Nontrivial On-site Soliton Solutions for Stationary Cubic-Quintic Discrete Nonlinear Schrodinger Equation

Haves Qausar, Marwan Ramli*, Said Munzir, Mahdhivan Syafwan, Hadi Susanto, and Vera Halfiani

***Abstract*—This paper discusses the stationary cubic-quintic discrete non linear Schrödinger (CQ-DNLS) equation. The solution of this equation is determined by using the Trust-region dogleg method and the obtained solution is called a soliton. In this paper we only focus on solitons with characteristics in the form of on-site which means soliton peaked and centered at one site. In order to get the desired solution, it is necessary to choose the initial value $u \in \mathbb{R}^{2N+1}$ whose characteristics are similar to on-site soliton and it is also important to choose an appropriate parameter value. Therefore, simulations were carried out by choosing parameter values $w, C \in \mathbb{R}$. Based on the exact calculation, on-site soliton can be obtained by selecting $C = 0$. In case $C \neq 0$, on-site soliton is obtained by using the Trust-region dogleg method, selecting the initial value $u_n = \mathrm{sech}(n)$, and choosing the parameter values for w and C in the intervals $0.1 \leq w \leq 0.41$ and $0.16 \leq C \leq 9.3$. There are differences in the shape of the on-site solitons regarding to the choices of parameter values; the greater the value of C is, the wider the soliton in the middle is. Also, the greater the value of w is, the higher the amplitude of the produced soliton is. For some parameter values, a comparison is made between the soliton generated by Trust-region dogleg and Newton method.**

***Index Terms*—Stationary CQ-DNLS equation, Soliton solution, On-site Soliton, Trust-region dogleg method, Newton method**

I. Introduction

THE discrete nonlinear Schrödinger equation (DNLS) is the basis of nonlinear lattice dynamic models [1]. This equation has a wide range of applications including the electric circuits [2], oscillations of nanomechanical [3], DNA double strand [4], biomolecular chains [5], nonlinear optics (coupled optical wave guides) [6], and matter waves [7].

A Bose Einstein condensate (BEC) is the fifth form of a matter when approaching 0 Kelvin. S.N Bose and A. Einsten theoretically predicted the phenomenon in 1925 [8]. The first successful experiment to prove the existence of BEC was carried out by Wolfgang Ketterle, Eric Cornell and Carl Wiemann. For this achievement, they were awarded the Nobel Prize in physics in 2001 [9].

The DNLS equation is interesting to study because it has a special solution known as soliton. This solution has a fixed speed and profile when propagating [10]. In the context of applications in optical fields, soliton can also be engineered as a carrier of information that can propagate in media with very long distances without experiencing significant interference [11]. This has become very important for the development of information technology in the future.

Soliton is the basis of self-supporting modes on DNLS systems [12]. The mobility [13,14] and collisions [14, 15] of solitons with the simplest self-focusing cubic (Kerr) nonlinearity have been studied in the 1-dimensional DNLS system. The more general equation containing discrete cubic nonlinearity which is called the Salerno model can be seen in [16]. Gomez-Gardeñes [17,18] studied the modification of Salerno model in one and two dimentional settings. However, the model has not added the quintic term. Recent experimental results [19-21] shows that the response to some materials becomes more suitable when quintic type nonlinearity is added. Therefore the DNLS equation with the cubic quintic type nonlinearity is interesting to study.

In order to simplify the work, we take the stationary form of the cubic-quintic DNLS equation [22]. The most significant difference is that the cubic-quintic DNLS equation is in the form of difference-differential equation while the stationary equation is in the form of difference equation. In this work we focus on determining solutions in the form of on-site soliton. We choose this solution because on-site solitons usually turns out to be stable [23]. Until now, the on-site soliton form produced by the stationary cubic-quintic DNLS equation has been determined using Newton method and studied in order to find the values of several parameters. [24]. Therefore, in this research we apply Trust-region dogleg method to find the solution. Furthermore, we observe the resulting soliton and compare it with the soliton produced by Newton Method.

The paper is organized as follows. In the next section, we introduce the stationary cubic-quintic DNLS model and use numerical methods namely Trust-region dogleg method to determine the soliton solutions. In section 3, the simulation is carried out by choosing an appropriate initial value and changing the value of the parameter to obtain a non-trivial on-site soliton solution. In Addition, we compare the results with the solitons produced by Newton method. In last section, we make some conclusions.

II. Materials and Methods

The stationary CQ-DNLS equation is given as follows.

$$wu_n - u_n^3 + 0.5u_n^5 - C(u_{n+1} - 2u_n + u_{n-1}) = 0, \quad u_n \in \mathbb{R}, \quad n = -N, -N+1, \ldots, N, \quad N \in \mathbb{Z}^+ \tag{1}$$

Manuscript received November 28, 2019; revised March 5, 2020. This research is funded by Penelitian PMDSU DRPM 2019 with contract number 123/UN11.2/PP /SP3/2019.

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where $u_{-N-1} = u_{N+1} = 0$ and w, C are real-valued parameters. Rewrite equation (1) in the form of a function.

$$\begin{aligned} f(u) =& w(u_{-N} + u_{-N+1} + \cdots + u_N) \\ & - (u_{-N}^3 + u_{-N+1}^3 + \cdots + u_N^3) \\ & + 0.5(u_{-N}^5 + 0.5u_{-N+1}^5 + \cdots + u_N^5) \\ & - C(-u_{-N} - u_N) = 0 \end{aligned} \quad (2)$$

and define the trust region.

$$B_k = \{u \in \mathbb{R}^{2N+1} | \, \|u - u_k\|_2 \leq \Delta_k\}, \quad (3)$$

where Δ_k represents the radius of the k-th iteration in the trust region. It should be noted that the trust region of each iteration can be different due to the use of different norm types, but in this paper we use Euclidean norm.

The Trust-region method will form a model function based on objective functions (2). This is because calculations through the model function will be simpler compared to counting directly on the objective function. The function model can be defined as follows:

$$m_k(u_k + s) = f(u_k) + g_k^T s + \frac{1}{2} s^T H_k s, \quad (4)$$

where g represents the gradient, H represents the Hessian matrix and s is called step.

The next important thing is to look for steps that are sufficient to reduce the model in the area of trust. In the selection of steps, there are two condition that must be met, those are

$$u_k + s_k \in B_k \text{ and } \|s_k\|_2 \leq \Delta_k, \quad (5)$$

because the steps are chosen based on the model of the function, this can cause the value of the objective function not to decrease (or even increase). Therefore it needs a rule that aims to ensure that the selection of the steps will make the value of the model function and objective functions reduced. The rule can be written as follows:

$$\frac{\text{Actual Reduction}}{\text{Predicted Reduction}} \overset{\text{def}}{=} \rho_k = \frac{f(u_k) - f(u_k + s_k)}{f(u_k) - m_k(u_k + s_k)}. \quad (6)$$

If the value produced by the predicted reduction is greater than the value generated by the actual reduction, it means that the model of the function decreases faster than the objective function, then the step is selected and the radius of the trust region will be expanded. However, if the value produced by the predicted value is smaller or equal to the value generated by the actual reduction, then the step value is not good enough to reduce the model of the function. In this case, the step value is not selected and the radius of the trust region will be reduced.

Trust-region Algorithm

1) Initialization
   - $k = 0$ and choose $\epsilon > 0$
   - Choose u_0 as initial value
   - Choose Δ_0 as initial trust region radius
   - Choose $\eta_1, \eta_2, \gamma_1, \gamma_2$ as parameters such that: $0 < \eta_1 \leq \eta_2 < 1$ and $0 < \gamma_1 < 1 \leq \gamma_2$
2) Define the model
   - $m_k(u_k + s) = f(u_k) + g_k^T s + \frac{1}{2} s^T H_k s$
3) Select a step value
   - The value of s_k must meet the following conditions: $u_k + s_k \in B_k$ and $\|s_k\|_2 \leq \Delta_k$
4) Acceptance of step
   - Compute $\rho_k = \dfrac{f(u_k) - f(u_k + s_k)}{f(u_k) - m_k(u_k + s_k)}$,
     If $\rho_k \geq \eta_1$, then $u_{k+1} = u_k + s_k$
     Else $u_{k+1} = u_k$
5) Trust-region radius update
   - $\Delta_{k+1} = \begin{cases} \gamma_1 \Delta_k, & \text{if } \rho_k < \eta_1, \\ \Delta_k, & \text{if } \rho_k \in [\eta_1, \eta_2), \\ \gamma_2 \Delta_k, & \text{if } \rho_k \geq \eta_2. \end{cases}$
6) Stopping criteria:
   - $\|g_k\|_2 < \epsilon$ and $\|s_k\|_2 < \epsilon$
7) Increase the value of k by 1 and go to step 2

The selection of the step that can minimize the value of the model function is an important step in the algorithm. Therefore we need an efficient way to be able to determine the step s_k quickly and accurately. One method that can be used to solve this is the Dogleg Method. This method will find the value of the step s_k based on the Trust-region subproblem defined as follows

$$m_k(s) = g_k^T s + \frac{1}{2} s^T H_k s, \quad (7)$$

subject to $\|s\|_2 \leq \Delta$.

The process of finding step values using the Dogleg method is efficient because it will search for two types of steps in the trust region. The two steps in question are the Newton step and the other step is the point in the descent direction which is often referred as Cauchy point [25]. However, if minimum point in the decent direction is outside the trust region, the step to be chosen is a point in the descent direction and intersects with the trust region boundary. Mathematically, for $\Delta_k \leq \|H_k^{-1} \nabla f(u_k)\|_2$ there are u_{k+1} such that $\|s_k\|_2 \leq \Delta_k$.

In order to get $\|s_k\|_2 \leq \Delta_k$, we define the Newton step as follows:

$$s^N = -H^{-1} g \quad (8)$$

and the step update in Newton's direction can be written mathematically as

$$s^{\hat{N}} = \eta s^N \quad (9)$$

where $\eta = 0.8\gamma + 0.2$ for $\gamma \leq \eta \leq 1$, and

$$\gamma = \frac{\|g_k\|_2^4}{(g^T H g)(g^T H^{-1} g)}. \quad (10)$$

This aims as a scaling factor to reduce the length of Newton steps. Then the dogleg step can be defined as follows.

$$s^D = s^C + \lambda(s^{\hat{N}} - s^C) \quad (11)$$

where $0 \leq \lambda \leq 1$ and s^C is Cauchy Point [25].

The objective of Trust-region Algorithm is to take enough steps to reduce the model for each iteration, a possible way to achieve it is to select the dogleg step that is located in the trust region's boundary. Mathematically, it can be written as

$$\|s_k^D\| \le \Delta_k. \quad (12)$$

From equations (11) and (12), the λ values can be determined such that it satisfies the following equation

$$\|s^C + \lambda(s^{\hat{N}} - s^C)\|_2^2 \le \Delta^2. \quad (13)$$

Then equation (13) can be written as quadratic equation as follows:

$$\|s^{\hat{N}} - s^C\|_2^2\lambda^2 + 2((s^{\hat{N}} - s^C)^T s^C)\lambda + \|s^C\|_2^2 - \Delta^2 \le 0. \quad (14)$$

Hence, we obtain the value of lambda as follows:

$$\lambda = \Bigg(-2\big((s^{\hat{N}} - s^C)^T s^C\big) \pm \Bigg[\Big(2\big((s^{\hat{N}} - s^C)^T s^C\big)\Big)^2 - 4\big(\|s^{\hat{N}} - s^C\|_2^2\big)\big(\|s^C\|_2^2 - \Delta^2\big)\Bigg]^{\frac{1}{2}}\Bigg) 2\big(\|s^C\|_2^2 - \Delta^2\big)^{-1}. \quad (15)$$

Since the value of $\lambda \in [0, 1]$, this causes the root of equation (14) to be positive.

Dogleg Algorithm

1) Compute s^N
   - If $s^N \le \Delta$ then go to step 4 of Trust-region Algorithm
   - Else if $s^N > \Delta$ then go to step 2
2) Compute s^C
   - If $s^C = \Delta$ then go to step 4 of Trust-region Algorithm
   - Else if $s^C < \Delta$ then go to step 3
3) Compute s^D
   - Go to step 4 of Trust-region Algorithm

III. RESULT AND DISCUSSION

The soliton solution for equation (1) can be determined with are exact and numerical approach (Trust-region dogleg method). The exact solution form can only be determined if the value $C = 0$, so equation (1) becomes

$$wu_n - u_n^3 + 0.5u_n^5 = 0 \quad (16)$$

and solutions that meet those equations are $u_n = 0$ and $u_n = \pm\sqrt{1 \pm \sqrt{1-2w}}$. In order to get an on-site soliton form, the value of $u_n \ge 0$ is chosen and the value of u_0 is greater than the others. Therefore, u_n is chosen under the following conditions.

$$u_n = \begin{cases} u_0 = \sqrt{1+\sqrt{1-2w}}, \\ u_{-1} = u_1 = \sqrt{1-\sqrt{1-2w}}, \\ u_i = 0, \quad i = \pm 2, \pm 3, \ldots, \pm N. \end{cases}$$

For example, if the value of $w = 0.1$ is taken, then the on-site soliton solution can be plotted as in Figure 1.

Figure 1 presents an exact on-site soliton solution for $w = 0.1$ and $C = 0$. The exact soliton solution only has three sites with nonzero values, namely Site u_{-1}, u_0, and u_1. By choosing $w = 0.1$, the resulting values for the three sites become $u_{-1} = u_1 = \sqrt{1-\sqrt{0.8}}$ and $u_0 = \sqrt{1+\sqrt{0.8}}$. In addition, the site u_0 is a site with a position in the middle and has the largest value which is often called the peak of soliton.

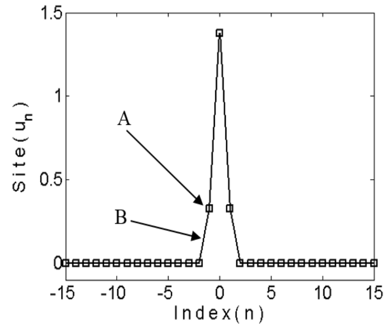

Fig. 1. An exact on-site soliton solution for $w = 0.1$ and $C = 0$. The square marker in the picture as indicated by arrow A represent the n-th site and arrow B represent the line connecting two adjacent sites.

In case $C \ne 0$, the solution with on-site soliton form of equation (1) will be found using the Trust-region dogleg numerical approach. In order to get the solution, it is necessary to choose an initial value that has a structural similarity with the soliton as well, therefore $u_n = \text{sech}(n)$ is chosen, with $n = -N, -N+1, \ldots, N$ and $N \in \mathbb{Z}^+$. The simulation process are carried out by varying the values of parameters w and C. Specifically, we define four cases:

1) $w \le 0$ and $C < 0$
2) $w \ge 0$ and $C < 0$
3) $w \le 0$ and $C > 0$
4) $w \ge 0$ and $C > 0$

Based on all of these cases, cases 1, 2 and 3 do not give a solution in the form of a on-site soliton, but in case 4 on-site soliton solution is obtained precisely in the interval:

$$0.1 \le w \le 0.41 \text{ and } 0.16 \le C \le 9.3. \quad (17)$$

Although the parameter values in (17) can produce an on-site soliton solution in equation (1), the resulting soliton is not the same. There are some differences caused by the magnitude of the parameters w and C. Therefore two cases are tested. The first case is to recognize a change in soliton by making changes to the value of parameter C and fixed w, for more details, see Figure 2.

Figure 2 presents nontrivial on-site soliton solutions for $w = 0.1$ and some values of C which are calculated numerically using the Trust-region dogleg method. From this figure, it can be seen that increasing the value of C will cause the soliton to widen in the middle. Furthermore, from the figure it can be seen the value of parameter C has no effect on the height of the soliton amplitude. As a result, regardless of the chosen value of C, the amplitude of the soliton shows the same magnitude.

The second case is to recognize a change in soliton by making changes to the value of parameter w while C is fixed, for more details, see Figure 3. Figure 3 represents non-trivial on-site soliton solutions for $C = 0.2$ and some values of w which are calculated numerically using the Trust-region dogleg method. From the figure, it can be seen that increasing the value of w will cause an increase in the height of the soliton amplitude. Furthermore, the value of parameter w has no effect on the width of the soliton in the middle.

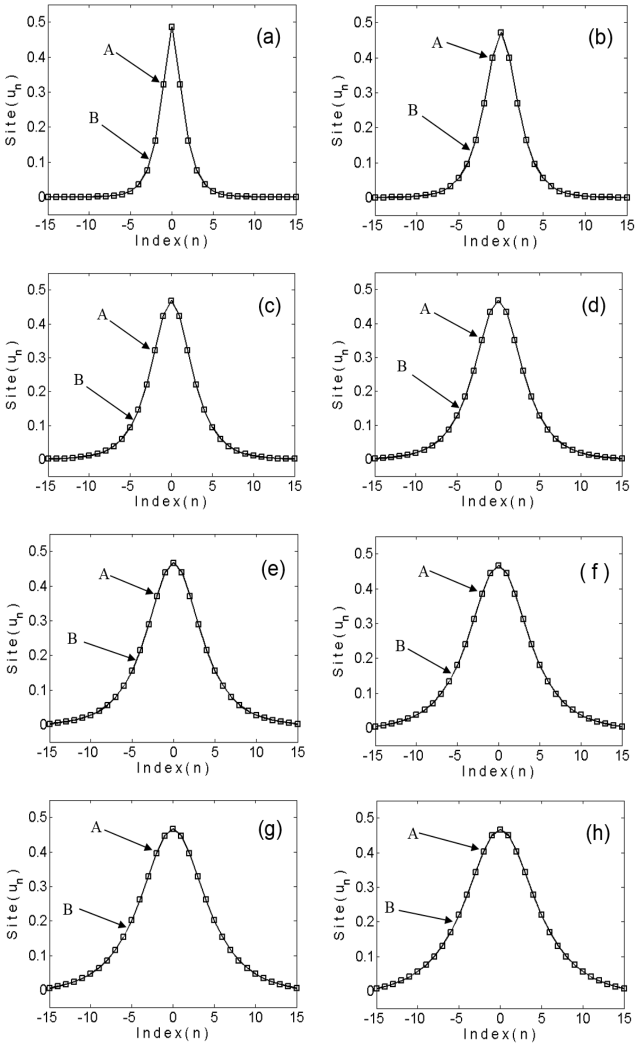

Fig. 2. Nontrivial on-site soliton solutions for $w = 0.1$ and some values of C (a) $C = 0.16$ (b) $C = 0.32$ (c) $C = 0.48$ (d) $C = 0.64$ (e) $C = 0.8$ (f) $C = 0.96$ (g) $C = 1.12$ (h) $C = 1.28$. The square marker in the picture as indicated by arrow A represents the n-th site and arrow B represents the line connecting two adjacent sites.

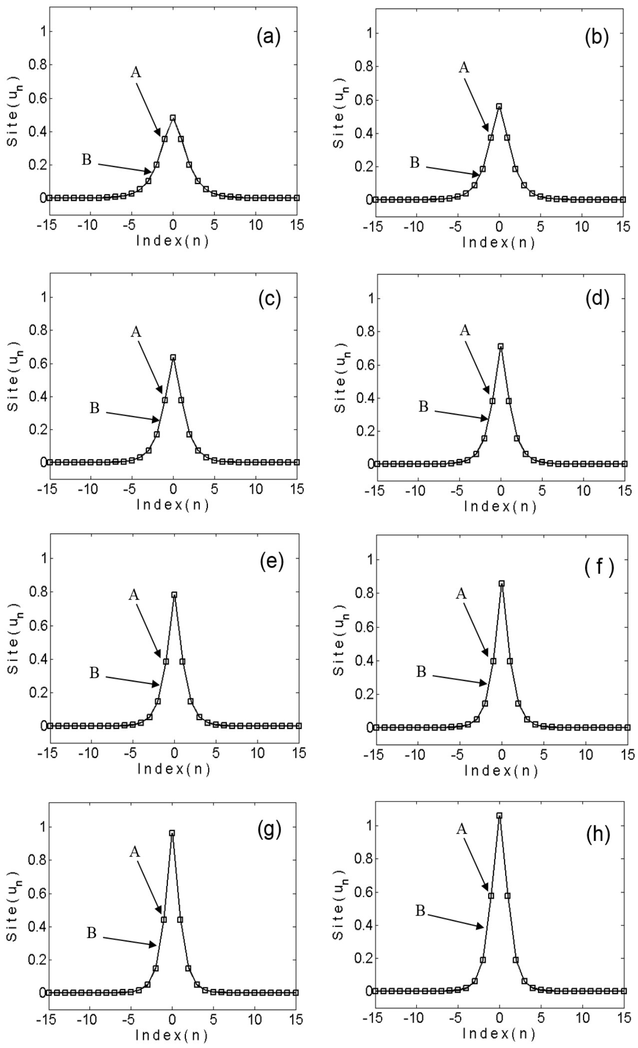

Fig. 3. Nontrivial on-site soliton solutions for $C = 0.2$ and some values of w (a) $w = 0.1$ (b) $w = 0.13$ (c) $w = 0.16$ (d) $w = 0.19$ (e) $w = 0.22$ (f) $w = 0.25$ (g) $w = 0.28$ (h) $w = 0.31$. The square marker in the picture as indicated by arrow A represents the n-th site and arrow B represents the line connecting two adjacent sites.

For some parameter values, a comparison between solitons obtained from equation (1) using the Trust-region dogleg method with solitons obtained from

$$\begin{aligned} \mu u_n + c(u_{n+1} + u_{n-1} - 2u_n) + 2u_n^3 - u_n^5 = 0, \\ u_n \in \mathbb{R}, \quad n = -N, -N+1, \ldots, N, \quad N \in \mathbb{Z}^+, \\ u_{-N-1} = u_{N+1} = 0 \end{aligned} \tag{18}$$

using Newton method which has been determined in [23]. An adjustments to equation (1) is made to equal the equation (18) by multiplying equation (1) by the constant '-2' hence the parameter relation between the two equations becomes $\mu = -2w$ and $c = 2C$.

Figure 4 presents a comparison between on-site soliton solutions calculated numerically using the Trust-region dogleg method and Newton method. From this figure, it can be seen that the solitons produced by the Trust-region dogleg method in Figure 4(b), 4(c) and 4(d) are very similar to the solitons produced by the Newton method. However, in Figure 4(a) the solution produced by the Trust-region dogleg method does not form on-site soliton but in the form of three-site [1]. This is because the minimum C value required to obtain an on-site soliton in the Trust-region dogleg method is 0.16. However, if the simulation is performed by taking $C \notin [0.16, 9.3]$, the on-site soliton is not formed.

IV. Conclusion

We have considered nontrivial on-site soliton solutions for stationary CQ-DNLS. An exact solution can be obtained by selecting $C = 0$. In case $C \neq 0$, on-site soliton can be obtained by using the Trust-region dogleg numerical approach and selecting the initial value $u_n = \operatorname{sech}(n)$, as will as choosing the parameters w and C in the intervals $0.1 \le w \le 0.41$ and $0.16 \le C \le 9.3$. The magnitude of the parameter values w and C also affects the shape of the soliton, the greater the value of C is, the wider the soliton in the middle is. Also, the greater the value of w is, the higher the amplitude of the produced soliton is. At intervals $0.1 \le w \le 0.41$ and $0.16 \le C \le 9.3$, the on-site soliton solution generated by the Trust-region dogleg method is very similar to Newton method except in case Figure 4(a), this is due to the selection of the $C \notin [0.16, 9.3]$. In addition, the

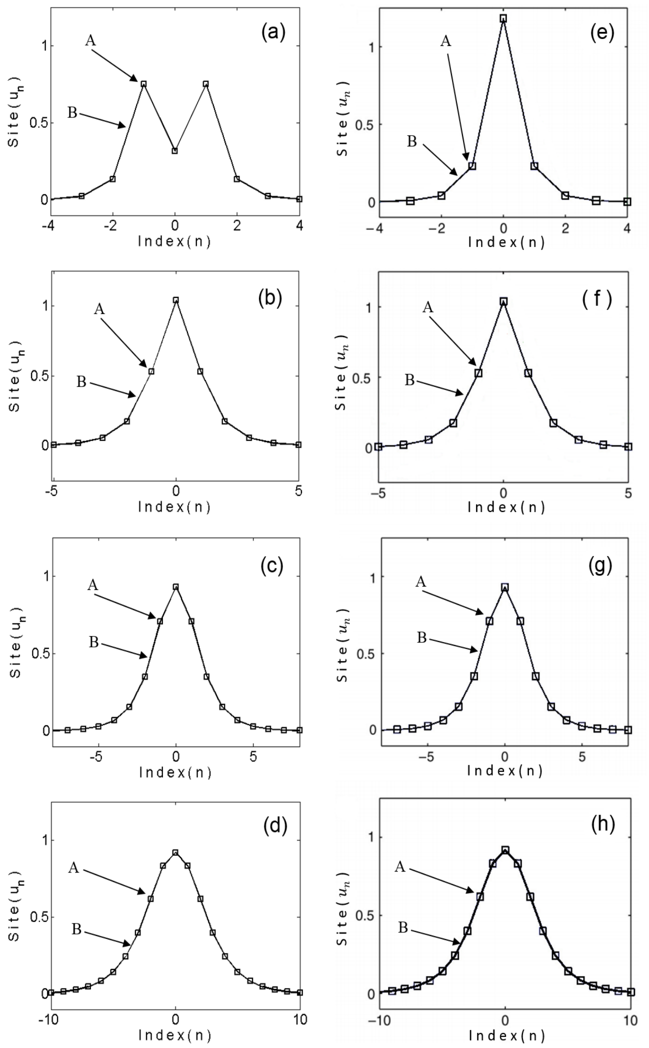

Fig. 4. (Left) Trust-region dogleg method for $w = 0.3$ and some values of C (a) $C = 0.075$, (b) $C = 0.2$, (c) $C = 0.4$, (d) $C = 1$. (Right) Newton method for $\mu = -0.6$ and some c (e) $c = 0.15$ (f) $c = 0.4$ (g) $c = 0.8$ (h) $c = 2$. The square marker in the picture as indicated by arrow A represent the n-th site and arrow B represent the line connecting two adjacent sites.

soliton deformation associated with the value of $C > 0$ can be reviewed using the Taylor series.

Acknowledgment

The authors thanks to Department of Mathematics, Universitas Syiah Kuala for the facilities provided of this research. Also thanks to the referees who helped so that the article could be published in this journal.

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