf{r},k) = \int_{\mathbf{s}\in\mathbb{R}^3} f(\mathbf{s}) \exp(-ik\hat{\mathbf{n}}\cdot\mathbf{s}) \exp(iks^2/2r) d^3\mathbf{s}$$

In order to write this result more succinctly (i.e. in terms of a convolution integral), we note that

$$\frac{1}{2r} \mid \mathbf{s} - \mathbf{r} \mid^2 = \frac{s^2}{2r} - \frac{\mathbf{s} \cdot \mathbf{r}}{r} + \frac{r}{2} = \frac{s^2}{2r} - \hat{\mathbf{n}} \cdot \mathbf{s} + \frac{r}{2}$$

TABLE ICONDITIONS ASSOCIATED WITH SOLUTIONS TO THE INHOMOGENEOUSHELMHOLTZ EQUATION FOR A SOURCE FUNCTION  $f(\mathbf{r})$  in the Near,INTERMEDIATE AND FAR FIELDS

Field	Condition	Solution for $u(\mathbf{r},k)$
Near Field	$\frac{r}{R} \sim 1$	$g(r,k)\otimes f(\mathbf{r}),\;f(\mathbf{r})\exists\forall\mathbf{r}\in\mathbb{R}^{3}$
Intermediate Field (Fresnel Zone)	$\left(\frac{r}{R}\right)^2 << 1$	$rac{\exp(ikR/2)}{4\pi R}\exp(ilpha r^2)\otimes f({f r})$
Far Field (Fourier Plane)	$\frac{r}{R} \ll 1$	$rac{\exp(ikR)}{4\pi R}\mathcal{F}_3[f(\mathbf{r})]$

and obtain

F

where

$$u(\mathbf{r},k) = \frac{\exp(ikr/2)}{4\pi r}F(\mathbf{r})$$

$$F(\mathbf{r}) = \int f(\mathbf{s}) \exp(ik ||\mathbf{r} - \mathbf{s}||^2 / 2r) d^3 \mathbf{s}$$

 $s \in \mathbb{R}^3$  or, using the convolution operator,

$$F(\mathbf{r}) = \exp(i\alpha r^2) \otimes f(\mathbf{r}) \tag{5}$$

where  $\alpha = k/2R$ ,  $R \notin \mathbb{R}^3$  which has dimensions of length<sup>-2</sup>. Note, that the statement  $R \notin \mathbb{R}^3$  is a necessary condition because in order to write the solution in the form of Equation (5) where  $\alpha$  is a constant, R must exist outside of the domain of integration, the convolution integral being of finite extent given that  $f(\mathbf{r}) \exists \forall \mathbf{r} \in \mathbb{R}^3$ . Equation (5) shows that the solution for  $u(\mathbf{r}, k)$  is determined by the convolution of the source function  $f(\mathbf{r})$  with the chirp function  $\exp(i\alpha r^2)$ under the condition  $r^2/R^2 << 1$ . Such a result is often referred to a solution in the Fresnel or intermediate zone. Thus, we obtain the results compounded in Table 1. These are standard (conditional) solutions to the Helmholtz equation upon transformation to an integral equation using the Green's function. They illustrate the natural 'evolution' of solutions that are characterized by the Fourier transform and the chirp transformation (i.e. the convolution of the source function with the chirp function) depending on the condition that is taken to be valid physically - the distance from the source of the wave-field  $u(\mathbf{r}, k)$ . However, as will be shown later in this paper, the Fourier transform can be written in terms of the chirp transform without loss of generality through the Bluestein decomposition. In this context, the chirp function is common to both (conditional) solutions and is thereby arguably a more fundamental characteristic of the physics of waves and vibrations than the Fourier transformation!

1) Inverse Solutions: From Equation (4), it is clear that in the Fourier plane, the ideal inverse solution is given by the inverse Fourier transform, i.e.

$$f(\mathbf{r}) = \mathcal{F}_3^{-1}[F(\mathbf{k})]$$

However, in the Fresnel zone, given Equation (5), the (ideal) inverse solution is

$$f(\mathbf{r}) = \left(\frac{\alpha}{\pi}\right)^3 \exp(-i\alpha r^2) \odot F(\mathbf{r}) \tag{6}$$

# This result is obtained from the noting that

$$\exp(-i\alpha r^{2}) \odot \exp(i\alpha r^{2})$$

$$\equiv \int_{-\infty}^{\infty} \exp(-i\alpha s^{2}) \exp(i\alpha | \mathbf{s} + \mathbf{r} |^{2}) d^{3}\mathbf{s}$$

$$= \exp(i\alpha r^{2}) \int_{-\infty}^{\infty} \exp(2i\alpha \mathbf{s} \cdot \mathbf{r}) d^{3}\mathbf{s}$$

$$= \exp(i\alpha r^{2}) (2\pi)^{3} \delta^{3} (2\alpha \mathbf{r})$$

$$= \exp(i\alpha r^{2}) \frac{(2\pi)^{3}}{(2\alpha)^{3}} \delta^{3}(\mathbf{r}) = \left(\frac{\pi}{\alpha}\right)^{3} \delta^{3}(\mathbf{r})$$

and thus, if

$$F(\mathbf{r}) = \exp(i\alpha r^2) \otimes f(\mathbf{r})$$

then

$$\exp(-i\alpha r^2) \odot F(\mathbf{r}) = \exp(-i\alpha r^2) \odot \exp(i\alpha r^2) \otimes f(\mathbf{r})$$
$$= \left(\frac{\pi}{\alpha}\right)^3 \delta^3(\mathbf{r}) \otimes f(\mathbf{r}) = \left(\frac{\pi}{\alpha}\right)^3 f(\mathbf{r})$$

Further, we note that if

$$F(\mathbf{r}) = \exp(i\alpha r^2) \otimes f(\mathbf{r}), \ \mathbf{r} \in \mathbb{R}^n$$

then

$$f(\mathbf{r}) = \left(\frac{\alpha}{\pi}\right)^n \exp(-i\alpha r^2) \odot F(\mathbf{r}) \tag{7}$$

2) Born Scattering Solutions: The same basic solutions can be constructed under the Born approximation to evaluate the (first order) scattered field generated by a scattering function  $\gamma(\mathbf{r})$  of compact support or non-relativistic quantum scattering potential  $V(\mathbf{r})$ . In this case, for an incident wavefield  $u_i(\mathbf{r}, k)$ ,  $f(\mathbf{r}) = k^2 \gamma(\mathbf{r}) u_i(\mathbf{r}, k)$  where  $\gamma(\mathbf{r}) \exists \forall \mathbf{r} \in \mathbb{R}^3$ and  $f(\mathbf{r}) = -V(\mathbf{r}) u_i(\mathbf{r}, k)$ , respectively [12]. Thus, for an incident plane wave (with unit amplitude) when  $u_i(\mathbf{r}, k) =$  $\exp(ik\hat{\mathbf{m}} \cdot \mathbf{r})$  where  $\hat{\mathbf{m}}$  is the unit vector pointing in the direction of propagation, in the Fresnel zone, we obtain the solution

$$u(\mathbf{r},k) = \frac{\exp(ikR/2)}{4\pi R} S(\mathbf{r},k)$$

where  $S(\mathbf{r}, k)$  is the intermediate field scattering amplitude given by

$$S(\mathbf{r},k) = k^2 \exp(i\alpha r^2) \otimes \gamma(\mathbf{r}) \exp(ik\hat{\mathbf{m}} \cdot \mathbf{r})$$
(8)

Further, as a special case, if  $u_i(\mathbf{r}, k) = g(r, k)$ , then we can construct a Green's function solution for the back-scattered near field given by [12]

$$u(\mathbf{r},k) = k^2 g^2(r,k) \otimes \gamma(\mathbf{r})$$

so that in the Fresnel zone the scattering amplitude is

$$S(\mathbf{r},k) = k^2 \exp(2i\alpha r^2) \otimes \gamma(\mathbf{r}) \tag{9}$$

with inverse solution

$$\gamma(\mathbf{r}) = \left(\frac{2\alpha}{\pi k^{\frac{2}{3}}}\right)^3 \exp(-2i\alpha r^2) \odot S(\mathbf{r},k))$$

These are examples of the solutions that are fundamental to the field of Fresnel optics, for example. Here,  $u(\mathbf{r}, k)$  is the scalar electric field associated with the propagation of an electromagnetic field in a non-conductive material characterized by scattering function [12]

$$\gamma(\mathbf{r}) = \epsilon_r(\mathbf{r}) - 1$$



Fig. 1. The (real) chirp function  $\operatorname{chirp}(x, y) = \cos[\alpha(x^2 + y^2)]$  for  $x \in [-1, 1], y = [-1, 1]$  and  $\alpha = 10^2$ . The image is an 8-bit map of the function  $[1 + \operatorname{chirp}(x, y)]/2$ .

where  $\epsilon_r(\mathbf{r})$  is the relative permittivity of a material with a constant magnetic permeability and where polarization effects are neglected.

Figure 1 shows the real component of a chirp function for  $r \in \mathbb{R}^2$  which is characteristic of the interference patterns associated with optical diffraction in the Fresnel zone, for example. This occurs when the diffraction pattern is observed at a distance z from an (infinitely thin) dielectric diffractor placed at z = 0. In this case, we can consider a model where the optical scattering function is given by  $\gamma(\mathbf{r}) = \gamma(x, y)\delta(z)$  and the incident Electric is given by  $u_i(\mathbf{r}, k) = E(x, y, 0) \exp(ikz)$  where E(x, y, 0) is the electric field amplitude at z = 0. The three-dimensional convolution model for the scattering amplitude in the Fresnel zone as compounded in Equation (8), becomes

$$S(x, y, z, k) = k^2 \exp[\alpha(x^2 + y^2)] \otimes_x \otimes_y E(x, y, 0), \ \alpha = \frac{k}{2z}$$

The pattern given in Figure 1 is a typical and iconic example associated with the study of Fresnel optics where the underlying characteristics are determined by a two-dimensional chirp.

For a conductive material, the scattering function is frequency dependent and given by [12]

$$\gamma(\mathbf{r}) = \epsilon_r(\mathbf{r}) - 1 - i\frac{z_0}{k}\sigma(\mathbf{r})$$
(10)

where  $\sigma(\mathbf{r})$  is the (variable) conductivity in Siemens per meter (Sm<sup>-1</sup>) and  $z_0 \sim 377$  Ohms is the impedance of free space.

3) Quantum Scattering: Unlike electromagnetism, in quantum scattering, it is only usually practically possible to measure the scattered field in the far-field. In this case, the far-field condition is particularly relevant and accurate, the scattering amplitude (under the Born approximation) being

given by

$$S[k(\hat{\mathbf{n}} - \hat{\mathbf{m}})] = -\mathcal{F}_3[V(\mathbf{r})\exp(ik\hat{\mathbf{m}}\cdot\mathbf{r})]$$
$$= -\int_{-\infty}^{\infty} V(\mathbf{r})\exp[-ik(\hat{\mathbf{n}} - \hat{\mathbf{m}})\cdot\mathbf{r}]$$

with (ideal) inverse solution

$$V(\mathbf{r}) = -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} S(\mathbf{K}) \exp(i\mathbf{K} \cdot \mathbf{r}) d^3 \mathbf{K}$$

where

$$\mathbf{K} = k(\hat{\mathbf{n}} - \hat{\mathbf{m}})$$

Thus, ignoring scaling constants, we observe the following results:

- In the far-field, the (Born) scattering amplitude is given by the Fourier transform of the scattering function and has an idealized inverse solution given by the inverse Fourier transform.
- In the Fresnel zone, the (Born) scattering amplitude is given by the convolution of the scattering function with the chirp function with an idealized inverse solution being given by the correlation of the scattering amplitude with the (complex conjugate) of the chirp function.
- B. The Beam approximation and the Paraxial Wave Equation Consider the homogenous Helmholtz equation for  $\mathbf{r} \in \mathbb{R}^3$ ,

$$(\nabla^2 + k^2)u(\mathbf{r}, k) = 0 \tag{11}$$

A unidirectional beam, taken to be propagating in the z-direction through a homogeneous medium, can be represented by the wave function

$$u(\mathbf{r},k) = \psi(\mathbf{r},k) \exp(ikz) \tag{12}$$

where it is assumed that:

- $\psi(\mathbf{r}, k)$  varies slowly in comparison with  $\exp(ikz)$ ;
- $\psi(\mathbf{r})$  is concentrated mainly around the axis (x, y) = (0, 0).

With these assumptions, an approximate partial differential equation for  $\psi({\bf r},k)$  can be obtained called the Paraxial Wave Equation (PWE) whose solution is characterized by a chirp function as shall now be shown.

By substituting Equation (12) into Equation (11) and differentiating, we assume that  $\psi(\mathbf{r}, k)$  varies very slowly with z. This assumption allows us to employ the condition

$$\left|\frac{\partial^2\psi}{\partial z^2}\right|<<2k\left|\frac{\partial\psi}{\partial z}\right|$$

and is equivalent to saying that the angle between the wave vector  $\mathbf{k}$  and the *z*-axis is small. Either way, this condition is the key to transforming the Helmholtz equation to the PWE which is given by

$$\nabla^2 \psi(\mathbf{r}, z, k) + 2ik \frac{\partial}{\partial z} \psi(\mathbf{r}, z, k) = 0, \ \mathbf{r} \in \mathbb{R}^2$$
(13)

The PWE has a number of applications in optics, for example, where it provides solutions that model the propagation of electromagnetic waves in the form of 'Gaussian beams'. Most lasers, for example, emit beams of this type which are modeled by the PWE. A Gaussian beam remains Gaussian at every point along its path of propagation through an optical system. Consequently, laser optics can be thought of in terms of a Gaussian beam of coherent light with a plane wave front.

Taking the two-dimensional Fourier transform of Equation (13) we obtain

$$-u^{2}\Psi(\mathbf{u},z,k) + 2ik\frac{\partial}{\partial z}\Psi(\mathbf{u},z,k) = 0$$

or, after rearranging,

$$\frac{\partial}{\partial z}\ln\Psi(\mathbf{u},z,k) = -i\frac{u^2}{2k} \tag{14}$$

where

$$egin{aligned} \Psi(\mathbf{u},z,k) &= \mathcal{F}_2[\psi(\mathbf{r},z,k)] \ &\equiv \int\limits_{-\infty}^{\infty} \psi(\mathbf{r},z,k) \exp(-i\mathbf{u}\cdot\mathbf{r}) d^2\mathbf{r} \end{aligned}$$

Equation (14) has the solution

$$\Psi(\mathbf{u}, z, k) = \Psi(\mathbf{u}, 0, k) \exp\left(-i\frac{u^2 z}{2k}\right)$$

so that with application of the convolution theorem and Equation (1), we obtained a solution for  $\psi(\mathbf{r}, z, k)$  given by

$$\psi(\mathbf{r}, z, k) = \psi(\mathbf{r}, 0, k) \otimes \exp(i\alpha r^2), \ \mathbf{r} \in \mathbb{R}^2$$

where  $\alpha = k/2z$ ,

$$\psi(\mathbf{r},0,k) := \frac{\alpha}{\pi} \exp(-i\pi/2)\psi(\mathbf{r},0,k)$$

and

$$\psi(\mathbf{r}, z, k) \leftrightarrow \Psi(\mathbf{u}, z, k)$$

Thus, we again obtain a solution that is characterized by a chirp function (specifically, convolution with a chirp function).

#### C. The Quantum Shutter Problem

Another example in physics which involves the chirp function is the 'quantum shutter' problem which leads to the principle of 'diffraction in time', a fundamental transient phenomenon in quantum mechanics, first studied in the early 1950's by Macros Moshinsky [13]. In this case, we consider a pencil-line beam of non-relativistic particles described by wave-function  $\psi(x, t)$  which satisfies the timedependent one-dimensional ( $\mathbf{r} \in \mathbb{R}^1$ ) Schrödinger's equation (for natural units  $m = \hbar = 1$  where m is the mass of the particle and  $\hbar$  is the Dirac constant)

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi(x,t) = 0 \tag{15}$$

with initial condition  $\psi_0(x) = \psi(x, t = 0)$ .

The beam is taken to be described by a right-traveling unit amplitude plane wave  $\exp(ikx)$  which is incident upon a closed shutter placed at x = 0. The shutter is taken to be a perfect absorber so that in the positive half-space,  $\psi(x,t) =$ 0, x > 0. The shutter is then opened instantaneously at t = 0after which the particle beam is free to travel into the positive half-space. The problem is to find the transient behavior of the particle beam once it has been made 'free' to travel in the

positive half-space after the shutter has been opened subject to the initial condition

$$\psi_0(x) = \begin{cases} \exp(ikx), & x \le 0; \\ 0, & x > 0. \end{cases}$$

If we consider this problem in regard to the propagation of photons, then the wave function u(x, t), is governed by the classical wave equation (with the wave-speed set to unity)

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u(x,t) = 0$$
(16)

subject to the initial conditions

$$u_0(x) = \begin{cases} \exp(ikx), & x \le 0; \\ 0, & x > 0 \end{cases} \text{ and } \frac{d}{dx}u_0(x) = 0.$$

Intuitively, one would consider the photons to propagate into the positive half-space after the shutter is opened so that for t > 0 this half-space is characterized by a linear wave traveling from left to right. This is verified by the Green's function solution to Equation (16) given by [12]

$$u(x,t) = \frac{1}{2}H(t - |x|) \otimes u_0(x)$$
(17)

where H(t - |x|) is the Heaviside step function,

$$H(t-|x|) = \begin{cases} 0, & t-|x| < 0; \\ 1, & t-|x| \ge 0. \end{cases}$$

However, in the case of Equation (15), the Green's function solution is [12]

$$\psi(x,t) = G(|x|,t) \otimes \psi_0(x) \tag{18}$$

where G(|x|, t) is the Green's function given by [12]

$$G(\mid x \mid, t) = \frac{\exp(i\pi/4)}{\sqrt{2\pi t}} \exp\left(\frac{ix^2}{2t}\right) H(t)$$

illustrating that  $\psi(x,t) \to 0$  as  $t \to \infty$ .

Compared to a beam of photons, the transient behavior associated with a beam of electrons, for example (subject to the instantaneous opening of a shutter), is determined by a chirp function. This is a direct result of Equation (15) being characterized by the (imaginary) time derivative operator  $i\partial/\partial t$  compared to Equation (16) which is characterized by a second order (real) time derivative operator  $\partial^2/\partial t^2$ . This is also the case for the multi-dimensional Schrödinger equation

$$\left(i\frac{\partial}{\partial t}+\frac{1}{2}\nabla^2\right)\psi(\mathbf{r},t)=0,\ \mathbf{r}\in\mathbb{R}^n,\ n=1,2,3$$

given that the Green's function for this case is [12]

$$G(\mathbf{r},t) = \frac{i}{2} \left(\frac{1}{2\pi i t}\right)^{\frac{n}{2}} \exp\left(\frac{i r^2}{2t}\right) H(t)$$

Moshinsky [13] studies the solution given by Equation (18) analytically using a change of variables to obtain an expression for the probability density given by

$$|\psi(x,t)|^2 = \frac{1}{2} \left( \left| C(\xi) + \frac{1}{2} \right|^2 + \left| S(\xi) + \frac{1}{2} \right|^2 \right)$$

where

$$\xi = \frac{(kt-x)}{\sqrt{\pi t}}$$



Fig. 2. From the top to bottom, respectively: (i) The initial condition showing the real component of the unit plane wave in the negative half-space and taken to represent a beam of non-relativistic particles that are incident on a closed shutter located at x = 0,  $\forall t < 0$ ; (ii) the real component of the Green's function at an instant in time t used to compute  $\psi(x, t)$ ,  $\forall x \in [-1, 1]$  via Equation (18); (iii) example of the probability density function  $|\psi(x, t)|^2$ , x > 0 at an instant in time; (iv) example of the wave function  $\psi(x, t)$ , t > 0 for a fixed point in space.

and  $C(\xi)$  and  $S(\xi)$  are the Fresnel integrals

$$C(\xi) = \int_{0}^{\xi} \cos\left(\frac{\pi u^2}{2}\right) du \text{ and } S(\xi) = \int_{0}^{\xi} \sin\left(\frac{\pi u^2}{2}\right) du$$

However, Equation (18) can be easily computed numerically via application of a convolution sum. Using this approach, Figure 2 shows some examples of the transient behavior for  $x \in [-1, 1]$  associated with the (non-relativistic) quantum shutter problem. This example illustrates the oscillatory behavior and decay of the density function compared to the optical case (i.e. a beam of photons being characterized by the classical wave equation) when, in comparison, the intensity function in the positive half-space would be a constant. The similarity of the expression for  $\psi(x,t)$  given in Equation (18) to the Fresnel zone solution given by Equation (5) has led the transient phenomenon associated with the quantum shutter problem to be dubbed 'diffraction in time', [14], [15]. The phenomenon compounded in Equation (18) is now recognized as ubiquitous in quantum dynamics [16], [17], experimental confirmation of this effect having been achieved in 1996, [18]. Figure 3 shows an example of the diffraction in time effect as a space-time map of  $|\psi(x,t)|^2$ .

An equivalent theoretical study for the relativistic case, when the wave function is given by the solution to the Klein-Gordon equation (for 0-spin particles) and the Dirac equation (for 1/2-spin particles) is given in [19]. A study of the threedimensional quantum shutter problem and diffraction in time is considered in [20]. A further study is required on time diffraction for the semi-relativistic case using the Fractional Schrödinger-Klein-Gordon Equation for modeling intermediate relativism [21] which will be published elsewhere [22].

#### IV. EXAMPLE APPLICATIONS OF THE CHIRP

The previous section has aimed to shed light on the fact that the chirp function is a fundamental characteristic



Fig. 3. 'Diffraction in Time'. An example of the transient phenomenon associated with Equation (18) showing a space-time (vertical-horizontal, respectively) gray-level map of  $|\psi(x,t)|^2$  where (x = 0, t = 0) is taken to be at the top left-hand corner of the image. In this case, the image has been histogram equalized to enhance the contrast of the gray-levels [12].

of physics. As an approximation to the Green's function the chirp transform is as fundamental to intermediate field analysis as the Fourier transform is to far field analysis. In this section, and, in the context of the study given in Section III, we explore some example applications of the chirp function.

#### A. Pulse-Echo Compression

Consider Equation (9) which we write in the form

$$S(x, y, z, k) = k^{2} \exp(ikx^{2}/R) \exp(iky^{2}/R) \exp(ikz^{2}/R)$$
$$\otimes_{x} \otimes_{y} \otimes_{z} \gamma(x, y, z)$$

noting that for  $y/x \ll 1$  and  $z/x \ll 1$ 

$$R = (x^{2} + y^{2} + z^{2})^{\frac{1}{2}} = x \left(1 + \frac{y^{2}}{x^{2}} + \frac{z^{2}}{x^{2}}\right)^{\frac{1}{2}} \simeq x$$

If we consider a model in which  $\gamma(x,y,z) := \gamma(x)\delta(y)\delta(z)$  then it is immediately clear that

$$S(x,k) = k^2 \exp(ikx^2/R) \otimes_x \gamma(x)$$

This model conditions the problem to one associated with a 'pencil line beam' where propagation is in the x-direction alone. In this case R = x and Equation (9) is reduced to

$$S(x,k) = k^2 \exp(ikx) \otimes_x \gamma(x)$$

and thus

$$S(k) \equiv S(0,k) = k^2 \mathcal{F}_1[\gamma(x)] = k^2 \Gamma(k)$$

The value of S(x, k) at x = 0 represents the back-scattered field measured at a fixed value, i.e. the origin x = 0.

For constant  $k = k_0$ , say, only a single spectral component is available on the spectrum of  $\gamma(x)$  which can not be reconstructed because the inverse Fourier transform can not be evaluated from a single complex number  $S(k_0)$ . To recover  $\gamma(x)$ , we require a spectrum S(k). This is achieved if a pulse is emitted and the echo (the back-scattered field) measured, a measurement that is compounded in the spectrum

$$S(k) = k^2 P(k) \Gamma(k)$$

where P(k) is the spectrum of the pulse  $p(x) \leftrightarrow P(k)$ . If P(k) is a base-band spectrum, then using the convolution theorem,

$$s(x) = -p(x) \otimes \frac{d^2}{dx^2} \gamma(x)$$

If  $P(k) := P(k + k_0)$  is a narrow side-band spectrum, with a carrier frequency  $k_0$  such that  $|k| << k_0$ , then

$$S(k - k_0) = k_0^2 P(k) \Gamma(k - k_0)$$

Demodulating to a base-band spectrum,

$$S(k) = S(k-k_0) \otimes \delta(k+k_0) = k_0^2 P(k) \Gamma(k-k_0) \otimes \delta(k+k_0)$$
$$= k_0^2 P(k+k_0) \Gamma(k)$$

Thus, upon application of the convolution theorem, we obtain (ignoring scaling by  $k_0^2$ )

$$s(x) = \exp(ik_0 x)p(x) \otimes \gamma(x)$$

The question then arises as to what form p(x) should have in order to provide an optimal estimate of  $\gamma(x)$  given s(x).

For an arbitrary function p(x), deconvolution algorithms are required which incorporate regularization methods to overcome the ill-conditioned nature of the problem. However, if a chirp is used so that

$$s(x) = \exp(i\alpha x^2) \otimes \gamma(x)$$

then

$$\gamma(x) = \frac{\alpha}{\pi} \exp(-i\alpha x^2) \odot f(x)$$

This form of deconvolution is predicated on the use of chirps of infinite extent. In practice, a chirp of compact support  $x \in [-X/2, X/2]$  is required, where X is the length of the (two-sided) chirplet. The effect of this on the deconvolution of  $\gamma(x)$  from s(x) is explored in Section VIII where it is shown that a band-limited version of  $\gamma(x)$  is recoverable from s(x) with a bandwidth determined by the product  $\alpha X$ , a process that is commonly known a pulse compression.

#### B. Synthetic Aperture Imaging

Consider Equation (9) for the case when  $\gamma(x,y,z) := \gamma(x,y)\delta(z)$  when we can write

$$S(x, y, k) = k^2 \exp(ikx^2/R) \exp(iky^2/R) \otimes_x \otimes_y \gamma(x, y)$$

This model conditions the problem to one associated with a 'pencil planar beam' propagating in the (x, y)-plane.

As in the previous application, with  $R \sim x$  we obtain

$$\begin{split} S(k,y) &\equiv S(0,y,k) = k^2 \exp(iky^2/R) \otimes_y \mathcal{F}_1[\gamma(x,y)] \\ &= k^2 \exp(iky^2/R) \otimes_y \Gamma(k,y) \end{split}$$

and introducing a narrow side-band pulse (with carrier frequency  $k_0 >> \mid k \mid$ )

$$S(k,y) = k_0^2 P(k+k_0) \exp(ik_0 y^2/R) \otimes_y \Gamma(k,y)$$

where R is taken to be a fixed value which defines the range at which the back-scattering interactions occur. Demodulation and application of the convolution theorem then yields

$$s(x,y) = k_0^2 p(x) \exp(ik_0 x) \exp(ik_0 y^2/R) \otimes_x \otimes_y \gamma(x,y)$$

If we consider p(x) to be the unit amplitude modulated chirp  $p(x) = \exp(i\alpha x^2) \exp(-ik_0 x)$  (for an arbitrary value of the chirp rate  $\alpha$ ), then we can write

$$s(x,y) = k_0^2 \exp(i\alpha x^2) \exp(ik_0 y^2/R) \otimes_x \otimes_y \gamma(x,y)$$
(19)

Hence, from Equation (7), we can construct an idealized inverse solution for  $r\in\mathbb{R}^2$  given by

$$\gamma(x,y) = \frac{\alpha}{k_0 R \pi^2} \exp(-i\alpha x^2) \exp(-ik_0 y^2/R) \odot_x \odot_y s(x,y)$$

Again, this result is predicated on the use of chirps of infinite extent in both x and y, and, in practice, chirps of compact support  $x \in [-X/2, X/2]$  and y = [-Y/2, Y/2] are used, where X is the length of the (two-sided) chirplet (equal to the order of the range) and Y is the width of the beam at the range R.

While there are a number of technical issues associated with its development and implementation, the analysis above is the theoretical basis for Synthetic Aperture Radar (SAR) imaging of the earths surface (e.g. [23], [24] and [25]). The model is highly idealized in regard to the scattering function  $\gamma(x, y, z)$  and the interaction of an electric field with a conductive dielectric in which polarization effects are neglected. The 'image' is typically a display the function  $|\gamma(x, y)|$ .

As the pulse generating platform (which may be airborne or space-borne) moves along the y coordinate, like pulses  $p(x) = \exp(ik_0x)\exp(i\alpha x^2)$  are periodically emitted, propagating toward the ground at a specific incidence angle and the back-scattered signal recorded. Each signal is quadrature demodulated to produce a base-band (complex) signal. Providing the scattering interactions take place at a range R that is in the Fresnel zone, Equation (19) provides a space-continuous model for the recorded data. Changes in the incidence angle, carrier frequency, bandwidth and polarization as well as other operational parameters effect the characteristics of a SAR image. Typical carrier frequencies for X-band SARs, for example (with wavelengths of  $\sim 3$ cm), are  $\sim 10$  GHz. The bandwidth of the range chirp is typically  $\sim 100$  MHz but again varies from one system to another, details of which lie beyond the scope or focus of this work.

Figure 4 shows an example of a SAR image. In this case, the bright features are due to back-scattering from ships and the dark region from the lack of back-scattering from a relatively calm sea surface. Taking the sea surface to be a dielectric ( $\epsilon_r \sim 80$ ) with very low conductivity ( $\sim 5 \text{ Sm}^{-1}$ ), and, given that from Equation (10),

$$\gamma(x,y) = \epsilon_r - 1 - i\frac{z_0}{k_0}\sigma(x,y)$$

the conductivity of the area imaged is dominated by the material from which the ships are composed, the conductivity of iron being  $\sim 10^7 \, {\rm Sm}^{-1}$ . Figure 4 is an example of the high Radar Cross Section (which is a measure of image intensity) generated by the back-scattering of microwaves from highly conductive objects. There are many other complex features that occur in such SAR images due to the multi-faceted nature of the interaction of microwaves with rough conductive dielectric surfaces some of which need to be modeled using statistical methods. However, the basic method of processing the data received in 'range' x and 'azimuth' y to obtain such images remains the same, and, as discussed above, is fundamentally dependent on the properties of the chirp function.

SAR is an example of a two-dimensional imaging system and the planer based model discussed in this section is



Fig. 4. A SAR image of an area of the sea near a busy port showing a number of ships, seen as bright spots in the image. In this example, the sea is relatively calm and hence, the ships can be easily detected against the dark background [26].

predicated on a model for the scattering function given by  $\gamma(x, y)\delta(z)$ , a model that is intrinsically limited given that, in practice, back-scattering occurs in three-dimensions. For three-dimensional synthetic aperture imaging, we can construct the three-dimension function

$$s(x, y, z) =$$

$$\exp(i\alpha x^2) \exp\left[i\frac{k_0}{R}(y^2 + z^2)\right] \otimes_x \otimes_y \otimes_z \gamma(x, y, z)$$

The (idealized) inverse solution is then obtained by a triple correlation with the complex conjugates of the chirp functions.

## C. Optical Fiber Communications

In optical fibers, chirping can occur that limits the performance of light pulse-based communications. For a given pulse spectrum, the minimum pulse duration is obtained when there is no chirping, a condition that is equivalent to a constant instantaneous frequency. Chirped pulses may broaden or compress, but pulse broadening is the most performance reducing effect. This is because a broadened pulse spills energy into the next bit position, and, if the broadening is significant, this can cause a bit error to occur when a binary 0 is replaced with a binary 1.

Chirps can occur for two principal reasons: (i) pulse generation; (ii) pulse propagation. The first reason is due to the use of semiconductor laser diodes to generate (Gaussian) light pulses. The refractive index of the semiconductor material depends on the current density so that as the laser is modulated and the current density changes so does the refractive index. The change in refractive index changes Optical Path Length (OPL) given by

$$OPL = \int_{P} n(x) dx$$

where n(x) is the refractive index over the path P. This effect shifts the central frequency of the laser diode and the

spectrum of an optical pulse becomes broadened as a result, due to temporally varying phase changes. The phenomenon can be compensated for by using light pulse generators with a continuous mode operation.

The second reason for the generation of chirps during light pulse propagation is due to the effects of chromatic dispersion and non-linearities generating self-phase modulation arising from the Kerr effect. Pulses subject to selfphase modulation and spectral broadening increases with propagation distance as does the rate of change of the instantaneous frequency. This leads to the generation and propagation of chirps with an increasing amount of chromatic dispersion applied to an initially un-chirped pulse which increases the amount of chromatic dispersion. Chromatic dispersion in the fiber yields a frequency-dependent time delay, and, in conjunction with the chirp, leads to signal degradation. Therefore, larger transmission distances require a low degree of chirping. The chirp of a pulse can be removed or reversed by propagating it through optical components with suitable chromatic dispersion characteristics. Because the deterioration of pulse propagation is so important in optical fiber communications technology, developing mathematical models for this non-linear effect is important.

A common phenomenological model for a Gaussian chirped pulse is given by

$$E(t) = \exp(i\omega_0 t) \exp\left[-\frac{(1+i\alpha)}{2}\frac{t^2}{T^2}\right]$$

where E(t) is the time-varying Electric field,  $\omega_0$  is the carrier (angular) frequency for constants  $\alpha$  (chirp rate) and T (pulse length). This pulse has a linear variation with the instantaneous frequency of  $\omega_0 + 2\alpha t$ . For  $\alpha > 0$  an up-chirp (positive chirp) is obtained when the instantaneous frequency increases linearly and for  $\alpha < 0$ , a down-chirp (negative chirp) is obtained when the instantaneous frequency decreases linearly. The chirp is either positive or negative depending on whether  $\alpha$  is positive or negative, [27].

In order to more accurately model the propagation of such pulses in an optical fibre it is necessary to resort to differential equations that are consistent with the physical mechanisms that influence the propagation. One commonly used model is the non-linear Schrödinger equation [28], [29]

$$\left[i\frac{\partial}{\partial x} - \beta\frac{\partial^2}{\partial t^2} + \gamma \mid E(x,t)\mid^2\right]E(x,t) = 0$$

where x is the propagation distance,  $\beta$  is the group velocity (second-order) dispersion factor and  $\gamma$  determines the selfphase modulation. For a standard telecommunication optical fibre operating at a (carrier) wavelength of 1550 nm, typical values for these coefficients are  $\beta = -20 \text{ ps}^2 \text{km}^{-1}$ ,  $\gamma \simeq 2 \text{ W}^{-1} \text{km}^{-1}$  and  $T \sim 100 \text{ps}$ , [27]. In the linear regime [30]

$$E(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\beta\omega^2 x) \exp(i\omega t) \widetilde{E}(0,\omega) d\omega$$

where  $E(0,\omega)$  is the Fourier transform of the input pulse. Hence, once again, we see that the chirp is a fundamental characteristic of a physical phenomena; in this case, the dispersion of light pulses in an optical fibre, [32].



Fig. 5. Chirplets used to encode the bit-pairs 00, 11, 01 and 10, [39].

## D. Cryptography and Information Hiding

In cryptology, the chirplet has been used for key exchange [33], for the self-authentication of digital signals [34] and in high resilience watermarking, e.g. [35] and [39]. With regard to bit stream encoding for embedding information in digital signals, for example, 'Chirp Coding' (e.g. [36], [37] and [38]) is one of the most robust techniques with regard to distortion through additive noise. The technique has been successfully applied to audio signal authentication and self-authentication problems for Digital Rights Management in the audio post-production industry as has its application in image watermarking and image authentication, [39]. In the latter case, the four two-dimensional chirplets shown in Figure 5 are used to encode the bit-pairs 00, 11, 01 and 10, for example. The result is then embedded in an image and the code(s) recovered via correlation with the relevant chirplet.

This approach provides a highly robust system for watermarking images using a block partitioning approach subject to a self-alignment strategy and bit error correction. The applications include the copyright protection of images and Digital Rights Management for image libraries. Moreover, the method is highly effective with print-scan and/or edisplay-scan image authentication devices for use with edocuments where QR codes can be covertly embedded in images of the document holder, for example. This requires that an embedding procedure is developed that is highly robust to blur, noise, geometric distortions such as rotation, shift and barrel and the partial removal of image segments, all of which are resilient to the method and its practical realization in a real operating environment.

#### E. Evolution

There are many examples of the use of chirps by animals for communication, navigation and hunting. This includes marine mammals such a Whales and Dolphin and land based animals, one of the most common examples that we are most familiar with being the wide range of chirps used by birds. For example, of the 900 known species of bat, nearly half use active ultrasonic (from 20-200 kHz) echolocation to 'see' with sound [31]. Their SOund Navigation And Ranging (SONAR) system uses special morphological and physiological adaptations to emit both single-frequency and frequency-modulated pulses (a series of clicks which are extremely short ~50–100  $\mu$ s) are produced hundreds of times a second. The frequency modulated components can be both narrow-band or broad-band and a combination of these components are used. A constant-frequency component allows the ultrasonic pulse to travel farther and last longer than frequency-modulated components, which are used to determine the location and the texture of a target.

Such pulse components are generated by contracting the larynx although a few species of bat click with their tongues. These ultrasonic pulses are generally emitted through the mouth with some species using their nostrils. The use of ultrasound means that less energy is required to emit a pulse (the intensity of the sound being  $\sim$  50 - 120 dB) and the scattering of each pulse from an object provides a 'cleaner' localization of the object without interference from refraction or diffraction. Moreover ultrasound attenuates and disperses quickly, so the a bat can differentiate between one pulse and another given that a previous pulse has the potential to echo in the local area. The large ears and brain cells in bats are especially tuned to the frequencies of the sounds they emit and the echoes that result. A concentration of receptor cells in their inner ear makes bats extremely sensitive to frequency changes. In regard to detecting in flight objects such as insects, all bats tend to increase the number of clicks per second and frequency modulate their pulse to form a chirplet. This provides a marked increase in the local resolution of the 'sound-scape' generated by a bat.

As with all other physical and behavioral characteristics in animals, the use of chirplets by bats has evolved through the process of natural selection. In this context, there is an important aspect of a chirp compared to other phase-only functions that could be constructed. This is that from an evolutionary point of view, a chirp is arguably the simplest way to broaden the spectrum of an audio pulse given the physical ability for many animals to generate sounds at a constant pitch which can then be chirped through muscular contraction. A 'wolf whistle', for example, is often used by the Third Chimpanzee [40] (subject to cultural conventions) to express sexual attraction or admiration through a rising and falling pitch, thereby representing a form of communication associated with the pro-creation of the species (a comment that is not meant to be taken too literally by the reader!).

#### F. Discussion

The examples given in Sections III have been presented to emphasize the importance that the chirp function plays in the physical world. The example applications presented in this section have been chosen to demonstrate how this function plays a critical role in the recovery of information from scattering interactions and the propagation light through optical fibers, for example. However, these are just a few of the applications in which the characteristics of chirp are of fundamental importance.

In this context, and, in regard to the principal remit of this paper, one of the most important aspects of the review considered relates to Equation (5) and Equation (6), the latter equation providing a unique and exact inverse solution to Equation (5). We note that this is only possible because the correlation of a chirp function with its conjugate self yields a delta function, a property that is applicable in any dimension, i.e.

$$\exp(-i\alpha r^2) \odot \exp(i\alpha r^2) = \left(\frac{\pi}{\alpha}\right)^n \delta^n(\mathbf{r}), \ \mathbf{r} \in \mathbb{R}^n$$

If we consider an arbitrary function  $p(\mathbf{r})$  say, which maps to a function  $s(\mathbf{r})$  via the convolution operation

$$s(\mathbf{r}) = p(\mathbf{r}) \otimes f(\mathbf{r}) \tag{20}$$

then, from the convolution theorem,

$$S(\mathbf{k}) = P(\mathbf{k})F(\mathbf{k})$$

where  $S(\mathbf{k}) \leftrightarrow s(\mathbf{r})$  and  $P(\mathbf{k}) \leftrightarrow p(\mathbf{r})$ . It is then clear that to obtain a unique and exact inverse solution we require that

$$p^*(\mathbf{r}) \odot p(\mathbf{r}) = \delta^n(\mathbf{r}) \Rightarrow |P(\mathbf{k})|^2 = 1$$

But this is only possible if and only if  $P(\mathbf{k}) = \exp[i\Theta(\mathbf{k})]$ ; in other words  $P(\mathbf{k})$  must be a phase-only spectrum. We are therefore required to study the class of functions  $p(\mathbf{r})$ that have a phase-only spectrum - a 'phase-only function'. It is clear that one such function is the chirp function that, from Equation (1), is an example of a function with a selfcharacteristic Fourier transform. In the following section we make a study of such functions and hypothesize (through the presentation of a single conjecture) that, in the context of the above discussion, the chirp function is unique.

If we extend the expression for  $s(\mathbf{r})$  given in Equation (20) to the form

$$s(\mathbf{r}) = p(\mathbf{r}) \otimes f(\mathbf{r}) + n(\mathbf{r})$$
(21)

where  $n(\mathbf{r})$  is a stochastic 'noise' function with some Probability Density Function  $\Pr[n(\mathbf{r})]$ , then it is clear that we can construct the inverse solution

$$f(\mathbf{r}) = p^*(\mathbf{r}) \odot s(\mathbf{r}) - p^*(\mathbf{r}) \odot n(\mathbf{r})$$

given that

$$p^*(\mathbf{r}) \odot p(\mathbf{r}) = \delta^n(\mathbf{r})$$

A sufficient and necessary condition for the two functions  $p^*(\mathbf{r})$  and  $n(\mathbf{r})$  to be 'uncorrelated' is that  $p^*(\mathbf{r}) \odot n(\mathbf{r}) = 0$ , i.e. their cross-correlation function is zero. Cross-correlation is a measure of the similarity between two functions, and, given that  $n(\mathbf{r})$  is a stochastic function and  $p^*(\mathbf{r})$  is a deterministic function, there can, in principle, be no matching features between the two functions. Hence, with the condition that  $p^*(\mathbf{r}) \odot n(\mathbf{r}) = 0$ ,

$$f(\mathbf{r}) = p^*(\mathbf{r}) \odot s(\mathbf{r})$$

This result assumes that the noise function  $n(\mathbf{r})$  is some zero mean random variable and is Ergodic (i.e. has the same 'stochastic signature' averaged over  $\mathbf{r} \in \mathbb{R}^n$ ).

The general problem of generating an inverse solution to Equation (21) for arbitrary functions  $p(\mathbf{r})$  is the basis for the development of many algorithms in signal and image processing [5], [12]. In this case, the function  $p(\mathbf{r})$  is a characteristic of some linear, time or space invariant system that, in general, we do not necessarily have control over and can only estimate and/or model from knowledge of the physics

of the system. Regularization methods are then required to solve the corresponding deconvolution problem. However, in regard to the communication of information subject to the communications model compounded in Equation (21) we can, in principle, choose a function  $p(\mathbf{r})$  which optimizes the simplicity of recovering  $f(\mathbf{r})$  from knowledge of  $s(\mathbf{r})$ .

In addition to using a phase only function, we could also consider a power spectrum normalized function when

$$p(\mathbf{r}) \leftrightarrow \frac{Q(\mathbf{k})}{\mid Q(\mathbf{k}) \mid^2}, \mid Q(\mathbf{k}) \mid^2 > 0$$

with the condition that

$$q^*(\mathbf{r}) \odot n(\mathbf{r}) = 0$$

In this case, the exact solution for  $f(\mathbf{r})$  is given by

$$f(\mathbf{r}) = q^*(\mathbf{r}) \odot s(\mathbf{r})$$

However, compared to the use of a phase-only function, this approach requires the power spectrum to be positive definite and in this context, application of a phase-only function is unrestricted given that the power spectrum of a (unit amplitude) phase-only function is always 1.

Power spectrum normalization and phase-only functions can be used to encrypt and hide encrypted data using a stochastic data generating function (a cipher) to compute  $Q(\mathbf{k})$  and the phase-spectrum  $P(\mathbf{k})$ , [41] and [42], respectively. In the former case, a no-keys protocol can be implemented using a three-way pass [43]. However, the chirp function is a phase-only function which is self-characteristic in the sense, that, ignoring scaling, the a chirp in real space yields a chirp in Fourier space. In this respect, for  $p(\mathbf{r}) = \exp(i\alpha r^2)$ , we can state the following fundamental result: If

$$s(\mathbf{r}) = \exp(i\alpha r^2) \otimes f(\mathbf{r}) + n(\mathbf{r})$$
(22)

then

$$f(\mathbf{r}) = \left(\frac{\alpha}{\pi}\right)^n \exp(-i\alpha r^2) \odot s(\mathbf{r})$$
(23)

Coupled with the study given in the following section, this result demonstrates a unique property of the chirp function. It is this uniqueness that is a central theme of this work and the ideas and results thereof, especially in regard to Section IX studied in light of Conjectures VI.1, VI.2 and VI.3 presented in the Section VI.

## V. Self-Characteristic Functions of the Fourier Transform

The purpose of this section is to revisit the selfcharacteristic functions of the Fourier transform in order to present a short background before presenting a fundamental conjecture associated with the chirp function, namely, that the chirp function is the only phase-only function to have a conjugate eigenfunction upon Fourier transformation.

It is well known that many Fourier transforms of different functions in *n*-dimensions exist, and, that in some cases, the Fourier transform  $F(\mathbf{k})$  is characteristic of the function  $f(\mathbf{r})$  in some way. Such functions are said to be self-characteristic. Further, for a (real or complex) constant  $\lambda$ , some functions  $f(\mathbf{r})$  yield a Fourier transform such that

$$F(\mathbf{k}) = \mathcal{F}_n[f(\mathbf{r})] = \lambda f(\mathbf{k}) \tag{24}$$

Such a function is said to be an 'Eigenfunction' of the Fourier transform operator with eigenvalue  $\lambda \in \mathbb{C}$  and falls into two classes:

- non-period eigenfunctions classified by Equation (24);
- periodic functions classified by Equation (24) subject to the *p*-periodic equation

$$f(\mathbf{r} + p) = f(\mathbf{r})$$

## A. Examples of Non-period Eigenfunctions

The most 'celebrated' example of a non-period eigenfunction of the Fourier transform is the Gaussian function where, for  $\mathbf{r}\in\mathbb{R}^1$ 

$$\mathcal{F}_1[\exp(-x^2/2)] = \sqrt{2\pi} \exp(-k^2/2)$$

Writing this result in the form

$$\mathcal{F}_1[f(x)] = \lambda f(k) \tag{25}$$

where  $\lambda = \sqrt{2\pi}$  we can define  $f(x) = \exp(-x^2/2)$  is an eigenfunction of the operator  $\mathcal{F}_1$  with eigenvalue  $\lambda$ . However, it is well known that this result can be extended to the case when (for n = 1, 2, 3, ...)

$$f_n(x) = \exp(-x^2/2)H_n(x)$$

where  $H_n(x)$  is the  $n^{\text{th}}$ -order Hermite polynomial given by

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$$

which are solutions to the eigenvalue problem defined by the differential equation

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right)H_n(x) = -2\lambda_n H_n(x)$$

In this case, there are n-eigenfunction satisfying the eigenequation

$$\mathcal{F}_1[f_n(x)] = \lambda_n f_n(k), \ \lambda_n = (-i)^n \sqrt{2\pi}$$

There are a number of other such function that satisfy the Equation (25) such as the following (expressed in terms of Fourier transform pairs) [6]:

 $\frac{1}{\sqrt{|x|}} \leftrightarrow \sqrt{\frac{2\pi}{|k|}}$ 

and

$$\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}x\right) \leftrightarrow \sqrt{2\pi}\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}k\right)$$

Further, noting that if we express the Fourier transform pair in terms of the unitary definition (as discussed in Section II for  $x \in \mathbb{R}^1$ ), i.e.

$$F(\nu) = \int_{-\infty}^{\infty} \exp(-2\pi i\nu x) f(x) dx$$

and

$$f(x) = \int_{-\infty}^{\infty} \exp(2\pi i\nu x) F(\nu) d\nu$$

when the convolution and product theorems are

$$g(x) \otimes f(x) \leftrightarrow G(\nu)F(\nu) \text{ and } g(x)f(x) \leftrightarrow G(\nu) \otimes F(\nu)$$

 $\nu$ =

respectively, then, for a function f(x) with Fourier transform  $F(\nu)$ 

$$h(x) = f(x) + F(x)$$

is an eigenfunction of the Fourier transform since

$$H(\nu) = F(\nu) + f(\nu)$$

where  $H(\nu) \leftrightarrow h(x)$ . Similarly,

$$h(x) = f^2(x) + F(x) \otimes F(x)$$

is an eigenfunction of the Fourier transform since, using the convolution and product theorems,

$$H(\nu) = F(\nu) \otimes F(\nu) + f^2(\nu)$$

This result can be extended by induction for the *n*-dimensional case and for *p* products and *p* convolutions, i.e. given that  $f = f_m \forall m$  and  $F = F_m \forall m$ , then if

$$h(\mathbf{r}) = \prod_{m=1}^{p} f_m(\mathbf{r}) + \prod_{m=1}^{p} F_m(\mathbf{r}),$$
$$H(\boldsymbol{\nu}) = \prod_{m=1}^{p} F_m(\boldsymbol{\nu}) + \prod_{m=1}^{p} f_m(\boldsymbol{\nu})$$

where

$$F(\boldsymbol{\nu}) = \int_{-\infty}^{\infty} f(\mathbf{r}) \exp(-i2\pi\boldsymbol{\nu} \cdot \mathbf{r}) d^{n}\mathbf{r}$$

and

$$\bigotimes_{n=1}^{p} F_{m}(\mathbf{r}) \equiv F_{1}(\mathbf{r}) \otimes F_{2}(\mathbf{r}) \otimes ... \otimes F_{p}(\mathbf{r})$$

assuming that  $f(\mathbf{r})$  and  $F(\mathbf{k})$  can be convolved *p*-times. A further generalization of the result is as follows: If

$$h(\mathbf{r}) = \sum_{n=1}^{p} \prod_{m=1}^{n} f_m(\mathbf{r}) + \sum_{n=1}^{p} \prod_{m=1}^{n} F_m(\mathbf{r})$$

then

$$H(\boldsymbol{\nu}) = \sum_{n=1}^{p} \prod_{m=1}^{n} F_m(\boldsymbol{\nu}) + \sum_{n=1}^{p} \prod_{m=1}^{n} f_m(\boldsymbol{\nu})$$

# B. Examples of Periodic Eigenfunctions

In the same way that the Gaussian function is an iconic example of a non-periodic eigenfunction of the Fourier transform, one of the best known examples of a periodic eigenfunction is the Dirac comb function given by (for  $\mathbf{r} \in \mathbb{R}^1$ )

$$\mathrm{III}(x) = \sum_{m=-\infty}^{\infty} \delta(x-m)$$

whose (unitary) Fourier transform is

$$\mathcal{F}_1[\mathrm{III}(x)] = \sum_{m=-\infty}^{\infty} \delta(\nu - m)$$

This result is fundamental to the proof of the Sampling Theorem, and, is an example of the Poisson sum formula where for certain functions f(x)

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{\nu=-\infty}^{\infty} F(\nu)$$

which relates the periodic summation of a function to values of the function's (continuous) Fourier transform. Here, the periodic summation of a function is completely defined by discrete samples of the original function's Fourier transform. This can be shown through a distributional formulation using Equation (3) as follows:

$$\sum_{n=-\infty}^{\infty} F(\nu) = \sum_{\nu=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x) \exp(-2\pi i\nu x) dx \right)$$
$$= \int_{-\infty}^{\infty} dx f(x) \sum_{\nu=-\infty}^{\infty} \exp(-2\pi i\nu x)$$
$$= \int_{-\infty}^{\infty} dx f(x) \sum_{m=-\infty}^{\infty} \delta(x-m)$$
$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \delta(x-m) dx$$
$$= \sum_{m=-\infty}^{\infty} f(m)$$

#### C. Examples of Self-characteristic Functions

Self-characteristic (and self-similar) functions of the Fourier transform are a much larger class of functions than those that can be formally classified as eigenfunctions of the transform. Some of these functions are generalisations of the eigenfunctions which include a scaling factor a say, for which the scaling theorem applies, i.e.

$$\mathcal{F}_n[f(a\mathbf{r})] = \frac{1}{\mid a \mid} F\left(\frac{\mathbf{k}}{a}\right) \tag{26}$$

Examples of such functions for  $\mathbf{r} \in \mathbb{R}^1$  include the following (non-unitary) Fourier transform pairs [6]

$$\exp(-ax^{2}) \leftrightarrow \frac{\pi}{a} \exp\left(-\frac{k^{2}}{4a}\right)$$
$$\cos(ax^{2}) \leftrightarrow \sqrt{\frac{\pi}{a}} \cos\left(\frac{k^{2}}{4a} - \frac{\pi}{4}\right)$$
$$\sin(ax^{2}) \leftrightarrow -\sqrt{\frac{\pi}{a}} \sin\left(\frac{k^{2}}{4a} - \frac{\pi}{4}\right)$$
$$\exp\left(-\frac{a^{2}x^{2}}{2}\right) H_{n}(ax) \leftrightarrow \frac{(-i)^{n}\sqrt{2\pi}}{a} \exp\left(-\frac{k^{2}}{2a^{2}}\right) H_{n}\left(\frac{k}{a}\right)$$
$$\operatorname{sech}(ax) \leftrightarrow \frac{\pi}{a} \operatorname{sech}\left(\frac{\pi k}{2a}\right)$$
$$\operatorname{III}_{X}(x) = \sum_{n=1}^{\infty} \delta(x - mX) \leftrightarrow \frac{2\pi}{X} \sum_{n=1}^{\infty} \delta\left(x - \frac{2\pi m}{X}\right)$$

Further, there are a number of functions that have similar scaling characteristics such as the following [6]:

$$\frac{1}{\mid x \mid^{\alpha}} \leftrightarrow -\frac{2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(1-\alpha)}{\mid k \mid^{1-\alpha}}$$

and

$$\frac{1}{(\mp ix)^{\alpha}} \leftrightarrow \frac{2\pi}{\Gamma(\alpha)} \frac{H(\pm k)}{(\pm k)^{1-\alpha}}$$

where  $0 < \alpha < 1$  and H(k) is the Heaviside step function

$$H(k) = \begin{cases} 1, & k \ge 0; \\ 0, & k < 0. \end{cases} \Rightarrow \frac{d}{dx} H(k) = \delta(x)$$

and, for  $\mathbf{r} \in \mathbb{R}^n$ ,

$$\frac{1}{|\mathbf{r}|^{a}}, \ 0 < \operatorname{Re}[a] < n \ \leftrightarrow \frac{\lambda_{n,a}}{|\mathbf{k}|^{n-a}}$$
(27)

where

$$\lambda_{n,a} = \frac{(2\pi)^n \Gamma\left(\frac{n-a}{2}\right)}{2^a \pi^{\frac{n}{2}} \Gamma\left(\frac{a}{2}\right)}$$

Relationship (27) is important in applications associated with self-affine fields and the solution to fractional differential equations such as the fractional Poisson equation

$$\nabla^a \phi(\mathbf{r}) = \rho(\mathbf{r}) \tag{28}$$

with a solution of the form

$$\phi(\mathbf{r}) = \frac{\lambda_{n,n-a}^{-1}}{|\mathbf{r}|^{n-a}} \otimes \rho(\mathbf{r})$$

obtained through application of the Fourier transform and the  $f(\mathbf{k})$  and the inverse Fourier transform of  $F(\mathbf{k})$  as Reisz definition of a fractional Laplacian, namely,

$$abla^a \phi(\mathbf{r}) \leftrightarrow \mid \mathbf{k} \mid^a \Phi(\mathbf{k}), \ \Phi(\mathbf{k}) = \mathcal{F}_n[\phi(\mathbf{r})]$$

The relationship between a, the Topological Dimension nand the Fractal Dimension  $D_F$  is given by the equation [44]

$$D_F = 1 - a + \frac{3n}{2}$$

Thus, for example, in the case when  $\rho$  is a stochastic field with a uniformly distributed power spectral density function, for  $\mathbf{r} \in \mathbb{R}^2$ , Equation (28) defines a Mandelbrot surface where  $a = 4 - D_F$ ,  $2 < D_F < 3$ , 1 < a < 2.

## VI. SELF-CHARACTERISTIC PHASE-ONLY FUNCTIONS OF THE FOURIER TRANSFORM

From the scaling theorem compounded in Equation (26), it is clear that if a = 1,

$$\mathcal{F}_n[f(\mathbf{r})] = F(\mathbf{k})$$

and if  $f(\mathbf{r})$  is an eigenfunction then, by definition, for eigenvalue  $\lambda$  (which may, in general, be real or complex),  $\mathcal{F}_n[f(\mathbf{r})] = \lambda f(\mathbf{k})$ . However, if a = i, then from Equation (26),

$$\mathcal{F}_n[f(i\mathbf{r})] = F(-i\mathbf{k})$$

Hence, for an eigenfunction  $f(\mathbf{r})$  (obtained for the case when a = 1) it is not possible to define an eigenfunction for the case when a = i, but rather a conjugate eigen-equation defined by the equation

$$\mathcal{F}_n[f(i\mathbf{r})] = \lambda f^*(i\mathbf{k})$$

This result illustrates that there can be no phase-only eigenfunctions of the type  $f(\mathbf{r}) = \exp[\pm i\theta(\mathbf{r})]$  for a phase function  $\theta(\mathbf{r})$ ; only conjugate phase-only eigenfunctions. In this context, we now consider the uniqueness of the case when  $\theta(\mathbf{r}) = \pm ar^2$  which is compounded in the following theorem.

Theorem VI.1. For a real constant a, there exists a Fourier pair of self-characteristic quadratic phase-only functions compounded in the result

$$\exp(\pm iar^2) \leftrightarrow \lambda_{\pm} \exp(\mp ik^2/4a), \ \lambda_{\pm} = (1\pm i)^n \left(\frac{\pi}{2a}\right)^{\frac{n}{2}}$$

*Proof.* A proof of this result can be obtained by expressing the Fourier transform pairs in terms of convolution integrals through application of the Bluestein decomposition [45]. Thus, noting that

$$\mathbf{k} \cdot \mathbf{r} = -\frac{|\mathbf{k} - a\mathbf{r}|^2}{2a} + \frac{k^2}{2a} + \frac{ar^2}{2}$$

where  $r \equiv |\mathbf{r}|$  and  $k \equiv |\mathbf{k}|$ , we can write, without loss of generality, the Fourier transform of  $f(\mathbf{r})$  in the form

$$F(\mathbf{k}) = \exp\left(-ik^2/2a\right)$$
$$\times \int_{-\infty}^{\infty} \exp\left(i \mid \mathbf{k} - a\mathbf{r} \mid^2/2a\right) \exp\left(-iar^2/2\right) f(\mathbf{r}) d^n \mathbf{r}$$

 $\mathbf{T}(\mathbf{1})$ 

$$f(\mathbf{r}) = \exp\left(iar^2/2\right) \frac{1}{(2\pi)^n}$$
$$\times \int_{-\infty}^{\infty} \exp\left(-i \mid \mathbf{k} - a\mathbf{r} \mid^2/2a\right) \exp\left(ik^2/2a\right) F(\mathbf{k}) d^n \mathbf{k}$$

We can therefore write the Fourier transform pair in the convolutional form

$$\int_{-\infty}^{\infty} \exp\left(i \mid \mathbf{k} - a\mathbf{r} \mid^{2} / 2a\right) \exp\left(-iar^{2} / 2\right) f(\mathbf{r}) d^{n}\mathbf{r}$$
$$= \exp\left(ik^{2} / 2a\right) F(\mathbf{k})$$
(29)

and

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \exp\left(-i \mid \mathbf{k} - a\mathbf{r} \mid^2 / 2a\right) \exp\left(ik^2 / 2a\right) F(\mathbf{k}) d^n \mathbf{k}$$

$$= \exp\left(-iar^2/2\right)f(\mathbf{r}) \tag{30}$$

Consider which phase-only functions of the form  $f(\mathbf{r}) =$  $\exp[\pm i\phi(\mathbf{r})]$  will generate a self-characteristic transformation of the type  $F(\mathbf{k}) = \lambda \exp[\mp i\phi(\mathbf{r})]$  for (complex) constant  $\lambda$ . This is equivalent to asking what is the form of the possible phase functions  $\phi_m(\mathbf{r}), m = 1, 2, \dots$  such that

$$\mathcal{F}_n[\exp[\pm i\phi_m(\mathbf{r})] = \lambda_m \exp[\mp i\phi_m(\mathbf{k})]$$
(31)

Consider the case when

$$f(\mathbf{r}) = \exp(iar^2/2) \tag{32}$$

It is then clear that  $\exp\left(-iar^2/2\right)f(\mathbf{r}) = 1$  and Equation (29) is reduced to

$$F(\mathbf{k}) = \lambda \exp\left(-ik^2/2a\right) \tag{33}$$

where

$$\lambda = \int_{-\infty}^{\infty} \exp(i \mid \mathbf{k} - a\mathbf{r} \mid^2 / 2a) d^n \mathbf{r}$$
$$= \int_{-\infty}^{\infty} \exp[i(k_1 - ar_1)^2 / 2a] dr_1 \times \dots$$
$$= \sqrt{\frac{2}{a}} \int_{-\infty}^{\infty} \exp(i\xi^2) d\xi \times \dots, \xi = \frac{k_1 - ar_1}{\sqrt{2a}}$$
$$= \sqrt{\frac{2}{a}} \sqrt{\frac{\pi}{2}} (1+i) \times \dots$$
$$= (1+i)^n \left(\frac{\pi}{a}\right)^{\frac{n}{2}} \equiv \exp\left(i\frac{\pi n}{4}\right) \left(\frac{\pi}{a}\right)^{\frac{n}{2}}$$

Thus we obtain the result

 $\exp(\pm iar^2/2) \leftrightarrow \lambda_{\pm} \exp(\mp ik^2/2a), \ \lambda_{\pm} = (1\pm i)^n \left(\frac{\pi}{a}\right)^{\frac{n}{2}}$ 

Corollary VI.1. If

$$\phi_{\pm}(\mathbf{r}) = \sum_{m=1}^{\infty} \exp(\pm imar^2/2), \ \mathbf{r} \in \mathbb{R}^n$$

then

$$\mathcal{F}_n[\phi_{\pm}(\mathbf{r})] = \lambda_{\pm}\psi_{\mp}(\mathbf{k})$$

where

$$\psi_{\mp}(\mathbf{k}) = \sum_{m=1}^{\infty} \frac{1}{m^{n/2}} \exp(\mp ik^2/2ma)$$

**Corollary VI.2.** Let  $f(\mathbf{r})$  be of compact support so that within some finite domain  $\mathbb{R}^n$ ,  $f(\mathbf{r}) \exists \forall \mathbf{r} \in \mathbb{R}^n$ . Since

$$|\mathbf{k} - a\mathbf{r}|^2 = k^2 \left( 1 - 2a \frac{\mathbf{k} \cdot \mathbf{r}}{k^2} + a^2 \frac{r^2}{k^2} \right),$$

then if ar/k << 1,

 $\infty$ 

$$\int_{-\infty}^{\infty} \exp\left(i \mid \mathbf{k} - a\mathbf{r} \mid^{2}/2a\right) \exp\left(-iar^{2}/2\right) f(\mathbf{r})d^{n}\mathbf{r}$$
$$\sim \exp(ik^{2}/2a) \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{r}) \exp(-iar^{2}/2) f(\mathbf{r})d^{n}\mathbf{r}$$

and hence from Equation (29), we can write

$$F(\mathbf{k}) \sim \mathcal{F}_n[f(\mathbf{r})\exp(-iar^2/2)]$$

This result shows that the high frequency spectrum associated with a function of compact support  $f(\mathbf{r})$  is similar to the high frequency spectrum of the function  $f(\mathbf{r}) \exp(-iar^2/2)$  when ar/k << 1.

**Corollary VI.3.** If we define the n-dimensional chirp transform (Fresnel transform for  $\mathbf{r} \in \mathbb{R}^2$ ) in the form

$$\mathcal{C}[f(\mathbf{r})] = \int_{-\infty}^{\infty} \exp(-i |\mathbf{k} - a\mathbf{r}|^2 / 2a) f(\mathbf{r}) d^n \mathbf{r}$$

*it is clear that we can write*  $C_n[f(\mathbf{r})]$ 

$$= \exp(-ik^2/2a) \int_{-\infty}^{\infty} \exp(-i\mathbf{k} \cdot \mathbf{r}) \exp(-iar^2/2) f(\mathbf{r}) d^n \mathbf{r}$$
$$= \exp(-ik^2/2a) \mathcal{F}_n[f(\mathbf{r}) \exp(-iar^2/2)]$$

and hence from Corollary VI.2

$$\mathcal{C}_n[f(\mathbf{r})](\mathbf{k}) \sim F(\mathbf{k}) \exp(-ik^2/2a), \ ar/k \ll 1$$

which shows that (for a function of compact support) the *n*-dimensional chirp transform in the high frequency range is the same as the Fourier transform filtered with the phase spectrum  $\exp(-ik^2/2a)$  at the same high frequency range.

**Corollary VI.4.** With  $a := \pm ia$ , we obtain the well known result for a Gaussian function, namely.

$$\exp(-ar^2/2) \leftrightarrow \lambda \exp(-k^2/2a), \ \lambda = \left(\frac{2\pi}{a}\right)^{\frac{n}{2}}$$

For the case when a = 1

$$\exp(\pm ir^2/2) \leftrightarrow \lambda_{\pm} \exp(\mp ik^2/2), \ \lambda_{\pm} = (1\pm i)^n \ (\pi)^{\frac{\tau}{2}}$$

and Form Equations (29) and (30) we can write the Fourier transform pair in the form

$$\exp\left(ir^2/2\right) \otimes \exp\left(-ir^2/2\right) f(\mathbf{r}) = \exp\left(ir^2/2\right) F(\mathbf{r})$$

and

$$\frac{1}{(2\pi)^n} \exp\left(-ir^2/2\right) \otimes \exp\left(ir^2/2\right) F(\mathbf{r})$$
$$= \exp\left(-ir^2/2\right) f(\mathbf{r})$$

where the independent vectors  $\mathbf{r}$  and  $\mathbf{k}$  are taken to be interchangeable.

**Remark VI.1.** There appears to be no other exact convolutional decomposition of  $|\mathbf{k} - a\mathbf{r}|^m$ , m = 2, 3, ... available in order express  $\mathbf{k} \cdot \mathbf{r}$  in terms of an expansion of  $|\mathbf{k} - a\mathbf{r}|^m$ other than for the case when n = 2. For the case when n = 1, the binomial expansion

$$\mid \mathbf{k} - a\mathbf{r} \mid = [k^2 + a^2r^2 - 2a\mathbf{k} \cdot \mathbf{r}]^{1/2} = k + \frac{a^2r^2}{2k} - \frac{a}{k}\mathbf{k} \cdot \mathbf{r} + \dots$$

provides a decomposition for  $\mathbf{k} \cdot \mathbf{r}$  that again can not facilitate a convolutional decomposition. It is therefore apparent that an exact convolutional representation of a Fourier transform is only available for the case when m = 2 and only m = 2. Thus, the Bluestein decomposition of a Fourier transform appears to be unique, and, by inference, the results obtained through application of this decomposition appear to be unique.

**Remark VI.2.** The result (for a = 1)

$$\mathcal{F}_n[\exp(\pm ir^2/2)] = \lambda_{\pm} \exp(\mp ik^2/2), \ \lambda_{\pm} = (1\pm i)^n (\pi)^{\frac{n}{2}}$$

is well known and can be derived directly from the Fourier transform itself. The approach taken here, which is based on re-writing the Fourier transform in terms of a convolution integral, has been taken in order to provide evidence for the uniqueness of this result in regard to asking the question: How many Fourier transform (conjugate) eigenfunctions are of a phase-only type? However, we have not proved that  $\exp(\pm ir^2/2)$  is a universally unique phase-only conjugate eigenfunction and that it is not possible for any other such functions to exist.

**Remark VI.3.** As discussed in Section V, there are many examples of amplitude-only functions that are eigenfunctions of the Fourier transform, both periodic and non-periodic.

However, in the context of the functions  $f(\mathbf{r})$  and  $F(\mathbf{k})$  considered in Equations (32) and (33), it is apparent that, from the Bluestein decomposition of the Fourier transform compounded in Equations (29) and (30), there can be no other phase-only conjugate eigenfunctions of the Fourier transform as defined by Equation (31), i.e. there appears to be no other phase-only function that has the same property (albeit of a conjugate type). This leads to the following conjectures.

**Conjecture VI.1.** The phase-only function  $\exp[i\theta(r)]$  has a conjugate eigenfunction of its Fourier transform if  $\theta(r) = \pm r^2/2$ , and, more generally, if  $\theta(r) = \pm (c + r^2/2)$  for a real constant c.

**Conjecture VI.2.** There are no phase-only eigenfunctions of the Fourier transform, i.e. if  $f(\mathbf{r})$  is a phase-only function, then for any complex or otherwise eigenvalue  $\lambda$ 

$$\mathcal{F}_n[f(\mathbf{r})] \neq \lambda f(\mathbf{k})$$

**Conjecture VI.3.** There is one and only one phase-only conjugate eigenfunction of the Fourier transform, namely  $f(\mathbf{r}) = \exp(-ir^2/2)$  for which the following conjugate eigen-equations are applicable:

$$\mathcal{F}_n[f(\mathbf{r})] = \lambda f^*(\mathbf{k}) \text{ and } \mathcal{F}_n[f^*(\mathbf{r})] = \lambda^* f(\mathbf{k})$$

where

$$\lambda = (1+i)^n \, (\pi)^{\frac{n}{2}}$$

**Remark VI.4.** These conjectures are representative of at least one, but nevertheless, a fundamental property associated with the chirp function which is unique. However, the proof given above only applies to the case when  $\theta(r) = \pm (c + r^2/2)$  and it has not been proven that there can be no other phase function except for  $\pm (c + r^2/2)$  which has this property. In order to convert this uniqueness conjecture into a uniqueness theorem it must be proved that  $\exp[i\theta(r)]$  has a conjugate eigenfunction of its Fourier transform **if and only if**  $\theta(r) = \pm (c + r^2/2)$ .

Coupled with the result compounded in Equations (22) and (23), Conjecture VI.3 is the foundation upon which we consider the chirplet modulation scheme discussed in Sections VII and VIII. As discussed in Section I, although the chirplet transform and chirplet modulation are well known, with a widely published range of applications, in light of the Conjecture VI.3, their use in communications engineering may not only provide an optimal but a universally unique solution to the exchange of information in any and all cases. In order to verify this statement, Conjecture VI.3 needs to be converted into a provable universal uniqueness theorem, a problem that lies beyond the scope of this paper and is left for future consideration by the author or otherwise.

#### VII. THE CHIRP FUNCTION AND THE COMMUNICATION OF INFORMATION

We return to the n-dimensional linear systems model compounded in the convolution equation

$$s(\mathbf{r}) = p(\mathbf{r}) \otimes f(\mathbf{r}) + n(\mathbf{r}), \ \mathbf{r} \in \mathbb{R}^{n}$$

where  $p(\mathbf{r})$  is characterized by a phase-only spectrum and where the noise function  $n(\mathbf{r})$  has a spectrum with both magnitude and phase. If we consider the function  $f(\mathbf{r})$  to be some 'information function' then  $s(\mathbf{r})$  is the output information signal.

In order to put the problem in to the more general context of information exchange, consider the case when Alice wishes to communicate with Bob by sending an information function  $f(\mathbf{r})$  in the knowledge that the information will be significantly perturbed by the noise associated with the communications environment, i.e. in the knowledge that upon reception by Bob, the Signal-to-Noise Ratio (SNR) is very low. In other words, Bob has no a priori knowledge of the information content - the function  $f(\mathbf{r})$  - but, like Alice, understands that upon reception of the 'signal'  $s(\mathbf{r})$ ,  $\|p(\mathbf{r}) \otimes f(\mathbf{r})\|_p \ll \|n(\mathbf{r})\|_p$ . In this context, let us assume that both Alice and Bob understand that if the model for the 'signal'  $s(\mathbf{r})$  is assumed to be of the form  $s(\mathbf{r}) =$  $f(\mathbf{r}) + n(\mathbf{r})$ , then for very low SNR's (now taken to be given by  $||f(\mathbf{r})||_p \ll ||n(\mathbf{r})||_p$ ), the information function is not recoverable, irrespective of Alice and Bob having knowledge of  $\Pr[n(\mathbf{r})]$  obtained by taking measurements of the background noise.

Consider the case when both Alice and Bob realize that in order to give the other the opportunity to recover the function  $f(\mathbf{r})$ , some transformation on  $f(\mathbf{r})$ , compounded in an (*n*-dimensional) operator  $\mathcal{T}_n$ , is required so that the model for the signal becomes

$$s(\mathbf{r}) = \mathcal{T}_n[f(\mathbf{r})] + n(\mathbf{r})$$

In terms of receiving information, the ideal transformation is one that supports the property

$$\mathcal{T}_n^{-1}[s(\mathbf{r})] = f(\mathbf{r})$$

This requires that

$$\mathcal{T}_n^{-1}[\mathcal{T}_n[f(\mathbf{r})]] = f(\mathbf{r}) \text{ and } \mathcal{T}_n^{-1}[n(\mathbf{r})] = 0$$

or at least

$$\mathcal{T}_n^{-1}[n(\mathbf{r})] = \delta^n(\mathbf{r})$$

when the information function is recovered uniquely except at  $f(\mathbf{r} = \mathbf{0})$  which remains undefined.

As discussed in Section IV.F, if the operator  $\mathcal{T}_n$  is of the form  $p(\mathbf{r}) \otimes$  where  $p(\mathbf{r})$  is characterized by a phase-only spectrum then

 $p^*(\mathbf{r}) \odot p(\mathbf{r}) = \delta^n(\mathbf{r})$ 

and

$$f(\mathbf{r}) = p^*(\mathbf{r}) \odot s(\mathbf{r}) - p^*(\mathbf{r}) \odot n(\mathbf{r})$$

Thus  $f(\mathbf{r})$  is recovered subject to a 'perturbation' by the correlation function  $p^*(\mathbf{r}) \odot n(\mathbf{r})$ , a perturbation whose influence is determined by: (i) the value of the SNR associated with the signal  $s(\mathbf{r})$ ; (ii) the extent to which  $p^*(\mathbf{r})$  remains uncorrelated with  $n(\mathbf{r})$ ,  $\forall \mathbf{r}$ , the idealized case being  $p^*(\mathbf{r}) \odot n(\mathbf{r}) = 0$ .

Given Conjectures VI.1, VI.2 and VI.3, the only function  $p(\mathbf{r})$  which has a self-characteristic phase-only spectrum is the chirp function  $\exp(\pm i\alpha r^2)$ . Moreover, in the following theorem, we show that if  $p^*(\mathbf{r})$  has a phase-only spectrum, then the perturbation of  $p^*(\mathbf{r}) \odot s(\mathbf{r})$  by  $p^*(\mathbf{r}) \odot n(\mathbf{r})$  is 'smaller' than for the case when  $p^*(\mathbf{r})$  has a spectrum characterized by both amplitude and and phase.

fe

**Theorem VII.1.** Let  $q(\mathbf{r})$  be an integrable function and  $n(\mathbf{r})$  be an integrable stochastic function, both of which are of compact support with band-limited spectra composed of amplitude and phase functions. If  $p(\mathbf{r})$  has the same phase spectrum as  $q(\mathbf{r})$  but a unit amplitude spectrum, then

$$\|p^*(\mathbf{r}) \odot n(\mathbf{r})\|_p < \|q^*(\mathbf{r}) \odot n(\mathbf{r})\|_p$$

*Proof.* The inequality given in the above theorem transforms into Fourier space as

$$|P^*(\mathbf{k})N(\mathbf{k})||_p < ||Q^*(\mathbf{k})N(\mathbf{k})||_p$$

where each spectrum is assumed to be band-limited so that, for any complex spectrum  $F(\mathbf{k})$ , say, with amplitude and phase spectra denoted by  $A_F(\mathbf{k})$  and  $\Theta_F(\mathbf{k})$ , respectively, we can write

$$\begin{aligned} \|A_F(\mathbf{k}) \exp[i\Theta_F(\mathbf{k})]A_N \exp[i\Theta_N(\mathbf{k})]\|_p \\ &\leq \|A_F(\mathbf{k}) \exp[i\Theta_F(\mathbf{k})]\|_p \|A_N(\mathbf{k}) \exp[i\Theta_N(\mathbf{k})]\|_p \\ &\leq \|A_F(\mathbf{k})\|_p \|A_N(\mathbf{k})\|_p \end{aligned}$$

Thus

$$||Q^*(\mathbf{k})N(\mathbf{k})||_p \le ||A_Q(\mathbf{k})||_p ||A_N(\mathbf{k})||_p$$

and if  $P(\mathbf{k}) = \exp[i\Theta_P(\mathbf{k})]$  then

$$||P^*(\mathbf{k})N(\mathbf{k})||_p \le ||1||_p ||A_N(\mathbf{k})||_p < ||A_Q(\mathbf{k})||_p ||A_N(\mathbf{k})||_p$$

This result yields the following conjecture:

**Conjecture VII.1.** The optimal solution to the problem of communicating information through additive noisy transmission environments is to apply a phase-only convolutional transform to the information function and correlate the received signal with an identical conjugate phase-function. Further, because of Conjectures VI.1, VI.2 and VI.3, a chirp provides a unique phase-only transform because a chirp has a self-characteristic Fourier transform.

If the propagation of this information through a noisy environment occurs over a long period of time where noise from different sources contributes additively on a continuous basis, then, upon reception of the signal, the noise function may be assumed to be a zero mean averaged Ergodic random field which, through the Central Limit Theorem will be normally distributed.

#### VIII. THE CHIRPLET TRANSFORM FOR FUNCTIONS OF COMPACT SUPPORT

So far in this paper, we have considered results that are consistent with the use of the (two-sided) chirp function in the infinite domain, studying forward and inverse solutions that are idealized. In this section we revisit the principal results associated with a chirplet, namely, a chirp function of compact support. Thus, consider the *n*-dimensional finite chirplet transform of a function  $f(\mathbf{r})$  that is of compact support, i.e.  $f(\mathbf{r}) \exists \forall \mathbf{r} \in \mathbb{R}^n$ , defined as

$$\mathcal{C}_n[f(\mathbf{r})] = \int_{\mathbb{R}^n} \exp(i\alpha \mid \mathbf{r} - \mathbf{s} \mid^2) f(\mathbf{s}) d^n \mathbf{s} \equiv \exp(i\alpha r^2) \otimes f(\mathbf{r})$$

where  $\alpha$  is a real constant and the convolution integral is finite.

**Theorem VIII.1.** The autocorrelation function  $c(\mathbf{r})$  of

$$p(\mathbf{r}) = \exp(i\alpha r^2), \ \mathbf{r} \in \mathbb{R}^n$$

or 
$$r_n \in [-R_n/2, R_n/2]$$
 is given by  
 $c(\mathbf{r}) = \exp(-i\alpha r^2) \otimes \exp(i\alpha r^2)$   
 $= \exp(-i\alpha r^2) \prod_{m=1}^n R_m \operatorname{sinc}(\alpha R_m r_m)$ 
(34)

where  $\operatorname{sinc}(x) \equiv \sin(x)/x$ .

Proof. The correlation function is given by

$$c(\mathbf{r}) = \int_{\mathbb{R}^n} \exp(-i\alpha |\mathbf{r} + \mathbf{s}|^2) \exp(i\alpha s^2) d^n \mathbf{s}$$
  
=  $\exp(-i\alpha r^2) \int_{\mathbb{R}^n} \exp(-2i\alpha \mathbf{r} \cdot \mathbf{s}) d^n \mathbf{s}$   
=  $\exp(-i\alpha r^2) \int_{-R_1/2}^{R_1/2} \exp(-2i\alpha r_1 s_1) ds_1 \times \dots$   
=  $\exp(-i\alpha r^2) \prod_{m=1}^n R_m \operatorname{sinc}(\alpha R_m r_m)$ 

Corollary VIII.1. Noting that

$$R_1 \operatorname{sinc}(\alpha R_1 r_1) \leftrightarrow \frac{\pi}{\alpha} \operatorname{rect}(k_1)$$

where

$$\operatorname{rect}(k_1) = \begin{cases} 1, & |k_1| \le \alpha R_1; \\ 0, & |k_1| > \alpha R_1. \end{cases}$$

and

$$\exp(-i\alpha r^2) \leftrightarrow \exp(-in\pi/2) \left(\frac{\pi}{2\alpha}\right)^{\frac{n}{2}} \exp(ik^2/4\alpha)$$

then from the convolution theorem

$$c(\mathbf{r}) \leftrightarrow \left[\frac{\pi \exp(-i\pi)}{2\alpha}\right]^{\frac{n}{2}} \exp\left(i\frac{k^2}{4\alpha}\right) \otimes \prod_{m=1}^n \left(\frac{\pi}{\alpha}\right)^m \operatorname{rect}(k_m)$$

**Corollary VIII.2.** For  $\alpha R_m >> 1$ ,

$$c(\mathbf{r}) \sim \prod_{m=1}^{n} R_m \operatorname{sinc}(\alpha R_m r_m)$$

and hence

$$\mathcal{F}_n[c(\mathbf{r})] \sim \prod_{m=1}^n \left(\frac{\pi}{\alpha}\right)^m \operatorname{rect}(k_m), \; \alpha R_m >> 1$$

Thus, when  $\alpha R_m >> 1 \forall m$ , the bandwidth of the autocorrelation function is determined by  $\alpha R_m$ . Further,

$$c(\mathbf{r}) = \prod_{m=1}^{n} R_m \operatorname{sinc}(\alpha R_m r_m) = \delta^m(\mathbf{r}), \ R_m \to \infty$$

illustrating that as the spatial extent of the chirp function increases, the autocorrelation function approaches a delta function and its spectrum  $C(\mathbf{k}) \rightarrow 1 \forall \mathbf{k}$  as  $R_m \rightarrow \infty \forall m$ . Also, for a chirp function of finite support, increasing the value of the chirp parameter  $\alpha$  linearly increases the bandwidth of the autocorrelation function.

**Corollary VIII.3.** If for  $\alpha R_m >> 1$ ,

$$c(\mathbf{r}) = p^*(\mathbf{r}) \odot p(\mathbf{r}) \leftrightarrow \prod_{m=1}^n \left(\frac{\pi}{\alpha}\right)^m \operatorname{rect}(k_m)$$

then since

$$p^*(\mathbf{r}) \odot p(\mathbf{r}) \leftrightarrow \mid P(\mathbf{k}) \mid^2$$

the spectrum  $P(\mathbf{k})$  of  $p(\mathbf{r})$  is also a band-limited spectrum and we may infer that

$$P(\mathbf{k}) = \prod_{m=1}^{n} \left(\frac{\pi}{\alpha}\right)^{m/2} \operatorname{rect}(k_m)$$

IX. CHIRPLET MODULATION OF A BINARY STRING

Consider the case when  $\mathbf{r} \in \mathbb{R}^1$  and  $r_1 = t$  so that we can work in the time domain when

$$f(t) \leftrightarrow F(\omega)$$

where  $\omega$  is the 'angular frequency' and

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt$$

If we chirplet transform an information function of time  $f(t) \exists \forall t \in [-T/2, T/2]$  and transmit the result through an environment characterized by an additive (zero-mean) noise function n(t), then the measured signal can be taken to be given by

$$s(t) = p(t) \otimes f(t) + n(t) \tag{35}$$

where

$$p(t) = \exp(i\alpha t^2), \ t = [-T/2, T/2]$$

By correlating s(t) with  $p^*(t)$  we obtain an estimate  $\hat{f}(t)$  say for f(t) given by

$$\hat{f}(t) = T\operatorname{sinc}(\alpha T t) \otimes f(t)$$
  
=  $p^*(t) \odot s(t) - p^*(t) \odot n(t), \ \alpha T >> 1$  (36)

We note that

$$T\operatorname{sinc}(\alpha Tt) \otimes f(t) \leftrightarrow \frac{\pi}{\alpha} \operatorname{rect}(\omega) F(\omega)$$

where

$$\operatorname{rect}(\omega) = \begin{cases} 1, & |\omega| \le \alpha T; \\ 0, & |\omega| > \alpha T. \end{cases}$$

The information function is therefore recovered with a bandwidth  $\alpha T$  subject to a distortion compounded in the correlation function  $p^*(t) \odot n(t)$ . In other words  $\hat{f}(t)$  is a band- and noise-limited version of f(t). Further, as discussed in Section IV.F, since we can expect the cross-correlation of  $p^*(t)$  with n(t) to generate a relatively small perturbation, we can write

$$\hat{f}(t) \sim p^*(t) \odot s(t), \ t \in [-T,T]$$

given that

$$f(t) = p^*(t) \odot s(t), \ t \in (-\infty, \infty)$$

For digital communications, the basic idea is to replace a single bit with a chirplet where the difference between a 0 and a 1 in a binary string is differentiated by the polarity of the chirp that is applied. Alternatively, a 0 and 1 could be differentiated by replacing them with a up-chirplet (increasing frequency) and down-chirplet (decreasing frequency), respectively (or visa verse). In this work, we consider the former case alone. Either way, the result produces a string of chirplets or a chirplet stream, which, after being frequency modulated using a carrier frequency  $\omega_0$ , say, is taken to be transmitted using a bandwidth  $\omega_0 \pm \alpha T$ . In this case, after frequency demodulation, the noise function n(t) in

Equation (35) is taken to be determined by the noise in the transmission frequency band  $\omega_0 \pm \alpha T$ .

Chirplet modulation of each bit as opposed to chirplet transforming a binary string provides the potential for generating greater 're-constructive power'. This approach comes at the 'computational cost' of extending the length N, say, of the binary string to a chirp-stream of length  $N \times T$ .

There are two approaches that can be considered: twosided chirplet modulation when a binary string is modulated using the chirplet  $p(t) = \exp(i\alpha t^2)$ , t = [-T/2, T/2]and single-sided chirplet modulation when the string is modulated using the chirplet  $p(t) = \exp(i\alpha t^2)$ , t = [0, T].

#### A. Continuous Time Chirplet Modulation

Consider an information function f(t) to be composed of a time limited sequence of M delta function which may have positive or negative polarity, i.e.

$$f(t) = III_{\pm}(t) = \sum_{m=-M/2}^{M/2-1} \delta_{\pm}(t - mT)$$

where  $\delta_+(t) = \delta(t)$  and  $\delta_-(t) = -\delta(t)$ . Two-sided chirplet modulation involves computing a chirplet stream function which is obtained by convolving  $\text{III}_{\pm}(t)$  with the two-sided chirplet  $p(t) = \exp(i\alpha t^2)$ , |t| < T/2, generating the signal

$$s(t) = \exp(i\alpha t^2) \otimes \operatorname{III}_{\pm}(t) = \sum_{m=-M/2}^{M/2-1} p_{\pm}(t - mT)$$

where  $p_+(t) = \exp(i\alpha t^2)$  and  $p_-(t) = -\exp(i\alpha t^2)$ . After frequency modulation, transmission and demodulation back to base-band, we assume that the received signal (now distorted by transmission noise) is given by

$$s(t) = \exp(i\alpha t^2) \otimes f(t) = \sum_{m=-M/2}^{M/2-1} p_{\pm}(t-mT) + n(t)$$

The estimate  $\hat{f}(t)$  of f(t), obtained through chirplet demodulation, is then given by correlating s(t) with  $p_+(t)$  or  $p_+(t)$  to give the band- and noise-limited reconstruction

$$\hat{f}(t) = \exp(i\alpha t^2) \odot s(t) \sim T\operatorname{sinc}(\alpha T t) \otimes f(t)$$

The information function considered is a continuous time function representation of a binary string, each bit in the string being distinguished by  $\delta_+(t)$  or  $\delta_-(t)$  for |t| < T/2 with a shift of mT. In practice (i.e. for digital signal processing), a discrete time approach is required which is considered in the following section.

#### B. Discrete Time Chirplet Modulation

Let  $f(t_n) \equiv \{0,1\}^N$  denote a binary string composed of N bits. The function  $f(t_n)$ , n = 1, 2, ..., N is a binary representation of information where each 1 in the string is a Kronecker delta function

$$\delta(t_n - t_m) = \begin{cases} 1, & m = n; \\ 0, & m \neq n. \end{cases}$$

Two-sided chirplet modulating, we replace the binary string with a chirplet stream consisting of a concatenation of the functions  $p_{\pm}(t_n) = \pm \exp[i\alpha(t_n - T/2)^2]$ ,  $t_n \in [0, T]$  where

 $p_+$  'encodes' 1 and  $p_-$  encodes 0 (or vice versa). Thus we consider a bit-to-chirplet stream transformation which can be written in the form

 $\{0,1\}^N \equiv \{1,0,1,\ldots\} \to \{\operatorname{cat}[p_+(t_n)]\}^{N \times T}$ 

where

$$\operatorname{cat}[p_{\pm}(t_n)] = \begin{cases} p_{+}(t_n), & t_n \in [0,T]; \\ p_{-}(t_n), & t_n \in [T,2T]; \\ p_{+}(t_n), & t_n \in [2T,3T]; \\ \vdots \\ p_{\pm}(t_n), & t_n \in [(N-1)T,NT]. \end{cases}$$

and cat denotes the concatenation of functions  $p_{\pm}(t_n)$ . The  $j^{\text{th}}$  bit,  $f(t_j)$ , j = 1, 2, ..., N, in the binary string  $\{0, 1\}^N$  therefore transforms to the  $j^{\text{th}}$  chirplet in the chirplet stream  $\operatorname{cat}[p_{\pm}(t_j)], j = 1, 2, ..., NT$ , the output being taken to be the real component of  $\operatorname{cat}[p_{\pm}(t_n)]$ .

'Chirp coding' a bit stream in this way allows each bit to be recovered by correlating the stream with the complex conjugate of an identical chirp function  $p_{\pm}(t_n)$ . Thus we can write the demodulated function as

$$d(t_n) = p_{\pm}^*(t_n) \odot \operatorname{Re}\{\operatorname{cat}[p_{\pm}(t_n)]\} \sim \operatorname{cat}[\delta_{\pm}(t_n - t_m)]$$

where

$$\delta_{\pm}(t_n - t_m) = \begin{cases} \pm 1, & m = n; \\ 0, & m \neq n. \end{cases}$$

A further process is then required to convert the Kronecker  $\delta$ -stream back to the original binary string, i.e.

$$\{\operatorname{cat}[\delta_{\pm}(t_n - t_m)]\}^{N \times T} \to \{0, 1\}^N$$

Since each reconstructed Kronecker delta will be positive or negative at positions nT/2, this can be accomplished through the following process:  $\forall j \in [1, N]$ :

$$\begin{cases} \text{if } \operatorname{Re}\{d[t_n = (2j-1)T/2]\} \ge 0, \quad f(t_j) = 1; \\ \text{if } \operatorname{Re}\{d[t_n = (2j-1)T/2]\} < 0, \quad f(t_j) = 0. \end{cases}$$

from which the binary string  $\{0,1\}^N$  is recovered.

#### C. Nyquist Sampled Chirplets

The algorithms considered in the following section are designed to test discrete-time modulation and investigate the Bit-Error-Rate (BER) for the reconstruction of binary strings with different values of the SNR. The reconstruction depends critically on knowledge of  $\alpha$  and T. In this sense, these parameters represent the keys required to recover the binary string, subject to the bit errors generated by low SNR's.

In practice, the value of T is the array length or period used to compute the chirplet. Given T, for discrete-time modulation, a value of  $\alpha$  must be chosen that avoids aliasing. To obtain a non-aliasing criterion for the maximum value of  $\alpha$  given T, we note that, from Equation (36), the autocorrelation of  $p(t_n) = \exp[i\alpha(t_n - T/2)^2]$ ,  $t_n \in [-T/2, T/2]$ has a bandwidth  $\sim \alpha T$  which is therefore the bandwidth of  $p(t_n)$  (from Corollary VIII.3). If  $t_n = n\Delta$  where  $\Delta$  is the time sampling interval, then, from the Sampling Theorem [5]

$$\Delta \le \frac{1}{2\alpha T}$$

A Nyquist sampled chirplet is therefore characterized by a chirp rate of

$$\alpha = \frac{1}{2\Delta T}$$

with an over-sampled chirplet obtained when

$$\alpha < \frac{1}{2\Delta T}$$

Thus, computation of a Nyquist sampled chirplet with  $\Delta = 1$ , requires a value of  $\alpha$  given by

$$\alpha = \frac{1}{2T}$$

This relationship for computing  $\alpha$  given T is used in the algorithms presented in the following section.

## X. CHIRP MODULATION AND DEMODULATION ALGORITHMS

To study the numerical performance associated with discrete time chirplet modulation, we consider the transmission of data distorted by additive noise and record the real component given by the signal

$$s(t_n) = \text{SNR} \times \text{Re}\{\text{cat}[p_{\pm}(t_n)]\} + r(t_n)$$
(37)

where

$$||r(t)||_{\infty} = 1$$

for different values of the SNR and the chirplet period T, where  $r(t_n)$  is a simulated random number stream. In this context, we present the principal steps associated with two algorithms for chirplet modulation and demodulation using a two-sided Nyquist sampled chirplet. It is assumed that the noise is zero mean Gaussian noise.

#### A. Modulation Algorithm

**Data inputs:** Binary string  $f_n$ , T - chirplet period (type: even integer), SNR - Signal-to-Noise Ratio (type: real double).

#### Data processing function(s): None.

**Step 1:** Read binary string  $f_n$ , n = 1, 2, ..., N computing the string length N.

**Step 2:** Convert the binary string to a bi-polar Kronecker delta array  $K_n$ , n = 1, 2, ..., N where each '0' in the input string is assigned value -1 and each '1' in the string is assigned value +1.

Step 3: Compute the chirplet array:  $p_m = \exp[i\alpha(m-T/2)^2], m = 1, 2, ..., T$  where  $\alpha = 1/2T$ 

**Step 4:** Multiply each value of  $K_n$  by  $p_m$  and concatenate the results to generate a 'chirplet stream' array  $c_k$  of length  $N \times T$ , returning the real component and normalizing the result.

**Step 5:** Compute a zero mean distributed array of random numbers  $r_k$  of size  $N \times T$  and normalize the result.

**Step 6:** Compute the array  $s_k = \text{SNR} \times c_k + r_k$ 

based on Equation (37).

**Data Output:** Write the array  $s_k$  to file (type: real double).

#### B. Demodulation Algorithm

**Data inputs:** Chirp stream array  $s_k$  (type: real double), T - chirplet period (type: even integer)

## Data processing function(s): Correlation.

**Step 1:** Read signal array  $s_k$ ,  $k = 1, 2, ..., N \times T$  and compute the array length NT.

Step 2: Compute the chirplet array:  $p_m = \exp[i\alpha(m-T/2)^2], m = 1, 2, ..., T$  where  $\alpha = 1/2T$ 

**Step 3:** Demodulate chirp stream by correlating array  $s_k$  with chirplet array  $p_m$  to compute array  $d_k$ ,  $k = 1, 2, ..., N \times T$ 

**Step 4:** Recover bit-stream  $f_n$  by evaluating the polarity of the array  $d_k$  at [(2k-1)T/2] - 1 such that if  $d_k \ge 0$  then  $f_n = 1$  and if  $d_k < 0$  then  $f_n = 0$ .

**Data output:** Write bit array  $f_n$  to file (type: Decimal Integer).

The step-by-step processes presented above are implemented in the MATLAB functions CM (Chirplet Modulation) and DM (Chirp Demodulation) given in Appendix A.A and Appendix A.B, respectively. These functions have been commented to provide a narrative on their composition which reflects the processes given using functions that are intrinsic to MATLAB. For generality, and, in order to provide a comparative study, both functions provide an option for the implementation of single-sided or two-sided modulation. Neither function undertakes checks for the validity of the input data or data processing errors and no graphical outputs are included. Function CM assumes the existence of a binary string file 'bstring.txt' and outputs a file consisting of a chirplet-modulated stream 'cm.txt'. Function DM inputs 'cm.txt' and outputs the de-modulated bit stream data to the file 'cd.txt'.

#### C. Examples Results

Figure 6 shows the real component of the two-sided chirp  $p_n$  used for modulation, the chirplet stream  $c_k$ , the noise distorted signal  $s_k$  for SNR=1 and the demodulated signal  $d_k$  for the simple input binary string  $f_n \equiv \{010\}$ . This result is the output from executing the MATLAB functions given in Appendix A for T=100, SNR=1 and opt=2, i.e. CM(100,1,2) and DM(100,2).

The Bit-Error-Rate (BER) in this case is zero and based on the equation

$$BER = \frac{1}{N} \sum_{n=1}^{N} |f_n - \hat{f}_n|$$

where  $f_n$  denotes the input binary string to function CM and and  $\hat{f}_n$  denotes the output string obtained after demodulation using function DM. This is because of the relatively



Fig. 6. Example of a two-sided chirplet (top-left), the chirplet modulation of  $\{010\}$  (top-right), the chirplet modulated signal with additive noise (bottom-left) and the signal after demodulation (bottom-right) for CM(100,1,2) and DM(100,2).



Fig. 7. Examples of a single-sided chirplet function (top-left), the chiplet modulation of  $\{010\}$  (top-right), the chirplet modulated signal with additive noise (bottom-left) and the signal after demodulation (bottom-right) for CM(100,1,1) and DM(100,1).

high value of the SNR used (i.e. SNR=1), which, using the conventional decibel scale  $SNR_{dB}$  where  $SNR_{dB} = 10 \log_{10}(SNR)$  is zero. For comparison, Figure 7 shows the equivalent result for single-sided modulation using the same input parameters, the BER also being zero, but the reconstruction having lower resolution. This is because the spectrum of a single-sided chirplet is, in effect, a single-sided spectrum with a band-width that is half that of the spectrum of a two-sided chirplet as illustrated in Figure 8.

To illustrate the effect of changing the input data and parameters, Figure 9 shows the demodulation of the longer binary string {0100001101001101} (ASCII binary for 'CM') using single-and two-sided chirplets for a larger chirplet period T = 1000 and for a lower SNR when SNR = 0.1 (SNR<sub>dB</sub> = -10 dB). In both cases the BER is zero.

Figure 10 shows the BER associated with the demodulation of a chirp stream generated by single- and two-sided chirplet modulation for T=1000 over a range of  $SNR_{dB}$ .



Fig. 8. Power spectra of the two-sided chirplet given in Figure 6 (solid line) and the single-sided chirplet given in Figure 7 (dashed line).



Fig. 9. Example of chirplet modulating {0100001101001101}: Singlesided chirplet (top-left), two-sided chirplet (top-right) and demodulated signals (lower-left and lower-right, respectively) for CM(1000,0.1,1), DM(1000,1) and CM(1000,0.1,2), DM(1000,2), respectively.

The binary string used in this case is the ASCII binary representation of the sentence 'BER test for chirplet modulation method.' and illustrates that as the SNR decreases (from the order of -15 dB to -30 dB), the BER increases but that from -15 dB to 0 dB, the BER is zero. Figure 10 also illustrates the greater accuracy (in terms of BER for a given SNR) associated with the application of single-sided chirplet modulation over the range of -15 dB to -30 dB. Note that the BER .v. SNR<sub>dB</sub> profile given in Figure 10 is specific to the value of T that is used. The general characteristics of this profile for different values of T lie beyond the scope of this work but can be investigated by interested readers using the functions provided in Appendix A.

## XI. CHIRPLET MODULATION USING BANDWIDTH FACTORIZATION

In order for Alice and Bob to communicate information in the form of a binary string using algorithms CM and CD, for example, the value of T must be known to both.



Fig. 10. Variations in the BER (vertical axis) with  $\rm SNR_{dB}$  (horizontal axis) for single-sided (solid line) and two-sided (dashed line) chirplet modulation.

In a cryptographic context, this can be undertaken using a key exchange algorithm where T is the key. Two such well-known and commonly used algorithms are the Diffie-Hellman algorithm [46] and the RSA algorithm [47]. The difference between these two approaches in terms of their 'security', is that the Diffie-Hellman method relies on the computational difficulties associated with evaluating discrete logarithms to high accuracy whereas the RSA algorithm relies on the factorization of two large prime numbers. Both the Diffie-Hellman and the RSA algorithms can be used to exchange keys which can then be used for symmetric encryption for which there are numerous legacy techniques and relative new methods [35]. This includes elliptic curve cryptography [48] which is a logarithm-family form of cryptography based on a different finite field with modular arithmetic.

Many other key exchange methods are available including those that are based the three-pass or no-keys protocol which includes the Adi Shamir three-pass protocol [49] and the Massey-Omura method [50]. This protocol requires that: (i) The encryption algorithm is commutative and strong enough so that the ciphertext cannot be broken using a known algorithm attack based on an intercept of any pass, particularly the single encrypted first and third passes; (ii) the keys used must be of a sufficient length to make an exhaustive attack impracticable on any pass; (iii) if the encrypted information is intercepted for each of the three passes, it is not possible to determine the plaintext from the three intercepts (assumed to be partial or complete intercepts in each case). It is Condition (iii) that yields the greatest vulnerability and any encryption system that exploits this protocol must be based on algorithms that exhibit some 'computational difficulty' in this respect. For example, in the case of the Massey-Omura algorithm, the security relies on the difficulty of computing discrete logarithms in a finite field [51]. Since the publication of this protocol in 1980, many key-exchange algorithms have been developed whose security relies on some form of computational difficulty such as the the exponentiation in a Galois field and the difficulty associated with computing inverse matrices. Variations have also been designed to make the protocol quantum safe [52]. This includes the application of phase-only based encryption using a three-pass protocol, [43] where the computational difficulties of breaking the ciphertext are compounded in: (i) the inability to solve the one-dimensional phase retrieval problem due to the Fundamental Theorem of Algebra; (ii) the inability to uniquely solve an under-determined cubic polynomial using a three-intercept cryptanalysis.

In this section, we consider a novel approach in which Alice and Bob can compute T based on knowledge of the bandwidth  $\alpha T$  of the communications system alone.

#### A. Key Exchange based on Knowledge of the Bandwidth

Suppose an analogue signal is received which, after demodulation to base-band, is known to be the result of chirplet modulation but where the value of the chirp rate  $\alpha$  and chirp length T is unknown. What values of  $\alpha$  and T should be used in order to construct the chirplet  $\exp(i\alpha t^2)$ ,  $t \in [-T/2, T/2]$ given that the angular bandwidth  $\alpha T$  and hence the bandwidth  $B = \alpha T/2\pi$  Hz is known?

This problem requires a solution other than undertaking an exhaustive search, i.e. searching through different values of  $\alpha$  and T and checking to see if the demodulation provides an output binary string that is 'intelligible' according to some test for intelligibility .v. randomness as discussed in Section XII. Given that we are required to compute two numbers from their product  $\alpha T$ , we consider a solution to this problem based on prime number factorization where the prime numbers p and q, say, are required to be evaluated from the semi-prime pq, a problem that is fundamental to breaking the RSA algorithm, for example [47].

Assuming that it is possible to undertake prime number factorization efficiently, consider a scenario where Alice wishes to communicate with Bob using chirplet modulation of a binary string. Instead of considering a separate key exchange protocol where Alice exchanges the keys  $\alpha$  and T to Bob prior to initiating the communication, she considers a maximum prime number factorization method (to be discussed). It must then be assumed that this method is known to Bob together with the bandwidth to be used (a pre-requisite in any communications system).

Given the algorithms considered in Section X, we are required to implement discrete time chirp modulation. As discussed in Section IX.C, we consider the case when the chirplet is Nyquist sampled and  $\alpha = 1/2T$ , noting that this restriction can be relaxed for the case when both  $\alpha$  and T are used independently. In this case, we consider the evaluation of T alone, determined form prime numbers whose product is as close to the maximum length of the string used to define the decimal precision of the bandwidth. This method is predicated on the assumption that both Alice and Bob know the angular bandwidth and have access to numerical processors with the same floating point accuracy.

#### B. Prime Number Factorization of the Angular Bandwidth

Let B be the side-band bandwidth in Hz of a communications channel available for Alice to communicate with Bob and let us write this number in the form

where *B* denotes the base 10 digits 
$$(0, 1, 2, ..., 9)$$
 and  $B_1B_2B_3...B_{n+1}$  is the significand consisting of  $n+1$  digits, the value of *n* being taken to be determined by the decimal accuracy available. We then find the largest pair, and only the pair, of prime numbers *p* and *q* such that the semi-prime  $pq$  is given by

$$pq = B_1 B_2 B_3 \dots B_m, \ m \le n$$

where (m-n) is a minimum.

In principle, this can be achieved by systematically reducing the number of digits  $B_1B_2B_3...B_n$  (from right to left) one digit at time and decomposing the integer obtained into a product of prime numbers (given that any positive integer can be decomposed into a product of prime numbers) until the first two prime number factorization is achieved. We then compute the decimal number

 $d = \frac{p}{q}, \ p > q \text{ or } d = \frac{q}{p}, \ q > p$ 

so that

$$d = d_1 d_2 d_3 \dots$$
, where  $d_1 \ge 1$ 

Finally, applying an upper bound to the value of T given by  $T \leq 10^L$ , say, where L is a positive integer, we set

$$T = d_1 d_2 d_3 \dots d_L$$

which gives the period used to compute the chirplet.

The larger the value of L the greater the computational time required to modulate and demodulate. A binary string of size N yields a chirp-stream of size  $NT \leq NL$ . For demodulation, the direct correlation process used in function CD requires  $\sim NL^2$  floatig point multiplications (and additions). While this can be reduced to  $\sim NL \log_2 L$  by using a Fast Fourier Transform, it is clear that the value of L needs to be kept to a minimum in order to reduce the computational overheads. This issue depends upon the computational speed coupled with floating point accuracy available to Alice and Bob.

#### C. Numerical Example

To illustrate the scheme considered in the previous section, suppose that Alice and Bob wish to communicate through a component of the electromagnetic spectrum that is composed of wavelengths between 18 - 21 cm inclusively (in the microwave range), which is equivalent to a frequency range of 1.42 - 1.67 GHz giving an available bandwidth of 0.25 GHz. Let us assume that Alice and Bob can process data with a maximum floating point precision of 16 digits, for example. In this case

$$\alpha T = 2\pi \times 0.25 = 1.570796326794897$$

and application of prime number factorization for the integers 1570796326794897 and 157079632679489 then yields the following results:

$$1570796326794897 = 2 \times 3 \times 751 \times 348601049;$$

 $157079632679489 = 13 \times 12083048667653,$ 

where it is noted that

$$B = B_1 \cdot B_2 B_3 \dots B_n = B_1 B_2 B_3 \dots B_n \times 10^{-(n-1)}$$

$$1.3 \times 1.2083048667653 = 1.570796326794890$$

The first two-prime number factorization of the largest integer less than 1570796326794897 is thus obtained, after which the process is terminated, the integer 157079632679489 being the first semi-prime to be computed in this example.

Since 13 < 12083048667653 we compute

 $d = 12083048667653/13 = 9.294652821271538 \times 10^{11}$ 

Finally, with L = 3 say, T = 929 and  $\alpha = 1/2T = 5.3821 \times 10^{-4}$ . Note that with L = 4, T = 9294 and  $\alpha = 5.3798 \times 10^{-5}$  which may impose significant computational overheads upon the process especially if the size of the binary string is large. In this case, subject to L having been set by both parties (which depends on the floating point accuracy available to both), Bob only needs to know the operational bandwidth of the communications channel (which needs to be known by default) to implement chirp demodulation and, of course, the prime number factorization rule that Alice has applied (as discussed in Section XI.B).

A key issue in regard to this approach is that the angular bandwidth can be defined with arbitrarily high decimal point precisions. This is because, provided B is taken to be a rational number, then the angular bandwidth  $\Omega = 2\pi B$  is an irrational number because  $\pi$  is an irrational number - a decimal number that does not terminate. Thus the angular bandwidth can be specified to any decimal place accuracy and is limited only by the floating point accuracy available to Alice and Bob. If we increase the size of at least one of the two primes (whose product is a semi-prime) then upon refloatation, the (angular) bandwidth is recovered to a decimal place accuracy that is equal to or less than that considered by Alice. This approach provides the potential for using Shor's algorithm [53] to generate the prime numbers when the angular band-width is specified as a decimal number with enough decimal digits to make the semi-prime representation unable to be factored using conventional digital computing. For example, no prime number factors have, to date, been found for the RSA algorithms RSA-240 (which has 240 decimal digits - 795 bits) through to RSA-2048 (which has 617 decimal digits - 2048 bits).

Prime number factorization is of particular significance in regard to quantum computing and the application of Shor's algorithm and developments thereof [54]. It is therefore conceivable that, at some future date, Alice and Bob could use chirplet modulation using semi-prime representations of  $\Omega = 2\pi B$  with 1000++ digits, say, if and only if they both have access to quantum computers to implement Shor's algorithm to compute  $\alpha$  and T. The security of such a communications protocol is self-evident especially when the computational facilities are available to significantly increase the value of L that can be applied and limited to a select few.

## XII. INTELLIGIBILITY OF DEMODULATED BINARY STRINGS

The approach considered in the previous section assumes that Alice and Bob have a prior knowledge on the following:

- (i) the bandwidth of the communications channel;
- (ii) the two prime-number factorization protocol of the bandwidth used to compute the chirp rate  $\alpha$  and chirp length T;

- (iii) the value of the upper bound L used to compute the chirplet period that is to be applied;
- (iv) the binary string chirp modulation/demodulation algorithms presented in Section X, albeit simple examples.

In regard to the last point, it is assumed the input binary string is not encrypted. If encryption is applied, then both Alice and Bob need to know *a priori* the encryption algorithm(s) and key(s) that are required. In order to perform a decrypt, the demodulation must be known to generate an output that is subject to a minimal, and, ideally, a zero BER. This requires the noise generated in the communication of the chirp stream to be as low as possible, i.e. the bandwidth chosen should have minimal noise characteristics.

The chirp demodulation algorithm given in Appendix A.B, has been designed to always outputs a binary string. This is because the bits are recovered by checking the polarity of the demodulated array at the positions in the array corresponding to the centers of the chirplets which occur at points [(2k + 1)T/2] - 1, k = 0, 1, 2, 3, ... making the value of T (an even number) critically important and thereby, the primary key to the demodulation process. The binary string generated by this function is therefore subject to BERs generated by the following:

- errors in the computation of T;
- an accurate computation of T but low values of the SNR leading to high BER;
- both of the above.

Errors in the computation of T may be related to an incorrect choice if the bandwidth subject to application of the prime number factorization of the angular bandwidth. In this context, we now consider how the binary string can be evaluated in terms of its 'intelligibility' when the input is unknown and hence a BER analysis can not be performed.

The term 'intelligibility' usually applies to the clarity of speech and/or writing and whether they are clear enough to be understood. Here, the term refers to whether or not a binary stream generated by chirp demodulation is entirely random or otherwise or a mixture of both where, in the former case, it is assumed that there is no source of chirplet modulated information embedded in the noise and that consequently the binary string obtained through demodulation is noise driven. One of the keys to doing this is to analyze the output binary strings in terms of their information entropy. This is discussed in the following section.

#### A. Information and Entropy

The first and arguably the most important relationship between information and entropy was first established by Leo Szilard as a result of his solution to the 'Maxwell demon' thought experiment, named after James Clerk Maxwell. Maxwell first proposed this thought experiment as a result of his work on the properties of ideal gases in the 1860's. He considered a model where gas particles are free to move inside a stationary container whose interactions occur through elastic collisions in which they exchange energy and momentum with each other or with their thermal environment. This model is compounded in the Maxwell-Boltzmann Probability Density Function P(v) for the velocities v of identical gas particles with a mass m given by

$$p(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2k_B T}\right)$$

where T is the thermodynamic equilibrium temperature of the gas in °K and  $k_B \simeq 1.4 \times 10^{-23} \, \mathrm{JK^{-1}}$  (Joules per Kelvin) is the Boltzmann constant which relates the average kinetic energy of particles in an ideal gas with the temperature of that gas. The mode of this distribution gives the most probable speed of a particle, i.e.  $v_p = \sqrt{2k_BT/m}$ .

The thought experiment considers a 'demon' operating a frictionless shutter placed in the center of a container that is opened to allow particles with a velocity  $v < v_p$  to enter into one section of the container and particles with velocity  $v \ge v_p$  to enter into the other section where both the containing and shutter are perfectly thermally isolated. In this way, high and low velocity gas particles are separated into the two sections of the container while preserving their velocities, the equilibrium temperatures of the two sections being lower and higher than that of the original container.

For a classical thermodynamic process, work W can only take place when there is a temperature gradient, and, for an irreversible process, the Entropy S always increases, the change in entropy  $\Delta S$  being given by  $\Delta S = \Delta W/T$ . This is the basis for the second law of thermodynamics, a law that appears to be broken according to the thought experiment considered above because the entropy of the two sections is now different, and, given that there is an increase in temperature in one section, the entropy has been lowered without expending energy.

In Leo Szilard's 1922 doctoral dissertation and companion landmark paper [55], he showed how the paradox can be solved by taking into account the fact that in order for the demon to open and close the shutter to let particles of different velocities through, a decision must be made, a decision that is based on gathering information on the velocity of the particle before it is let through the shutter *a priori*. The information measured is taken to provide a 'balance' to the decrease in the physical entropy and is compounded in the 'Information Entropy'. In this context, Szilard's principal contribution was to consider that the demon must be an 'intelligent being' that can make a decision based on *a priori* information on the velocity of a particle, a critical issue, that Maxwell had failed to conceive of an include in his original thought experiment.

Szilard's original concept on information entropy has become the basis of information theory, showing that there is an increase of  $k_B \log_2 2$  units of entropy in any measurement. This concept was independently 'discovered' by Claude Shannon in 1949 [56] (to whom credit is usually given) and Andre Kolmogorov and Yakov Sinai, who developed a modified form in 1959 [57]. In developing a solution to a paradox in thermodynamics, Leo Szilard introduced an idea that is arguably the single most important icon of the information revolution of today. This is because information entropy provides the key for estimating the (average) minimum number of bits needed to encode a string of symbols, based on the frequency of those symbols.

In statistical mechanics, entropy is a measure of the number of ways in which a system may be arranged, often taken to be a measure of 'disorder' where the higher the entropy, the higher the disorder. Another way of interpreting this metric is in terms of it being a measure of the lack of information available on the exact state of a system. Shannon entropy is a measure of the information required to determine precisely a systems state from all possible states, and is expressed in binary digits, or 'bits'.

More generally, information is a measure of order, a universal measure applicable to any structure or any system. It quantifies the instructions that are needed to produce a certain organization. There are several ways in which one can quantify information but a specially convenient one is in terms of binary choices. In general, we compute the information inherent in any given arrangement from the number of choices we must make to arrive at that particular arrangement among all possible arrangements. Intuitively, the more arrangements that are possible, the more information that is required to achieve a particular arrangement.

1) Shannon Entropy: Consider a digital signal  $s_m$ , m = 1, 2, ..., M composed of M values. Let the probability that a specific value  $s_m$  occurs in the signal within a bin n be  $p_n$ , n = 1, 2, ..., N. The information associated with an outcome  $s_m$  within a bin n is  $-\log p_n$  which is a measure of the information required to specify  $s_m$  in terms of it being a member of a subset or 'bin' where  $p_n$  is the distribution of bins. Thus,  $p_n$  is the histogram of  $s_m$ . The mean value  $\mu$  say, of  $s_m$  is equal to the sum over every possible value weighted by the discrete probability distribution  $p_n$  of that value, i.e.

$$\mu = \sum_{n=1}^{N} n p_n$$

Similarly, the Shannon Information Entropy (usually denoted by S), is a measure of the mean (in this context, the 'expected value') of the information measure  $-\log p_n$  and is given by the dimensionless quantity

$$S = -\sum_{n=1}^{N} p_n \log p_n$$

The higher the entropy of a signal becomes the greater is ambiguity, and, in this context, the information entropy Sis a measure of the unpredictability or randomness of any message contained in the signal. This is typically determined by the noise that distorts the information contained in a signal. In general, the information entropy associated with the transmission of information in a signal tends to increase with time. This is due to the increase in noise that distorts the signal as it propagates, the sources of this noise being multifaceted and tending to Gaussian noise as a consequence of the Central Limit Theorem.

2) Boltzmann Entropy: The partner entity to the information entropy in physics has a dimension called 'Entropy' first introduced by Ludwig Boltzmann and J. Willard Gibbs as a measure of the dispersal of energy; in a sense, a measure of disorder, just as information is a measure of order. In fact, Boltzmann's entropy concept has the same mathematical roots as Shannon's information concept in terms of computing the probabilities of sorting objects into bins. In statistical mechanics, the Boltzmann Entropy is defined as

$$E = -k_B \sum_{n=1}^{N} p_n \ln p_n$$

Shannon's and Boltzmann's definitions of entropy are similar given that S and E differ only by their scaling factors.

In the definition of the Boltzmann entropy, the probabilities  $p_n$  refer to the energy levels of a 'classical system' (e.g. a collection of classical Newtonian particles). With the information entropy,  $p_n$  is not assigned *a priori* such specific roles and the expression can be applied to any physical system to provide a measure of order. Thus, information becomes a concept equivalent to physical entropy and any system can be described in terms of one or the other. An increase in information entropy implies a decrease of information.

3) Renyi Entropy: As with many other fundamental definitions in mathematics and physics, so the information entropy has a number of 'variations on a theme'. A generalization of the Shannon entropy is the Renyi Entropy  $H_{\alpha}$ (of order *a*) given by

$$H_{\alpha} = \frac{1}{1-\alpha} \log \sum_{n=1}^{N} p_n^{\alpha}$$

where  $\alpha \geq 0, \ \alpha \neq 1$  and

$$\lim_{\alpha \to 1} H_{\alpha} = \sum_{n=1}^{N} p_n \log p_n$$

which recovers the Shannon entropy. From this generalization, a number of complementary information entropy measures are obtained when  $\alpha = 0$  ('maximum entropy'),  $\alpha = 2$  ('collision entropy') and  $H_{\infty} = -\log[\max p_n]$ ('minimum entropy'), for example.

#### B. Binary Information Entropy

In the case of a binary string  $f_{\ell}$  composed of L bits (i.e. the bit-stream  $\{0,1\}^L$ ) the elements of the string can take on only two values, 0 and 1, which are mutually exclusive. In this case, the Binary Information Entropy (BIE) function denoted by H becomes

$$H(p) = -\sum_{n=1}^{2} p_n \log_2 p_n$$
  
=  $-p \log_2 p - (1-p) \log_2(1-p)$  bits

where, if we let p denote the probability of 1 occurring in the binary string, then the probability of obtaining a 0 in the same string is 1 - p. Similarly, if p is taken to denote the probability of 0 occurring in the string, then the probability of obtaining 1 is (1 - p). In either case,  $0 \log_2 0 \equiv 0$  and H(p) = H(1 - p).

#### C. On the Intelligibility of a Binary String using BIE

Given a binary string, our problem is to evaluate whether the string is a binary representation of noise or whether it contains intelligible information in terms of it having some degree of determinism. This could include any natural language that has evolved through use, application and repetition without conscious planning but binary coded in a planned a premeditated way, e.g. the ASCII or any other coding systems for that matter. The purpose is therefore to establish a method by which a finite binary string of arbitrary length L can be compared against another of equal length L in terms of the relative order and/or disorder of all of its bits. Applying a basic binary entropy test is not sufficient, and, for this reason a BiEntropy function has been developed which is based upon a weighted average of the Shannon entropy for all but the last binary derivative of the string [59]. This is one a many studies that have been undertaken to develop suitable tests and measures of order, disorder, randomness, irregularity and entropy based on the computation of a single metric. While desirable computationally, focusing on the use of a single metric for this purpose is restrictive and can be statistically insignificant because of its self-selecting data predication. For this reason, in the following section, we consider complementary approach to the problem which is based on the application of the Kullback-Leibler Divergence of Relative Entropy for a stream of data that yields a statistically significant result as apposed to a single metric. This provides the foundations for an application of a machine learning approach as discussed later on.

#### D. Application of Kullback-Leibler Divergence

Since intelligibility is a relative concept, a relative metric should be considered which provides a measure of how a binary string compares in some way with a string that is known to be the product of a genuinely random process. Further, this comparison needs to be undertaken on a statistical basis, measuring how one probability distribution associated with the binary string compares to a reference probability distribution in terms of its information content.

We consider a solution to this problem using the Kullback-Leibler Divergence or Relative (Binary) Entropy function given by

$$R = -\sum_{n=1}^{2} p_n \log_2\left(\frac{q_n}{p_n}\right)$$

where  $p_n$  is the binary histogram of binary string  $f_{\ell} \equiv \{0,1\}^L$  and  $q_n$  is the binary histogram of some reference binary string  $g_{\ell} \equiv \{0,1\}^L$ , both strings being of finite length L. Suppose string  $f_{\ell}$  is 'intelligible string' (e.g. a binary string representation of some text from a natural language) and  $g_{\ell}$  is a random string. We require the metric R to be significantly different in terms of its numerical value to the case when both  $f_{\ell}$  and  $g_{\ell}$  are random binary strings. Ideally, what is required is to establish a threshold for the value of R, below which  $f_{\ell}$  can be classified as intelligible say, and above which,  $f_{\ell}$  can be classified as random. However, this assumes that a binary decision making process can be applied which may not be statistically significant for all cases and is not accurate enough to consider any transition from  $f_{\ell}$  being random to intelligible.

Instead, we consider an analysis of the relative entropy based on an interpretation of the statistical difference between the case when  $f_{\ell}$  is intelligible and  $g_{\ell}$  is random and when both  $f_{\ell}$  and  $g_{\ell}$  are random strings. Thus we compute the Relative Entropy Signal

$$R_m = -\sum_{n=1}^{2} p_{nm} \log_2\left(\frac{q_{nm}}{p_{nm}}\right), \ m = 1, 2, ..., M$$
(38)

where  $q_{nm}$  denotes the  $m^{\text{th}}$  binary histogram of the  $m^{\text{th}}$  random bit-stream. We then consider the following cases

(which are referred to as such in regard to presenting the results that follow):

- (i)  $p_{nm}$  is the  $m^{\text{th}}$  binary histogram of a non-random binary string;
- (ii)  $p_{nm}$  is the  $m^{\text{th}}$  binary histogram of a genuinely random string.

In the results that follow, the non-random string is obtained by generating the binary representation of the text associated with the abstract of this paper, achieved using the ASCII text to binary converter available at [60] (with the delimiter string set to none). A sequence of random binary arrays are generated using the MATLAB uniform distributed random number generator function *rand* (which returns floating point numbers in the interval [0,1]) and applying a round transformation (to output an array consisting of 0's and 1's), each array having the same length L and each array being independent of the other in terms of their pattern of elements. For each array of random bits, Equation (38) is computed where  $p_{nm} = p_n$ ,  $\forall m$ .

Figure 11 shows example signals of the relative entropy given by Equation (38) for cases (i) and (ii) above with M = 3000 and the corresponding 100-bin histograms. It is immediately clear that:

- For Case (i), when the string is non-random,  $R_m$  has a Gaussian-type distribution.
- For Case (ii), when the string is random,  $R_m$  also has a Gaussian type distribution but with a mean value  $\sim 1$ that is significantly greater than the mean for Case (i). However, in both cases, application of the Jargue-Bera (JB) test for normality shows that the JB statistic is significantly larger than  $\chi^2$  and the null hypothesis must therefore rejected, i.e. the series  $R_m$  does not actually conform a normal distribution.

From Figure 11, a principal observation is that the statistical characteristics of  $R_m$  for cases (i) and (ii) above is significantly different. For example, the difference in the mean of the relative entropy signal for the two cases is at least two orders of magnitude and can therefore be used to differentiate between an intelligible and a random binary string. We thus consider additional statistical measures to the mean value alone. Figure 12, shows log-linear scatter plots for the mean, standard deviation (std), the median and the mode for cases (i) and (ii) using four natural languages. This result shows that an intelligible binary string can be clearly differentiated from a random string by computing the relative entropy signal given by Equation (38). The results given in Figure 12 illustrate that the statistical characteristics change significantly for at least four different languages using translations from the English obtain with [61]).

In each case, the four metrics considered adhere to the following conditions:

$$\begin{split} & \operatorname{mean}[R_m]_i < \operatorname{mean}[R_m]_r, \ \operatorname{std}[R_m]_i < \operatorname{std}[R_m]_r \\ & \operatorname{median}[R_m]_i < \operatorname{median}[R_m]_r, \ \operatorname{mode}[R_m]_i < \operatorname{mode}[R_m]_r \end{split}$$

where the subscripts i and r denote the use of an intelligible and random binary string, respectively. The difference in the standard deviations between the two cases is less significant than the other parameters. The natural languages used for this exercise have been chosen for their structural and semantic differences, a more comprehensive study in this regard using a broader spectrum of natural languages lying beyond the scope of this work. In this respect, the MATLAB code given in Appendix A.C function RET (Relative Entropy Test) - used to generate the four metrics considered is provided for readers to reproduce the results given and investigate the output for other natural languages and non-random binary strings in general.



Fig. 11. Plots of the  $R_m$  (left) given by Equation (38) and the corresponding 100-bin histograms (right) for Case (i) - above - and Case (ii) - below, respectively, with M = 3000.



Fig. 12. Log-linear scatter plots of the mean, standard deviation, the median and the mode (from left to right in each plot, i.e. log values for points 1-4, respectively) for Case (i) - circular symbols - and Case (ii) - square symbols. The top-left plot shows the result for English, the top-right plot is the result for Arabic, the lower-left plot for Chinese (traditional) and the lower-left plot for Greek.

Figure 13 shows the effect of using this relative entropy test to characterize chirplet demodulation for increasing values of the SNR using the function RET provided in Appendix A.C. In this case, only the mean values of  $R_m$  have been used to characterize the difference between a binary string as it undergoes a transition from being random to intelligible (for low to relative high SNR's, respectively). This is quantified by the BER associated with the demodulation of a chirplet stream with an increasing SNR.

Form Figure 13 it is clear that as the BER decreases (for  $SNR_{dB} \in [-20, -10]$  dB), the value of mean $[R_m]$  increase quasi-linearly until a threshold is obtained when,



Fig. 13. Plots of the BER (solid line) and the mean values (dashed line) of the relative entropies  $R_m$  for M = 1000 and  $\text{SNR}_{dB} \in [-20, -2]$  dB. This result has been obtained using two-sided chirplet modulation/demodulation with T = 100 and  $\text{SNR} \in [0.01, 0.5]$ . Note that the mean values shown have been re-scaled by normalization. Also note that the binary string used in this test case is the binary form of the text given in the abstract of this paper.

for SNR<sub>dB</sub>  $\in [-10, -2]$  dB, the BER is zero. This test provides a way of differentiating between noise-driven (or otherwise) binary strings output by the chirplet demodulation process when there is no reference input string to compute the BER. In the context of the results given in Figure 13, we define the intelligibility of a binary string to be proportional to mean[ $R_m$ ] with maximum intelligibility being achieved when mean[ $R_m$ ] is a maximum.

In order to achieve a statistically significant result of this type, it is necessary to use relatively long binary strings N >> 1 and values of the length of the relative entropy signal M >> 1, the results given in Figure 13 being obtained for  $N \times M = 13355 \times 1000 = 1.3355 \times 10^7$ . It is important to note, that this test on the intelligibility of a binary string is predicated on the term 'intelligibility' being associated with a natural language only. This is a limited definition of the test studied in this work. In general, the term 'intelligibility' should be applied to a binary string that can be considered to be the result of a process that is other than an entirely random processes. In a more general sense, the test compounded in Equation (38) is a measure of the lack of randomness of a binary string.

#### E. Machine Learning

Given that the demarcation between an intelligible and a random binary string can be determined by applying the relative entropy test as discussed Section XII.D, the potential exists to compute further statistical metrics and other parameters based on an analysis of the signatures given in Figure 11. These may include the statistical moments and spectral properties of  $R_m$ , for example, designed to develop a feature vector whose purpose is to provide a multi-class classification used to input into an Artificial Neural Network (ANN). Four components of such a feature vector could be the mean, standard deviation, median and mode of the relative binary entropy signal as considered in Section XII.D. The value of such an approach relative to the growth in Deep Learning using deep ANNs operating on the binary strings themselves remains to be quantified.

#### XIII. COMMUNICATING THROUGH THE 'WATERHOLE'

The numerical example given in Section XI.C is based on using a bandwidth of 0.25 GHz for the communication of a chirplet modulated binary string. Although any bandwidth could be used, this bandwidth has not been chosen arbitrarily; it is the bandwidth associated with the 'Waterhole', a term first coined in 1971 by Bernard Oliver [63]. The waterhole is a particularly quiet band of the radio wave spectrum; the quietest channel in the interstellar radio noise background. For this reason it has been theorized that the waterhole would be the optimal frequency band for communicating with extraterrestrial intelligent life.

With its origins dating back to 1959 and its incorporation as an Institute in 1984, SETI (Search for Extraterrestrial Intelligence) has, to date, not obtained any re-producible evidence of intelligent information based on the analysis of radio signals in the waterhole spectrum. Given the uniqueness conjecture associated with chirplet modulation for communicating binary strings through noisy channels, and, that the waterhole is a minimum noise radio spectrum, chirplet demodulation of SETI signals may provide a way forward.

There are of course many assumptions (known and unknown unknown's) that have to be made in regard to the processing and analysis of any SETI signal (in addition to using the waterhole). In this case, we assume that Alice (an extraterrestrial) is attempting to communicate with Bob (ourselves) using unencrypted binary strings through application of chirplet modulation whose demodulation is dependent on the key T.

For consistency with the material discussed in Section XI, we also suppose that Alice uses a value of T that has been determined by prime number factorization of the sub-prime determined by the waterhole bandwidth (as discussed in Section XI.B) where, as the size of the sub-prime increases, the need for application of Shor's algorithm using a quantum computer becomes increasingly necessary. However, irrespective of the approach taken (consistent with Section XI, or otherwise), for any value of T used in the chirplet demodulation of a SETI signal, the binary strings that are output must be tested for intelligibility in order to ascertain whether they are the product of chirplet demodulating cosmic noise or the demodulation of chirplet modulated binary strings that are the product of some intelligible 'language'. Such a language can not be assumed to be a natural language as we understand the term, only something that is differentiable from binary strings that are genuinely random using the relative entropy test discussed in Section XII.D.

All the arguments given above, and, any variations upon their themes, are of course entirely speculative. However, given that SETI has, to date, and, after some 60 years of trying, not developed an algorithm confirming the existence of intelligible signals, it is arguable that future signal analysis of the type discussed in this paper could and perhaps should be applied on a complementary basis. The most important argument for this application is predicated on Conjecture VII.1, coupled with the interpretation of the outputs from chirplet demodulation using some of the ideas and methodologies discussed in [64], for example. However, there is another issue to consider which is where to look for such signals (i.e. which type of planetary systems should we 'focus' our radio telescopes on) and in the rest of this section we consider a new hypothesis in regard to this question.

### A. The Drake Equation

The Drake equation for estimating the number of communicative civilization N is well known and given by

$$N = R_* f_p n_e f_\ell f_i f_c L \tag{39}$$

where  $R_*$  is the average rate of star formation in our galaxy (~ 2 per year),  $f_p$  is the fraction of those stars that have planets (~ 1),  $n_e$  is the average number of planets which can potentially support life per star that has planets ( $\sim 0.2$ ),  $f_{\ell}$  is the fraction of planets which could support life that actually develop life at some point ( $\sim 0.1$ ),  $f_i$  is the fraction of planets with life that actually go on to develop intelligent life and civilizations ( $\sim 1$ ),  $f_c$  is the fraction of civilizations which develop a technology that releases detectable signs of their existence through electromagnetic emissions, for example ( $\sim 0.2$ ) and L is the window of time for which such civilizations release detectable and intelligible signals into space ( $\sim 10^6$  years) before their extinction, e.g. [65] and [66]. The numbers associated with each of these factors (as given above in the parenthesis) are of course just estimates and may vary, some of them quite considerably. However, some of the terms can be estimated with greater confidence than others. The term  $R_*$  is relatively well known and the terms  $f_p$  and  $n_e$  can be estimated with some precision. The terms  $f_{\ell}$  and  $f_i$  depend on biology and evolution and the terms  $f_c$  and L can only be speculated upon. Using the values given above for each term, we obtain a value for N given by  $\sim 8000$ . This number can change radically, especially in regard to the values of  $f_i$  and L that are assumed. The two principal scenarios are that  $N \ll 1$ (the rare earth hypothesis) or that N >> 1 which states that intelligent, technologically advanced civilizations are (relatively) common and that we should therefore be able to discover and identify intelligible signals transmitted through the waterhole.

Since its launch in 2009, and, over the nine years of its operation, the Kepler Space Telescope (KST) surveyed a relatively small region of the Milky Way in an attempt to discover Earth-size exoplanets and to provide an estimate of how many of the billions of stars in the Milky Way have such planets. Analysis of the data gathered by the KST to date reveals that every star appears to have at least one exoplanet, which is the reason for letting  $f_p \sim 1$  in Equation (39), and, that the order of 23% of such planets are rocky planets. While it is still not clear as to how many of these rocky exoplanets might support life, least of all intelligent life, it is also clear that some the guesstimates given in association with Equation (39) could be significantly larger than previously considered, in particular the factors  $f_p$  and  $n_e$ . In this context, the KST has enhanced the principles of cosmic pluralism which is the philosophical belief in numerous planets or natural satellites in addition to the Earth which harbor extraterrestrial life and in some cases, intelligent life

with advanced technological civilizations, as promoted by such culturally diverse philosophers as Aneximander (610-546 BCE), Fakhr al-Din al-Razi (1150-1210 CE), Giordano Bruno (1548-1600 CE), Benjamin Franklin (1706 - 1790 CE), Carl Sagan (1934 – 1996 CE) as well as Johannes Kepler (1571-1630 CE) and Frank Drake (1930 - CE).

## B. Longevity of Intelligent Life: Single .v. Multiple Star Systems

There is a wide spectrum of issues that need to be considered in relation to estimating the values of the latter terms in Equation (39), specifically  $f_i$ ,  $f_c$  and L. In regard to developing an estimate for L (a measure for the longevity of an intelligent species radiating electromagnetic radiation containing intelligible information) many known unknowns need to be considered such as the effect of  $\gamma$ -ray bursts, geological catastrophes (such a super volcanism), global pandemics, ecological catastrophes, extreme climate change (which is of particular current concern) and other mass extinction events e.g. [67], [68], [69] and [70]. The evolution of life on Earth has depended on a multitude of such events which have come at the expense of many life forms on Earth at the time of an event, often leading to new windows of evolutionary opportunity. This includes the evolution of one life form due to the way a now extinct life form has altered its environment. In this context, and, from the considerable wealth of studies that have been carried out in this field, there appears to be one that has not been considered, and, in relation to Equation (39), concerns the estimate L in terms of the number of multiple star systems that could support the evolution of intelligent life.

The large majority of stars (more than  $\sim 80\%$ ) are actually binary, triple or even higher star systems, the most common of these systems consisting of binary stars, i.e. systems of only two stars orbiting each other. The solar systems associate with these stars is very diverse in terms of the number of planets, there period of orbit, eccentricities and so on, and, to date, there is not enough data to provide a statistically significant distribution from which the normality (if any) of such solar systems can be quantified. Our sun is therefore in a  $\sim 20\%$  minority and our civilization might therefore be considered to be a relatively rare example of the development of an intelligent species capable of emitting intelligible signals who have developed on a rocky planet orbiting one star. The reason for emphasizing this point is that there may be a correlation between the orbit of the Earth around a single star and the longevity of our species, given the development of a highly influential and socially cohesive force that is predicated on the existence of one star, namely, monotheism.

Monotheism was first established by a small Egyptian sect led by Akhenaten (Akhen-Aten) over a twenty year period after he became Pharaoh in 1353 BCE as Amenophis IV during the 18<sup>th</sup> Dynasty (1550-1307 BCE). Loosely coupled with the independent development of Zoroastrianism (which emerged in Persia as a prehistoric Indo-Iranian religious system in the second millennium BCE), Akhenaten's challenge significantly disturbed the status quo of a fundamentally polytheistic society and seeded all the basic monotheistic values and many of the associated practices that have emerged since, [71], [72].

Judging from the artwork at the time, Akhenaten's behavior may have been the result of him suffering from a genetic disorder known as Marfan Syndrome (due to a mutation of fibrillin 1 on chromosome 15) which is a disease of microfibril dysfunction often leading to the development of a psychotic illness involving acute self-centric behavior [73]. The point here is that Akhenatenism was directly influenced by the Aten - the sun disk - from which the name Akhen-Aten (which loosely translates from Middle Egyptian as 'worshiper of the sun' or more specifically, the 'sun-disk') is derived. Moreover, it is arguable that there is a correlation between the catalyst for, and, the initial development of monotheism and regions of the Earth where the weather conditions prohibit biological diversity due to a lack of water. This is particularly noticeable in regard to societies with an agricultural infrastructure that could and still can (at least for the present) support a population in a relatively small geographical location due to its disposition to a regular source of water, i.e. arid regions where the 'Aten' is a dominant daily feature in the sky and where there are rivers such as the Nile and Jordan, for example.

Given the influence of monotheism on society that has developed since 1300 BCE, it has been, and, continues to be a detrimental factor on social development, especially in comparison to the philosophical, social and scientific progress that emerged in ancient Greece starting with the philosopher Thales (624-546 BCE), whose intellectual disposition was predicated on polytheism and the political processes it inspired, namely, Democracy. It is arguable that the social influence of monotheism has and continues to actively hinder the scientific, technological and social development of the 'Greek mind', especially in regard to the narrow-band spectrum associated with the educational ethos of monotheistic communities. For example, in Europe, some 1500 years of scientific and technological stagnation occurred through the active suppression of the Greek view of life following the legalization of Christianity in the Roman Empire by Constantine through the Edict of Milan in February 313 AD. Even before the development of Christianity of Europe in the fourth century AD, whatever may have been the achievements under centuries of Roman republican and imperial rule, the advancement of science was not, by comparison with Greek culture, one of them. Although Greek science was accepted and applied by the Romans, little was added to it, but upon the introduction and proliferation of Christianity in Roman society, intellectual activity became centered on the theology associated with monotheistic practice rather than 'pagan science', even though a number of Greek and Roman pagan scientists and philosophers were given the status of 'righteous pagan'.

Ironically, it is due to primarily Islamic culture that much of the scientific progress associated with ancient Greek civilization was at least documented, studied, translated and above all tolerated, but not, by comparison with the Greek's, developed further. As a result of this, Greek culture was eventually re-born in Europe (mainly through the scholarship exercised in southern Spain) in the  $15^{th}$  century AD which led to the renaissance in southern Europe and later, the reformations in northern Europe. It is therefore interesting to contemplate how much more technologically advanced our civilization might be today, had there been a continuity and proliferation of Greek culture, philosophy and science from its inception some 2500 years ago and Rome had not adopted Christianity for what was, at the time, politically correct reasons associated with the unification of the Roman empire under one religion. However, had monotheism not been initiated by an individual suffering from Marfan Syndrome and developed as it has, it might have occurred at another time and evolved along a different path because our planet orbits a single star. In other words, monotheism might be an inevitability for an intelligent species that evolves on a planet sustained by just one star. On the basis, it is therefore arguable that one star can inhibit the continuity of scientific progress of an intelligent species leading to long term damage on societal cohesion, thereby limiting both the scientific progress and the longevity of the species. This 'damage' is predicated on the principle that monotheism deprives an intelligent species from a fundamental need, namely, customer choice; a depravity that polytheists do not suffer from such as in Hinduism and modern paganism or in non-theistic religions such as Buddhism. This lack of choice can then lead to the development of a 'blame culture' and severe intolerance, thereby limiting scientific progress from the strength acquired through intellectual diversity when differentiation between races is not a conscious issue as in ancient Greek and pre-Christian Roman societies, for example. Another way of appreciating this is to consider that the continuity of one idea is always subservient to the diffusion of many ideas. This may be considered to be a societal reflection of the Central Limit Theorem, driven by disruptive ideas, and, more recently, disruptive technologies, the relatively recent information technology revolution being of particular significance in this respect. An example of such a 'societal reflection' is the collapse of communism in the 1980s which is arguably an important epoch in regard to the longevity of our civilization to date.

In the context of the discussion above, let us assume that the longevity of an intelligent species is better served by their development on a planet (or planets) that orbits a binary or triple star system, the stability of such orbits being accepted to be significantly more complex and thereby leading to conditions that are not necessarily suitable for the development of life. On the basis of this assumption, observations of intelligible signals might be better served by selecting regions of the cosmos that are known or at least suspected to contain multiple star systems supporting (stable) orbits of rocky planets, especially binary Red Dwarf stars.

Red Dwarfs stars are the most common type of star in the Milky Way ( $\sim 75\%$ ), at least in the neighborhood of the Sun. Further, unlike the 10 billion year lifespan of the Sun, the lifespan of a Red Dwarf is of the order of trillions of years, thereby providing significantly greater time for intelligent species to develop, possibly over multiple cycles. In fact, the nearest star to the Sun - Proxima Centauri - is a main sequence type M5 Red Dwarf approximately 4.2 light years distance from the Earth. It is both a triple star system (consisting of three Red Dwarf stars) and a planetary system consisting of one planet slightly larger than the Earth, with a relatively stable orbit in the habitable zone.

The arguments presented above imply that the term  $n_e$  should be considered to be the average number of planets that can potentially support life per multiple star system (that

has planets) and leads to the following Hypothesis (which should be considered in a complementary sense to Conjecture VII.1):

**Conjecture XIII.1.** In the search for extraterrestrial intelligence, concentration should focus on the chirplet demodulation of (water-hole) signals directed at multiple Red Dwarf star systems that are suspected or known to have rocky planets with a stable orbit in a habitable zone.

This conjecture, and the reasons for stating it, can of course be challenged, especially in the context of the stability of planets orbiting multiple stars. Indeed, it could be argued that intelligent life is more likely to occur on planets orbiting relatively rare single stars because of the greater likelihood of stable orbits being achieved. In this context, and, coupled with the arguments given above, the value N in Equation (39) becomes lower. The issue as to which argument may have a greater validity depends on further experimental evidence into the percentage of stable planetary orbits associated with multiple star systems compared with the relative low percentage of single stars supporting planets in the habitable zone. Either way, it should be appreciated that 99.9...% of all the species that have ever evolved on the Earth are now extinct and the idea that an intelligent species such as ourselves should defy such a statistically significant result in an evolutionary context and geological time frame is null and void.

Another issue concerns the multiplicity of chirp modulated signals. In the chirp demodulation method discussed in Sections X and XI, it is assumed that there is just one chirp modulated transmission which is taken to be embedded in the noise that is analyzed. But suppose that technologically advanced civilizations exist throughout our local region of the milky-way and beyond and that in each case, these civilization establish Conjecture VII.1 (ideally with the rigorous proof that currently eludes the author) and consequently emit chirp modulated binary strings. If a signal is detected which happens to be a combination of emissions in the sense that the noise contains the sum of chirp modulated information from different sources in the same locality, then the demodulated output would become distorted especially if the chirp pulse length is the same for each source. However, this scenario assumes that there are parallel civilizations emitting (chirp modulated) binary information over a similar time frame (at least on the scale of a light year) which is arguably going to be very rare.

## XIV. CONCLUSIONS

In revisiting the chirp function we have shown the importance of the chirp in a range of physical problems and further, have developed the conjecture that there is one and only one phase-only function that has a conjugate eigenfunction upon Fourier transformation, namely, the function  $\exp(ir^2/2)$ . We have considered the problem of how to chirplet modulate a binary string (using a one-sided or two-sided chirplet) and briefly demonstrated the numerical performance of some exemplar MATLAB functions (as provided in the Appendix A). These functions have been designed to test the process using a Nyquist sampled chirplet with period T, showing that a zero BER can be achieved subject to relatively low values of the SNR as illustrated in Figure 10. We have further considered a protocol for computing T based on the factorization of a known bandwidth by treating the integer string associated with the bandwidth as a semi-prime, thereby illustrating a method of securing chirplet modulated information as discussed in Section XI. A relative entropy test has been considered in order to obtain a statistically significant metric that differentiates between a randomized binary string and a binary string that represents an intelligibility associated with a range of natural languages as studied in Section XII. In this context, Section XIII has presented some speculative arguments associated with a possible application of chirplet modulation for communicating through the 'waterhole'.

## A. Some Open Questions and Ideas for Further Investigation

1) An important result given in this work in compounded in Conjecture VII.1. This is based on the result that, from Theorem VI.1, for (complex) eigenvalue  $\lambda$ 

$$\mathcal{F}_n[\exp[\pm i\phi(\mathbf{r})] = \lambda \exp[\mp i\phi(\mathbf{k})/2]$$

if  $\phi(\mathbf{r}) = \exp(ir^2/2)$ . In the proof of Theorem VI.1, it has not been proved that this results is applicable under the statement 'if and only if' and hence Conjecture VI.3 remain a conjecture until proven otherwise. Thus, an open question is whether it is possible to prove that there can be no other phase-only conjugate eigenfunction of the Fourier transform, other than the chirp function, coupled with a study of the Hermite Polynomials discussed in Section V for the complex case.

- 2) In [74], it is demonstrated that at least four different sources of independent information can be embedded in an audio signal by applying chirplet modulation at four different frequency ranges. In the context of the algorithms considered in Section X, this is equivalent to changing the value of T in the chirplet modulation of N different binary strings and embedding the chirp streams in noise. It remains to be investigated as to whether this approach is applicable, and, if so, what is the maximum value of N that provides recovery upon chirplet demodulation subject to an acceptable BER. If applicable, this would provide the potential for using chirplet modulation to communicate multiple binary strings in parallel, increasing the information content and throughput by a factor of N.
- 3) A study is required of the minimum bandwidth that can be used for chirplet modulation subject to the noise characteristics of the bandwidth. An open question is therefore what bandwidths are available within current communication infrastructures and whether there is sufficient bandwidth left for the application of frequency hopping, for example.
- 4) No bit-error correction schemes have been considered in the case when the BER is non-zero and a study is therefore required to obtain a threshold for the BER below which bit-error correction algorithms are applicable. The relationship between this characteristic and a minimum bandwidth that can be applied is required in order to obtain an optimum criteria for the application of chirplet modulation in general.
- 5) The study given in Section X.C is based on the application of Gaussian noise. Although this is a common noise

type in many communication systems due to the 'effect' of the Central Limit Theorem, an equivalent study is required using different noise types.

- 6) The chirplet modulation/demodulation schemes considered in this work are based exclusively on the application of a linear chirplet, with an instantaneous frequency that varies linearly with time. A study of the characteristics and performance associated with the application of different chirp types (such as the quadratic and exponential chirplet, for example) for modulation/demodulation is therefore an area for further research.
- 7) The chirplet modulation method considered in Section IX is based on coding bits using 'up-chirplets' which are characterized by an increasing frequency for both t > 0 and t < 0. Another approach is to differentiate between a bit through application of an 'up-chirplet' and a 'down-chirplet', where, in the latter case, the chirplet is characterized by a (linear) decrease in frequency with time. In this case, correlation with an up-chirplet, for example, will recover one bit type (0 or 1) alone. The other bit types could then be inferred indirectly. The effect of implementing this approach on the BER for different SNRs can easily be studied by modifying the Chirplet Modulation and Chirplet Demodulation functions given in Appendix A for both single- and two-sided up/down-chirplets.
- 8) The relative entropy test developed in Section XII.D has only been explored for four diverse natural languages and a further study is required for a much broader spectrum of modern and ancient languages to ascertain the extent to which this test can differentiate between random or otherwise binary strings. Moreover, such a study requires an extension into the nature of intelligible signals that transcends natural languages alone.

## B. Final Comments

In this extended paper, the author has attempted to introduce the reader to the chirp function and some of its applications in a way that illustrates how its characteristics emerge from fundamental physical models. It is therefore, with the exception of the Fourier transform, unlike any other function which forms the kernel of an integral transforms. There is a fundamental relationship between the Fourier transform and the chirp function which is compounded in the Bluestein decomposition of the Fourier transform; the basis for Theorem VI.1. In this context, it is arguable that the chirp transform should be considered to be even more fundamental than the Fourier transform of a function  $f(\mathbf{r})$ , given that the Bluestein decomposition of a Fourier transform yields what is essentially a chirp transform of  $f(\mathbf{r}) \exp(-i\alpha r^2)$ (ignoring the additional complex exponential that occurs outside the integral). Coupled with the results compounded in Equations (22) and (23), Theorem VI.I is a principal component of the arguments leading to Conjecture VII.1. In this context, the paper has aimed to qualify an aspect of communications engineering that has been respected by engineers since the early 1950s, [1]. This has been the governing purpose for the composition of this work that, in

its development, has yielded two other results that are of note in terms of original contributions: (i) the prime number factorization of the angular bandwidth used to compute the chirplet period; (ii) application and extension of the Kullback-Leibler Divergence to evaluate the intelligibility of a binary string.

To the best of the authors knowledge, the application of chirplet demodulation in the analysis of SETI signals does not appear to have been considered and may therefore be worth including in the portfolio of signal processing algorithms currently being used. Thus, coupled with the highly speculative argument given Section XIII, which is a personal view of the author alone, the following is proposed in regard to the search for extraterrestrial intelligence:

- (i) Point our radio telescopes at binary Red Dwarf star systems and 'listen' to the noise received in the waterhole bandwidth.
- (ii) Demodulate the signals received to base-band and chirplet demodulate the result using different values of the chirp rate and chirplet period or, at least to start with, the chirplet period alone, the latter case assuming that chirplet modulation has been implemented for a Nyquist sampled chirplet - as considered in the paper (a restriction that can be relaxed).
- (iii) Test the output binary streams for intelligibility using the method discussed in XII.D, a method that is routinely used to differentiate between bit streams derived from real noise and those that appear to be noise but are in fact encrypted fields or else have some plaintext or encrypted information embedded in them.

## C. Final Remark

By thinking about the search for extraterrestrial intelligent signals as a problem in cryptography, it is possible that new ideas will emerge that are of value in the advancement of secure communications in general as given in Section IX, for example. Similarly, the developments taking place in cryptology may become increasingly applicable to SETI. In this paper, the author has failed to achieve one of the principal goals which is to replace Conjecture VII.1 with a Theorem VII.1. The extensive nature of this work is in some ways a reflection of the authors frustration in not having achieved this. Nevertheless, the author hopes that the material will serve readers with enough information and direction to encourage further investigation into the reoccurring physical significance of the chirp function coupled with the apparent uniqueness of the chirplet transform in regard to the interpretation of cosmic noise.

## APPENDIX A PROTOTYPE MATLAB FUNCTIONS FOR CHIRPLET MODULATION AND DEMODULATION

The functions given in this Appendix have not been tested formally and are provided only to give the reader a guide to the basic algorithms required to implement the computational procedures discussed in Section X and to help the reader appreciate the theoretical models presented. Where possible, the notation used for array variables and constants etc. are based on the mathematical notation used or are acronyms for function names. Note that the m-code has been condensed spatially in order to conform to the format of this publication while minimizing the number of pages required to present it. The software was developed and implemented using (64bit) MATLAB R2017b with double precision floating point arithmetic.

A. MATLAB Function for Chirplet Modulation

```
function []=CM(T,SNR,opt)
%INPUTS: (int) T - chirplet period.
%(double) SNR - Signal-to-Noise Ratio.
%opt=1 - one-sided modulation.
%opt=2 - two-sided modulation.
%Read binary string from default
%file name 'bstring.txt'.
fid=fopen('bstring.txt','r');
f=fread(fid); fclose(fid);
N=size(f,1);%Compute size of string.
%Convert binary string (composed of
%ASCII decimal integers 48 and 49)
%into an array composed of values
%-1 (equivalent to 0) and 1.
for n=1:N
temp=f(n); if temp==48, K(n) = -1;
else K(n) = 1; end
end
%Compute (complex) chirplet, checking
%to see if T is odd, and, if so, setting
%its value to next largest even number.
if mod(T,2)==1 T=T+1; end
%opt=1 for single-sided modulation.
if opt==1
for n=1:T
t(n)=n-1; end
end
%opt=2 for two-sided modulation.
if opt==2
for n=1:T
t(n) = n - (T/2); end
end
%Compute Nyquist sampled chirplet array.
alpha=1/(2*T);
p=complex(cos(alpha*t.*t), ...
sin(alpha*t.*t));
%Chirp modulate the bit array.
for m=1:N
    for n=1:T
c(n,m) = K(m) * p(n); end
end
%Concatenate c, returning reals
%and normalising the result.
c=real(c(:)); c=c/max(abs(c));
%Compute normally distributed random
%numbers and normalise the result.
NT=size(c,1); r=randn(1,NT)';
r=r/max(abs(r));
%Add noise to c using specified SNR
s=SNR*c+r; %and output to file.
dlmwrite('cm.txt',s);
```

B. MATLAB Function for Chirplet Demodulation
function []=CD(T,opt)

%INPUTS:%int T - chirplet period. %opt=1 - single-sided modulation. %opt=2 - two-sided modulation. %Read chirp modulated data s=dlmread('cm.txt'); %Compute (complex) chirplet %checking to see if T is an odd %number and if so, setting value %to the next highest even number. if mod(T, 2) == 1 T = T + 1; end %opt=1 for one-sided modulation. if opt==1 for n=1:T t(n) = (n-1); end end %opt=2 for two-sided modulation. if opt==2 for n=1:T t(n) = n - (T/2); end end %Compute Nyquist sampled chirplet. alpha=1/(2\*T); %Compute chirp rate. p=complex(cos(alpha\*t.\*t), ... sin(alpha\*t.\*t)); %Demodulate data by correlation with %the conjugate of the chirplet. This %is implemented through application %of the MATLAB convolution function %conv by the flipping data from left %to right and returning the central %components of the convolution that %is the 'same' size as the input data %The output is normalised and the %real component returned. d=conv(s,conj(fliplr(p)),'same'); d=real(d/max(abs(d))); %Recover bit array by checking %polarity of array d for points %at the centre of a chirplet %of length T which occur at points %(T/2)-1, 3\*(T/2)-1, ... NT=size(d,1); %Compute size of d. step=round(NT/T);%Compute step size. n=1; %Start array processing. for m=1:step M=2\*m-1; temp=d((M\*T/2)-1); if temp <= 0 f(n)=0; n=n+1; end if temp > 0 f(n)=1; n=n+1; end end %and write result to file. fid=fopen('cd.txt','w'); fprintf(fid,'%d',f); fclose(fid);

# C. MATLAB Function to compute Basic Statistics of the Relative Entropy Test

function [Mean,Std,Median,Mode]=RET(M); %INPUTS: (int) M - length of Relative %Entropy Signal (RES). %OUTPUTS: Mean - Mean of the RES. %Std - Standard Deviation of the RES. %Median - Median of the RES.

```
%Mode - Mode of the RES.
%Shuffle random number generator.
rng('shuffle');
%Read binary string from default file.
fid1=fopen('binary_string.txt','r');
bstring=fread(fid1); fclose(fid1);
%Compute length of string
L=size(bstring,1);
%and convert binary string to an array of
%bits B with elements equal to 0 and 1.
for n=1:L
temp=bstring(n); if temp==48, B(n)=0;
else B(n) = 1; end
end %Compute binary histogram
h=hist(B,2);%and evaluate
%probabilities of bits.
p(1)=h(1)/L; p(2)=h(2)/L;
%Compute relative entropy signal.
for m=1:M
%Return random bits using function rand,
RB=round(rand(1,L));%compute histogram
h=hist(RB,2);%and evaluate
%probabilties of bits.
q(2)=h(2)/L; q(1)=h(1)/L;
%Compute relative entropy.
RES(m) = -sum(p.*log2(q/p)); end
%Plot the RES in figure 1.
figure(1), plot(RES);
%Compute the 100-bin histgram of RES
h=hist(RES,100);%and plot a bar graph
%of the result in figure 2.
figure(2), bar(h);
%Compute the Mean, Standard Deviation,
%Median and Mode of the RES.
Mean=mean(RES); Std=std(RES);
Median=median(RES); Mode=mode(RES);
```

## ACKNOWLEDGMENTS

The author would like to acknowledge the support of the Science Foundation Ireland, Technological Institute Dublin, Warsaw University of Technology, University of Western Cape, University of KwaZulu-Natal, University of Wales and the Ministry of Defense (UK and Oman). The views expressed in Section XIII.B are those of the author alone.

#### REFERENCES

- S. Darlington, "Chirp Pulse Transmission", US Patent 2678997, May 18, 1954.
- [2] S. Darlington, "Chirp Pulse Equalization", US Patent 3618095, November 11, 1969.
- [3] IEEE Computer Society, IEEE Standard 802.15.4a-2007, 2007. https: //ieeexplore.ieee.org/document/4299496
- [4] S. Mann and S. Haykin, "The Chirplet Transform: Physical Considerations", IEEE Transactions on Signal Processing, vol. 43, no. 11, pp. 2745–2761, 1995.
- [5] J. M. Blackledge, Digital Signal Processing: Mathematical and Computational Methods, Software Development and Applications, Edition 2, Woodhead Publishing, 2016. ISBN: 9781904275268. https: //arrow.tudublin.ie/engschelebk/4/
- [6] Wikipedia, Fourier Transform, 2020. https://en.wikipedia.org/wiki/ Fourier\_transform
- [7] J. R. Klauder, A. C. Price, S. Darlington and W. J. Albersheim, "The Theory and Design of Chirp Radars", Bell System Technical Journal, vol. 39, issue 4, pp. 745-808, 1960. https://ieeexplore.ieee. org/document/6773600

- [8] C. Q. Cook and A. Ariel, "Theory of Chirped Photonic Crystals in Biological Broadband Reflectors", Optica, vol. 3, issue 12, pp. 1436-1439, 2016. https://doi.org/10.1364/OPTICA.3.001436.
- [9] P. Dombi, P. Racz, L. Veisz and P. Baum, "Conversion of Chirp in Fiber Compression", Optics Letters, vol. 39, no. 8, 2014. http: //real.mtak.hu/23578/1/Dombi\_FiChirpConv\_OL.pdf
- [10] V. Tiwari, S. Klimenko, V. Necula and G. Mitselmakher, "Reconstruction of Chirp Mass in the Search of Compact Binaries", Classical and Quantum Gravity, vol. 33, no. 1, 2016. https://arxiv.org/abs/1510. 02426
- [11] C. Cutler and E. E. Flanagan, "Gravitational Waves from Merging Compact Binaries: How Accurately can one Extract the Binary's Parameters from the Inspiral Waveform?", Physical Review D, Vol. 49, No. 6, pp. 2658–2697, 1994.
- [12] J. M. Blackledge, Digital Image Processing: Mathematical and Computational Methods, Woodhead Publishing Series in Electronic and Optical Materials, 2005. ISBN-13: 978-1898563495. https:// arrow.dit.ie/engschelebk/3/
- [13] M. Moshinsky, "Diffraction in Time", Physical Review, vol. 88, no. 625, 1952. http://www.academiadelasciencias.va/content/dam/ accademia/pdf/acta18/acta18-moshinsky.pdf
- [14] C, Brukner and A. Zeilinger, "Diffraction of Matter Waves in Space and in Time", Physical Review A, vol. 56, no. 5, pp. 3804-3824, 1997. https://www.univie.ac.at/qfp/publications3/pdffiles/1997-05.pdf
- [15] A. Goussev, "Diffraction in Time: An Exactly Solvable Model", Physical Review A 87, 053621, pp. 1-5, 2013. https://arxiv.org/pdf/ 1304.5528.pdf
- [16] M. Kleber, M. "Exact Solutions for Time-dependent Phenomena in Quantum Mechanics", Physics Reports, vol. 236, pp. 331–393, 1994.
- [17] A. del Campo, G. García-Calderón and J. G. Muga, "Quantum Transients", Physics Reports, vol. 476, pp. 1–50, 2009.
- [18] A. Szriftgiser, D. Guéry-Odelin, M. Arndt and J. Dalibard, "Atomic Wave Diffraction and Interference Using Temporal Slits", Physical Review Letters, vol. 77, pp. 4–7, 1996.
- [19] S. Godoy and K. Villa, "A Basis for Causal Scattering Waves, Relativistic Diffraction in Time Functions", Journal of Modern Physics, vol. 7, no. 10, pp. 1181-1191, 2016. https://file.scirp.org/pdf/JMP\_ 2016062216590098.pdf
- [20] M. Beau and T. C. Dorlas, "Three-dimensional Quantum Slit Diffraction and Diffraction in Time", Dublin Institute for Advanced Studies, DIAS-STP-14-04, pp. 1-38, 2014. https://www.dias.ie/wp-content/ uploads/2013/06/DIAS-STP-14-04.pdf
- [21] J. M. Blackledge and B. Babajanov, "The Fractional Schrödinger-Klein-Gordon Equation and Intermediate Relativism", International Journal for Pure and Applied Mathematics, vol. 3, no. 8, pp. 601 -615, 2013. https://arrow.tudublin.ie/engscheleart2/76/
- [22] J. M. Blackledge, "Time Diffraction for the Intermediate Relativistic Case", In Preparation, 2020.
- [23] K. S. Chen, Principles of Synthetic Aperture Radar Imaging: A System Simulation Approach (Signal and Image Processing of Earth Observations), CRC Press, 2016, ISBN-13: 978-1-4665-9315-2 (eBook - PDF).
- [24] H. Maitre, Processing of Synthetic Aperture Radar (SAR) Images, Wiley, 2010, ISBN: 047039370X, 9780470393703
- [25] C Oliver and S QueganUnderstanding Synthetic Aperture Radar Images, SciTech Publishing, 2004, ISBN: 1891121316.
- [26] Interpreting SAR Images, Centre for Remote Imaging, Sensing and Processing (CRISP), 2019. https://crisp.nus.edu.sg/~research
- [27] A. S. Ghammeid, A. M. Abobaker, P. R. Babu and K. Senthilnathan, "Effect of Initial Chirp on Gaussian Pulse in an Optical Fiber", International Journal of Latest Engineering Research and Applications (IJLERA), vol. 2, no. 2, pp. 49-53, 2017. ISSN: 2455-7137
- [28] G. P. Agrawal, Nonlinear Fiber Optics, Academic Press, 2004.
- [29] A. Hasegawa and Y. Kodama, Solitons in Optical Communications, Oxford University Press, 1995.
- [30] F. Deng and W. Hong, "Chirp-Induced Channel of an Airy Pulse in an Optical Fiber Close to Its Zero-Dispersion Point", IEEE Photonics Journal, vol. 8, no. 6, 2016.
- [31] Y. Yovel, M. Geva-Sagiv and N. Ulanovsky, "Click-based Echolocation in Bats: Not so Primitive After All", Journal of Comparative Physiology A (2011) vol. 197, pp. 515-530, 2011.
- [32] S. V. Smirnov, S. M. Kobtsev and S. V. Kukarin, "Linear Compression of Chirped Pulses in Optical Fibre with Large Step-index Mode Area", Optical Society of America, Vol 23, No. 4, pp. 3914-3919, 2015.
- [33] A. Al-Ismaili, Dynamic Block Encryption with Self-Authenticating Key Exchange, PhD Thesis, Loughborough University, 2006.
- [34] J. M. Blackledge and E. Coyle, "Self-Authentication of Audio Signals by Chirp Coding", Proc. DAFx09, 12(1), Como, Italy, 1-2 September, pp. 1-8, 2009.

- [35] J. M. Blackledge, Cryptography Using Steganography: New Algorithms and Applications, Lecture Notes, Center for Advanced Studies, Warsaw University of Technology, Warsaw, 2012, ISBN: 978-83-61993-05-6;https://arrow.dit.ie/engscheleart2/40/
- [36] J. M. Blackledge, "Digital Watermarking and Self-Authentication using Chirp Coding", ISAST Transactions on Electronics and Signal Processing, vol. 1, no. 1, pp. 61-71, 2007. https://arrow.tudublin.ie/ engscheleart2/35/
- [37] J. M. Blackledge and O. Farooq, "Audio Data Verification and Authentication using Frequency Modulation Based Watermarking", International Society for Advanced Science and Technology, Journal of Electronics and Signal Processing , vol. 3 (ISSN 1797-2329), no. 2, pp. 51 - 63, 2008. https://arrow.tudublin.ie/engscheleart2/31/
- [38] J. M. Blackledge, Self-Authentication of Audio Data for Copyright Protection, Technology to License, Dublin Institute of Technology, July, 2009
- [39] J. M. Blackledge and O. Iakovenko, "Resilient Digital Image Watermarking for Document Authentication", IAENG International Journal of Computer Science, vol. 41, no. 1, pp. 1-17, 2014, http://www. iaeng.org/IJCS/issues\_v41/issue\_1/IJCS\_41\_1\_01.pdf
- [40] J. Diamond The Third Chimpanzee: The Evolution and Future of the Human Animal, Hutchinson Radius, 1991, ISBN: 978-0-09-174268.
- [41] J. M. Blackledge, W. Govere, D. Sibanda, "Phase-Only Digital Encryption", IAENG International Journal of Applied Mathematics, vol. 49, no. 2, pp. 212-228, 2019. https://arrow.tudublin.ie/engscheleart2/ 193/
- [42] J. M. Blackledge, P. Tobin, J. Myeza and C. M. Adolfo, "Information Hiding with Data Diffusion using Convolutional Encoding for Super-encryption", International Journal for Pure and Applied Mathematics, vol. 7, no. 4, pp. 319-356, 2017. https://arrow.tudublin. ie/engscheleart2/127/
- [43] J. M. Blackledge, P. Tobin, W. Govere, D. Sibanda and C. M. Adolfo, "Phase-only Digital Encryption using a Three-pass Protocol", ISSC2019, IEEE UK and Ireland Signal Processing Chapter, Maynooth University, 17-18, June 2019. https://arrow.tudublin.ie/ engscheleart/297/
- [44] M. J. Turner, J. M. Blackledge and P. R. Andrews, Fractal Geometry in Digital Imaging, Academic Press, 1998, ISBN 13: 9780127039701.
- [45] L. Bluestein, "A Linear Filtering Approach to the Computation of Discrete Fourier Transform", IEEE Transactions on Audio and Electroacoustics, vol. 18, issue 4, pp. 451-455, 1970. https://ieeexplore. ieee.org/document/1162132
- [46] W. Diffie and M. E. Hellman, "New Directions in Cryptography", IEEE Transactions on Information Theory, vol. 22, no. 6, pp. 644-654, 1976.
- [47] R. Rivest, A. Shamir and L. Adleman, "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems", Communications of the ACM, vol. 21, no. 2, pp. 120-126, 1978.
- [48] N. Koblitz, "Elliptic Curve Cryptosystems", Mathematics of Computation, vol. 48, no. 177, pp. 203-209, 1987.
- [49] A. Menezes, P. Van Oorschot and S. Vanstone, Handbook of Applied Cryptography, CRC Press, 500-642, 1996. ISBN: 0-8493-8523-7.
- [50] J. L. Massey and J. K. Omura "Method and Apparatus for Maintaining the Privacy of Digital Messages Conveyed by Public Transmission", US Patent US4567600A, 1986. https://patents.google.com/ patent/US4567600
- [51] K. Sakurai and H. Shizuya, "A Structural Comparison of the Computational Difficulty of Breaking Discrete Log Cryptosystems", Journal of Cryptology, vol. 11, pp. 29-43, 1998.
- [52] ETSI (European Telecommunications Standards Institute) White Paper No. 8, Quantum Safe Cryptography and Security: An introduction, Benefits, Enablers and Challenges, June 2015. ISBN No. 979-10-92620-03-0. https://www.etsi.org/images/files/ETSIWhitePapers/ QuantumSafeWhitepaper.pdf
- [53] Wikipedia, Shor's Algorithm, 2020. https://en.wikipedia.org/wiki/ Shor%27s\_algorithm
- [54] D. J. Bernstein, N. Heninger, P. Lou and L. Valenta, "Post-quantum RSA", International Workshop on Post-Quantum Cryptography, pp. 311-329, 2017. https://cr.yp.to/papers/pqrsa-20170419.pdf
- [55] L. Szilard, "On the Decrease of Entropy in a Thermodynamics System by the Intervention of Intelligent Beings", Zeitschrift für Physik vol. 53, pp. 840-856, 1922. http://www.sns.ias.edu/~tlusty/ courses/InfoInBio/Papers/Szilard1929.pdf
- C. E. Shannon, "A Mathematical Theory of Communication", [56] The Bell System Technical Journal, vol. 27, pp. 379-423, 1948. http://math.harvard.edu/~ctm/home/text/others/shannon/entropy/ entropy.pdf
- [57] A. N. Kolmogorov, "Entropy per Unit Time as a Metric Invariant of Automorphism", Journal of the Russian Academy of Sciences, vol. 124, pp. 754-755, 1959.

- [58] Y. G. Sinai, "On the Notion of Entropy of a Dynamical System", Journal of the Russian Academy of Sciences, vol. 124, pp. 768-771, 1959.
- [59] G. J. Croll, "BiEntropy - The Approximate Entropy of a Finite Binary String", 2013. https://pdfs.semanticscholar.org/d8d6/ bf07050f9dcd026bfafe62c079b4e309dd7a.pdf
- Text to Binary translator, RapidTables, 2019. https://www.rapidtables. [60] com/convert/number/ascii-to-binary.html
- Google Translate, 2019. https://translate.google.co.uk/
- What Is the Water-Hole?, 2003. http://www.setileague.org/general/ [62] waterhol.htm
- [63] M. Ross, The Search for Extraterrestrials: Intercepting Alien Signals, Springer, New York, p. 95, 2009. ISBN 978-0-387-73453-8.
- [64] D Oberhaus, Extraterrestrial Languages, MIT Press, Cambridge Massachusetts, London England, 2019. ISBN: 9780262043069. [65] F. Drake and D. Sobel, "The Origin of the Drake Equation",
- Astronomical Society of the Pacific, no. 46, pp. 1-4, 2010. https: //astrosociety.org/wp-content/uploads/2013/02/ab2010-46.pdf
- [66] S. Scoles and S. A. Heatherly "The Drake Equation: 50 Years of Giving Direction to the Scientific Search for Life Beyond Earth", Astronomical Society of the Pacific, Universe in the Classroom No. 77, pp. 1-5, 2011. https://astrosociety.org/edu/publications/tnl/77/uitc77.pdf
- [67] D. Pamlin and S. Armstrong, Global Challenges: 12 Risks that Threaten Human Civilization, Global Challenges Foundation, 2015. https://api.globalchallenges.org/static/wp-content/uploads/ 12-Risks-with-infinite-impact.pdf
- [68] A. D. Barnosky, Has the Earth's Sixth Mass Extinction Already Arrived?, Nature, Vol. 471, 51-57, 2011, Macmillan Publishers Limited. https://ib.berkeley.edu/labs/barnosky/Barnosky%20et% 20al%20Sixth%20Extinction%20Nature.pdf
- [69] N. Bostrom, "The Future of Humanity", New Waves in Philosophy of Technology (Eds. J. K. Berg Olsen, E. Selinger and S. Riis), pp. 186-215, McMillan, 2009. https://nickbostrom.com/papers/future.pdf [70] M. Kaku, The Future of Humanity, Penguin, 2019.
- [71] S. Freud, Moses and Monotheism, Vintage Books, 1939.
- A Osman, Moses and Akhenaten, Graftin Books, 1990. [72]
- [73] P. N. Robinson and M. Winzen (Eds.), Marfan Syndrome: A Primer for Clinicians and Scientists, Springer, 2004.
- [74] J. M. Blackledge and O. Farooq, "Audio Data Verifcation and Authentication using Frequency Modulation based Watermarking", ISAST Transactions on Electronics and Signal Processing, vol. 3, no 2, pp. 51-63, 2008. https://arrow.dit.ie/engscheleart2/31/

Jonathan Blackledge holds a BSc with Honors in Physics from Imperial College, a Diploma of Imperial College in Plasma and High Temperature Physics, a PhD in Theoretical Physics from London University, a Doctorate in Mathematics from the University of Jyvaskyla, Finland and a Doctorate in Management from the Chartered Institute of Personnel and Development, London. He has published over 300 research papers and 14 teaching and research books, has filed 20 patents and technologies to license and has been supervisor to over 250 MSc/MPhil and 67 PhD research graduates. After a career in academia, the scientific civil service and the Ministry of Defense, in 2008, he was appointed the Science Foundation Ireland's Stokes Professor at the Technological University Dublin where he has been an Honorary Professor since 2012. With 30 years' experience of R & D in defense-related science and technology and the intelligence services, he is an Honorary Colonel of military intelligence with current research interests in post-quantum cryptology and electronic warfare. He holds visiting professorships with a number of UK and overseas universities including the University of Wales, Warsaw University of Technology, the University of Western Cape and the University of KwaZulu-Natal where he was previously Deputy Vice-Chancellor for Research. He is a Fellow of Institutes and Societies which include the Institute of Physics, the Institute of Mathematics and its Applications, the British Computer Society, the Institution of Engineering and Technology, the Institution of Mechanical Engineers, the Royal Statistical Society, the Energy Institute, the Royal Society of Arts, the Chartered Management Institute, the Institute of Leadership and Management and the Institute of Directors. In 2007 he was elected a Fellow of the City and Guilds London Institute for his contributions to work-placed based learning and apprenticeship training schemes and in the same year was elected a Freeman of the City of London for his contributions to economics.