Effect of Vessel Roughness in Casson's Mathematical Model of Blood Flow

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***Abstract*— We consider a Casson's model describing the blood flow through vessels with a rough microstructure. Precise analytical formulas and some apriori estimates on relation between velocity, viscosity and stress of flow were derived for stationary blood flow. We compare different flow states depending on relation between vessels' diameters and wavelength of vessel walls.**

***Index Terms*— Casson's flow, hemodynamics, non-Newtonian fluid, small parameter, variable viscosity.**

I. Introduction

HEMODYNAMIC research is an important branch of science. It plays a crucial role for biological and medical applications. The knowledge of blood flow behavior in thin capillaries could help to overcome different diseases caused by anomaly of vessels' geometry. Scientists are interested in effects and common influence of blood flow characteristics such as velocity, pressure gradient, stress of capillaries' walls, viscosity etc. [1-5,8,9]. It turned out that blood flow in thin capillaries couldn't be described only by Newtonian fluid model. The flow in the area near the capillaries' walls has other rheological properties. This is caused by a high concentration of red blood cells near the walls. A possible mathematical model to describe such flow is Casson's system of differential equations. The research carried out in [1-5,8,9] deals with either vessels of cylindrical geometry or with simply stenosed arteria.

The novelty of present work is to take into account all possible micro-geometrical properties of vessel walls. In this paper we derive analytical formulas and estimates for blood velocity in the presence of boundary roughness. We investigate effects of rough vessels to flow characteristics. Three possible types of roughness are analyzed. For each case we study the flow behavior when both vessel diameter and roughness wavelength are small. A specific property of the considered problem is a varying viscosity which depends on the vessel radius. Some general methods of estimation in hydrodynamic problems with variable viscosity were developed by authors in [6,7].

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In this paper we use numerical computations to validate theoretical results. It was discovered that high boundary oscillations cause velocity variations for vessels with sufficiently small diameters. An increase in viscosity and the yield stress implies a decrease of velocity. It was also noted that the smaller non-Newtonian region near the walls of the capillaries, the lower the blood flow velocity.

The paper is organized as follows: in section II we state the problem mathematically; section III deals with estimation of the crucial parameters of the blood flow; some analytical formulas for the described model are derived in section IV; finally, different types of vessel roughness and its effect on blood flow are compared in section V.

II. Statement of the Problem

Consider two-phase blood flow of variable viscosity in a thin cylindrical vessel $\Omega = \{(z,r),\ 0 \le z \le L,\ 0 \le r \le R(z)\}$ having a rough boundary $R(z)$. The flow is a Newtonian one close to the center of vessel, where the concentration of erythrocytes is high. We denote this domain by $\Omega^{Newt} = \{(z,r),\ 0 \le z \le L,\ 0 \le r \le R_p\}$, where $R_p(\tau_y)$ is the boundary between Newtonian and non-Newtonian flows. There is a limit stress τ_y, upon reaching which the blood begins to behave like a non-Newtonian fluid. It determines the border between two phases of flow: $\tau(R_p(\tau_y), z) = \tau_y$. According to a research in hemodynamics, blood is plasma with rare sprinkles of red blood cells away from the vessel center. Thus, there exists a layer $R_p(\tau_y) \le r \le R(z)$ with non-Newtonian rheological properties. To simplify our analysis we consider the stationary symmetric flow along the vessel's walls caused by constant pressure gradient. We assume that only one component of the velocity $\vec{u} = (u_r, u_\varphi, u_z)$ is non-zero: $u_r \equiv u_\varphi \equiv 0,\ u_z \ne 0.$ For simplicity of notations, we denote further $u_z(r)$ by $u(r)$. The considered flow can be modelled by dimensionless Casson's equations:

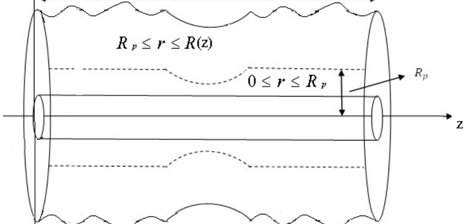

Fig. 1. Geometry of vessel

$$\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{-\frac{\partial u}{\partial r}\mu(r)}, \text{ if } \tau \ge \tau_y,\ R_p \le r \le R(z) \quad (2.1)$$

$$\frac{\partial u}{\partial r} = 0,\ if \quad \tau < \tau_y,\ 0 \le r \le R_p \qquad (2.2)$$

and by momentum equation which in our case reads as $\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial p}{\partial z} + \frac{2}{r}\frac{\partial}{\partial r}(r\tau) = 0$.

Here u is an unknown velocity of flow, p is a pressure, τ is a stress tensor, τ_y is the yield stress, $\mu(r)$ is Newtonian viscosity. Boundary conditions are naturally set as follows: the stress τ is finite at $r = 0$, $u = 0$ at $r = R(z)$, and p_0, p_L are given pressure on vertical vessel's walls.

III. APRIORI ESTIMATES

Below we derive some apriori estimates which help to understand the flow behavior.

Lemma (Friedrich's inequality)

The velocity satisfies the following estimate

$$\int_0^{R(z)} u^2 dr \le C(R_p, R(z)) \int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr, \qquad (3.1)$$

$$C = \max\{2R_p(R(z) - R_p), (R(z) - R_p)^2\}$$

Proof

Suppose $R_p \le r \le R(z)$. The boundary condition $u(R(z), z) = 0$ and Newton-Leibnitz formula give the formula for velocity via its gradient:

$$u(r,z) = u(r,z) - u(R(z),z) = -\int_r^{R(z)} \frac{\partial u}{\partial r} dr. \quad (3.2)$$

Squaring (3.2) and applying Hölder's inequality, one gets:

$$u^2(r,z) \le (R(z) - r)\int_r^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr \le (R(z) - R_p)\int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr.$$

Integrating the derived inequality over $[R_p, R(z)]$, we have

$$\int_{R_p}^{R(z)} u^2 dr \le (R(z) - R_p)^2 \int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr \quad (3.3)$$

Consider now $r \in [0, R_p]$. The following representation holds:

$$u(r,z) = u(R_p,z) + u(r,z) - u(R_p,z) = u(R_p,z) - \int_r^{R_p} \frac{\partial u}{\partial r} dr.$$

Squaring both sides of this identity , applying the estimate $(a+b)^2 \le 2a^2 + 2b^2$ and Hölder inequality, we get

$$u^2(r,z) \le 2u^2(R_p,z) + 2\int_r^{R_p} \left(\frac{\partial u}{\partial r}\right)^2 dr =$$

$$= 2u^2(R_p,z) \le 2(R(z) - R_p)\int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr.$$

Integrating once more over $[0, R_p]$, we conclude:

$$\int_0^{R_p} u^2 dr \le 2R_p(R(z) - R_p)\int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr. \quad (3.4)$$

Combining both (3.3) and (3.4), we get the desired estimate for the velocity in the whole flow domain:

$$\int_0^{R(z)} u^2 dr = \int_0^{R_p} u^2 dr + \int_{R_p}^{R(z)} u^2 dr \le C\int_{R_p}^{R(z)} \left(\frac{\partial u}{\partial r}\right)^2 dr, \quad (3.5)$$

where C is given in (3.1).

Corollary. *The Friedrich's inequality implies directly the following estimate of the velocity depending on viscosity and stress:*

$$\|u(r)\|_{L_2(0,R(z))} \le C\left\|\mu^{-1}(r)\left(\sqrt{\tau} - \sqrt{\tau_y}\right)^2\right\|_{L_2(R_p,R(z))} \quad (3.6)$$

$$C = \max\{2R_p(R(z) - R_p), (R(z) - R_p)^2\}.$$

It is clear from (3.6) that velocity of flow reduces when the viscosity increases. In addition, the velocity takes smaller values, when the stress tends to value τ_y . However, an increase of the non-Newtonian layer $[R_p, R(z)]$ theoretically could imply a velocity growth. Summing up, the parameter τ_y which is crucial for the size of non-Newtonian boundary layer, effects on the solution as follows: the smaller values of τ_y the greater the velocity. We confirm numerically this observation in Section V on Fig. 3.

IV. ANALYTICAL FORMULAS

Now we derive exact analytical formulas for flow characteristics. The boundary condition $u(R(z), z) = 0$ and Newton-Leibnitz formula implies

$$-u(r,z) = u(R(z),z) - u(r,z) = \int_r^{R(z)} \frac{\partial u}{\partial r} dr \qquad (4.1)$$

Hence, we have

$$u(r,z) = \int_r^{R(z)} -\frac{\partial u}{\partial r} dr \quad \text{for} \quad R_p \le r \le R(z). \qquad (4.2)$$

Applying (2.1), one gets for $\mathrm{R}_p \le r \le R(z)$:

$$-\frac{\partial u}{\partial r} = \frac{1}{\mu(r)}\left(\sqrt{\tau} - \sqrt{\tau_y}\right)^2 \quad (4.3)$$

Formulas (4.2) and (4.3) give the expression for velocity:

$$u = \int_r^{R(z)} \mu^{-1}(\rho)\left(\tau - 2\sqrt{\tau_y \tau} + \tau_y\right) d\rho \text{ if } \mathrm{R}_p \le r \le R(z). \quad (4.4)$$

Denote by $p_s = \frac{\partial p}{\partial z}$ the known pressure gradient and integrate the momentum equation:

$$0=\int_0^r \left(rp_s + \frac{\partial}{\partial r}(r\tau) \right) dr = \frac{r^2}{2} p_s + r\tau \Rightarrow \tau = -\frac{p_s}{2} r \Rightarrow R_p = -2\tau_y p_s^{-1}.$$

Substituting (2.1) in (4.3) we derive the dependence of velocity on pressure gradient for $R_p \le r \le R(z)$:

$$u(r,z) = \int_r^{R(z)} \mu^{-1}(\rho)\left(\tau_y + \sqrt{-0.5\tau_y \rho p_s} - 0.5\rho p_s\right) d\rho \,. \qquad (4.6)$$

The condition (2.2) implies that u is independent on variable r in layer $0 \le r \le R_p$. Moreover, a compatibility conditions for velocity must be valid on the boundary $r = R_p$. Hence, denoting $p = -0.5 p_s$, we have

$$u = \int_{-2\tau_y p_s}^{R(z)} \frac{1}{\mu(r)}\left(\tau_y + \sqrt{\tau_y pr} + rp\right) dr \text{ in } \Omega^{Newt}. \qquad (4.5)$$

Other important flow characteristics are the volumetric flow rate Q and the flow resistance λ given by

$$Q = 2\pi \int_0^{R(z)} rudr \quad \text{and} \quad \lambda = Q^{-1}(p_0 - p_L).$$

From (4.5) and definitions of Q and λ it is evident that an increase in viscosity leads to a decrease in the volumetric flow rate and, conversely, increases the resistance to blood.

V. Roughness Effect

Let us introduce a small parameter $0 < \varepsilon \ll 1$ characterizing vessel radius. Here parameter $\nu(\varepsilon)$ is the wavelength of walls which tends to zero as $\varepsilon \to 0$. Denote by k the limit ratio of the vessel radius and the wavelength: $k = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\nu(\varepsilon)}$.

Different types of roughness are possible depending on values of k (see Fig. 2): the case $0 < k < \infty$ corresponds to «middle oscillations» which means that the roughness period as small as vessel radius; the value $k = 0$ corresponds to «small oscillations» which means that the vessel radius is much smaller than the wavelength; if $k = \infty$ then one deals with a «high frequency regime» which corresponds to the case when the roughness period is much smaller than the vessel radius. Taking into account the derived analytical formula for blood velocity, one can observe the dependence on roughness in limits of the integration:

$$u_\varepsilon = \int_{\tau_y p^{-1}}^{\varepsilon h_0(z) + \varepsilon h_\varepsilon\left(\frac{z}{\nu(\varepsilon)}\right)} \frac{1}{\mu(r)}\left(rp - 2\sqrt{p\tau_y}\sqrt{r} + \tau_y\right) dr, \ 0 \le z \le 1,$$

Applying Lagrange formula, we can estimate the velocity depending on ε and $\nu(\varepsilon)$:

$$u_\varepsilon \le \varepsilon\left(h_0'(z) + \nu^{-1}(\varepsilon) h_\varepsilon'\right) \max \mu^{-1}(r)(\sqrt{rp} - \sqrt{\tau_y})^2. \ (5.1)$$

This estimate and boundedness of functions h_0', h_ε' imply the asymptotics

$$u_\varepsilon \approx k \max \mu^{-1}(r)(\sqrt{rp} - \sqrt{\tau_y})^2 \text{ as } \varepsilon \to 0. \quad (5.2)$$

Figures 3 a)-d) demonstrate the velocity behavior for all possible types of the roughness. To handle the numerical computations we chose the following particular cases of wavelength: $\nu(\varepsilon) = \varepsilon$, $\nu(\varepsilon) = \sqrt{\varepsilon}$, $\nu(\varepsilon) = \varepsilon^2$ which corresponds to $0 < k < \infty$, $k = 0$, $k = \infty$ respectively. The boundary $r = R(z)$ is modeled by $h_0(z) = 3$ and $h_\varepsilon\left(z\nu^{-1}(\varepsilon)\right) = \sin\left(z\nu^{-1}(\varepsilon)\right)$. The viscosity is assumed to be the polynomial function: $\mu(r) = r^\alpha$, $\alpha = \pm 4.1$.

Analyzing the roughness effect, one can observe that the velocity behavior is similar to the behavior of roughness. Small boundary oscillations give a slow change of velocity. The growth of velocity is more rapid in case of middle oscillations. Finally, high oscillations produce more significant effect: the graphs for velocity are rapidly oscillating functions as well. Comparing data on Fig. 3, one can see that an increase of yield stress τ_y gives a decrease of blood velocity what is agreed with estimate (3.6).

We study also the velocity behavior depending on viscosity for all roughness types. It was observed that high viscosity corresponds the low values of velocity, see plots on Fig. 3 c), d). This effect is explained by (3.6). Thinner vessels have higher velocity for the same given pressure. Plots 3a), 3b) and 3c), 3d) compare the velocity behavior for different limit stresses and constant values of ε: the higher stress τ_y the lower the velocity, what is proved by (3.6). Observe also that roughness leads to oscillations in velocity only for sufficiently small values of ε (compare Fig. 3a) plotted for $\varepsilon = 0.1$ and 2c) where $\varepsilon = 0.01$). Similar conclusions are valid for dependence of the volumetric flow rate on roughness, since Q is strictly proportional to the velocity. However, the behavior of blood resistance is the opposite one. The more oscillations of roughness, the greater the variation of resistance. Clearly, values of resistance are inversely proportional to volumetric flow rate. Small roughness regime gives a significant reduce in the resistance. It achieves the same minimum for both middle and high oscillation regimes. The dependence of blood resistance on yield stress is the direct one: smaller values of τ_y give smaller amplitude for λ. Thus, the thinner non-Newtonian layer the greater blood resistance. As a conclusion we can state that blood flow does not feel small oscillations of boundary. However, the roughness effects on velocity and resistance variation speed. High roughness of vessel walls effects significantly in very thin capillaries: the flow passes in the area below oscillating peaks.

The formula (5.2) can be used to estimate the varying viscosity as soon as the velocity and stress of flow are measured: $\mu \approx k u_\varepsilon^{-1}(\sqrt{rp} - \sqrt{\tau_y})^2$. The viscosity of the flow is proportional to k and inversely proportional to the velocity. The varying viscosity, which evidently depends on ε as well, is a reason why the velocity was higher for $k = \infty$ rather than for $k = 0$.

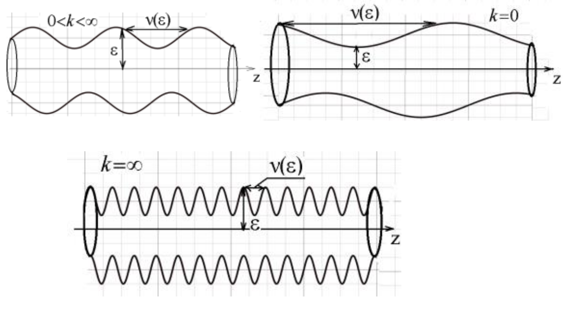

Fig. 2. Different roughness regimes

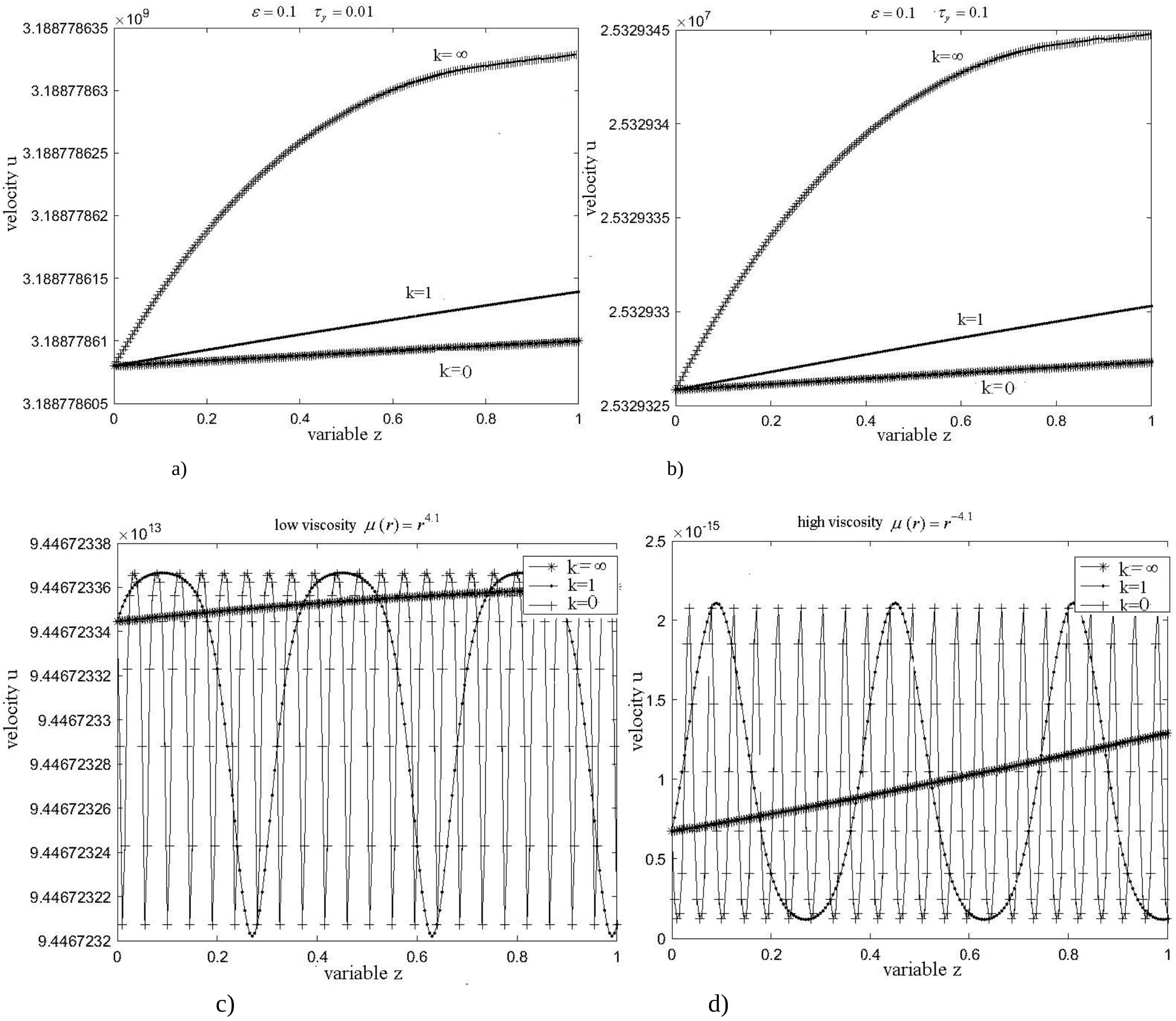

Fig. 3. Comparison of velocity profiles for different types of roughness at several values of yield stress and viscosity

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