Dynamic Behaviors of a Discrete May Type Cooperative System Incorporating Michaelis-Menten Type Harvesting

Zhenliang Zhu, Fengde Chen, Liyun Lai and Zhong Li

***Abstract*—A discrete May type cooperative model incorporating Michaelis-Menten type harvesting takes the form**

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{r_1 - b_1x(k) - \frac{a_1x(k)}{y(k)+k_1} \\ &\qquad -\frac{Eq}{m_1E+m_2x(k)}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{r_2 - b_2y(k) - \frac{a_2y(k)}{x(k)+k_2}\Big\} \end{aligned}$$

is proposed and studied in this paper. Sufficient conditions which ensure the permanence, extinction of the first species and the existence of a unique globally attractive interior equilibrium of the system are obtained, respectively. Numeric simulations are carried out to show the feasibility of the main results.

***Index Terms*—Global attractivity; Extinction; Cooperation; Equilibrium; Permanence.**

I. Introduction

THE aim of this paper is to investigate the dynamic behaviors of the following discrete May type cooperative model incorporating Michaelis-Menten type harvesting

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{r_1 - b_1x(k) - \frac{a_1x(k)}{y(k)+k_1} \\ &\qquad -\frac{Eq}{m_1E+m_2x(k)}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{r_2 - b_2y(k) - \frac{a_2y(k)}{x(k)+k_2}\Big\}, \end{aligned} \tag{1.1}$$

where $r_1, b_1, a_1, k_1, E, q, m_1, m_2, r_2, b_2, a_2, k_2$ are all positive constants.

In [1], May proposed the following two species cooperative system

$$\begin{aligned} \dot{x} &= x\Big(r_1 - b_1x - \frac{a_1x}{y+k_1}\Big), \\ \dot{y} &= y\Big(r_2 - b_2y - \frac{a_2y}{x+k_2}\Big). \end{aligned} \tag{1.2}$$

He showed that system (1.2) admits a unique positive equilibrium which is globally attractive. Since then, many scholars([3]-[40]) done works on this direction. For example, Roberts and Joharjee[2] argued that the beneficial effects of the indirect, interspecies interactions not being realised immediately, and should introduced the delay to system (1.2). They investigated the local stability property of the positive equilibrium of the following delayed system.

$$\begin{aligned} \dot{x} &= x\Big(r_1 - b_1x - \frac{a_1x}{y(t-\tau)+k_1}\Big), \\ \dot{y} &= y\Big(r_2 - b_2y - \frac{a_2y}{x(t-\tau)+k_2}\Big). \end{aligned}$$

Chen, Xie and Chen[10] proposed a stage structured May type cooperative system, they showed that the cooperation between the species is not the essential factor to ensure the permanence of the system, while the death rate of the mature species and the birth rate of the immature species are two of the most important factors lead to the permanence or extinction of the system. Other topics such as the the influence of feedback controls to the cooperative system ([6], [11]-[17]), the stability property of the equilibria of cooperative model ([3]-[6], [8]-[10],[18]), the existence of periodic solution or almost periodic solution ([7], [24]) and the persistent property of the cooperative system ([13]-[21]) are also well studied.

In [3], Wei and Li incorporating harvesting to system (1.2), this leads to the following model

$$\begin{aligned} \dot{x} &= x\Big(r_1 - b_1x - \frac{a_1x}{y+k_1}\Big) - Eqx, \\ \dot{y} &= y\Big(r_2 - b_2y - \frac{a_2y}{x+k_2}\Big). \end{aligned} \tag{1.3}$$

The authors of [3] investigated the persistent and stability property of the system (1.3). Recently, Xie, Chen and Xue[4] revisited the dynamic behaviors of the system (1.3). Their study indicates that the condition which ensure the existence of a unique positive equilibrium is enough to ensure the globally attractive of the positive equilibrium.

Chen, Wu and Xie[5] argued the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, corresponding to system (1.3), they proposed the following discrete cooperative model incorporating harvesting:

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{r_1 - Eq - b_1x(k) - \frac{a_1x(k)}{y(k)+k_1}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{r_2 - b_2y(k) - \frac{a_2y(k)}{x(k)+k_2}\Big\}, \end{aligned} \tag{1.4}$$

where $x(k), y(k)$ are the population density of the species x and y at k-generation. They did not investigate the extinction property of the system (1.4). Concerned with the stability property of the system (1.4), they obtained the following result.

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Theorem A. *Assume that*
(H_1) $r_i, b_i, a_i, E, q, i = 1, 2$ *are all positive constants,* $r_1 > Eq$,

and

(H_2) $0 < r_1 - qE \leq 1,\ r_2 \leq 1,\ (i = 1, 2)$

hold, then system (1.4) admits a unique positive equilibrium $E(x^*, y^*)$*, which is globally attractive.*

It brings to our attention that in system (1.3) and (1.4), the authors only considered the linear harvesting. Suck kind of harvesting embodies several unrealistic features and limitations. For example, in system (1.3), the authors took $h(E, x) = qEx$ as the fishing term, it easy to see that h tends to infinity as the effort E tends to infinity if the population x is finite and fixed, or as the population x tends to infinity if the effort E is finite and fixed. To overcome this drawback, recently, many scholars ([32], [33], [34], [35]) argued that the nonlinear harvesting, or named as Michaelis-Menten type harvesting is more suitable. If we adopt the Michaelis-Menten type harvesting to system (1.2), we will establish the following model

$$\begin{aligned} \dot{x} &= x\Big(r_1 - b_1 x - \frac{a_1 x}{y + k_1}\Big) - \frac{Eqx}{m_1 E + m_2 x}, \\ \dot{y} &= y\Big(r_2 - b_2 y - \frac{a_2 y}{x + k_2}\Big), \end{aligned} \tag{1.5}$$

where x and y denote the densities of two populations at time t. The parameters $r_1, r_2, a_1, a_2, b_1, b_2, k_1, k_2, E, q, m_1, m_2$ are all positive constants. Based on system (1.5), we could establish the discrete May type cooperative system with Michaelis-Menten type harvesting, i.e., system (1.1). To the best of the authors knowledge, this is the first time that the discrete population model with Michaelis-Menten type harvesting is proposed and studied.

As far as system (1.1) is concerned, the extinction, permanence and the global attractivity are the most important topics that need to be investigated.

The rest of the paper is arranged as follows. We will introduce some useful Lemmas in the next section, and then investigate the persistent property of the system (1.1) in Section 3. We will investigate the extinction property in Section 4, and investigate the stability property of the positive equilibrium in Section 5. Some numeric simulations which show the feasibility of the main results are presented in Section 6. We end this paper by a briefly discussion.

II. Lemmas

We first establish a Lemma, which ensure the existence of the positive equilibrium of the system (1.1).

Lemma 2.1. *Assume that*

$$b_1 k_2 > r_1 > \frac{q}{m_1} \tag{2.1}$$

holds, then system (1.1) admits a unique positive equilibrium.

Proof. The positive equilibrium of system (1.1) satisfies the equations

$$\begin{aligned} r_1 - b_1 x - \frac{a_1 x}{y + k_1} - \frac{Eq}{m_1 E + m_2 x} &= 0, \\ r_2 - b_2 y - \frac{a_2 y}{x + k_2} &= 0. \end{aligned} \tag{2.2}$$

From the second equation, we have

$$y = \frac{r_2\ (x + k_2)}{b_2 k_2 + b_2 x + a_2}. \tag{2.3}$$

Substituting (2.3) into the first equation of system (2.2) and simplifying, we finally obtain

$$A_0 x^3 + A_1 x^2 + A_2 x + A_3 = 0, \tag{2.4}$$

where

$$\begin{aligned} A_0 &= b_1 b_2 k_1 m_2 + a_1 b_2 m_2 + b_1 m_2 r_2 > 0, \\ A_1 &= E b_1 b_2 k_1 m_1 + E a_1 b_2 m_1 + E b_1 m_1 r_2 \\ &\quad + a_1 b_2 k_2 m_2 + a_2 b_1 k_1 m_2 + a_1 a_2 m_2 \\ &\quad + m_2 (b_2 k_1 + r_2)(b_1 k_2 - r_1), \\ A_2 &= E a_1 b_2 k_2 m_1 + E a_2 b_1 k_1 m_1 \\ &\quad - b_2 k_1 k_2 m_2 r_1 + E a_1 a_2 m_1 + E b_2 k_1 q \\ &\quad - E m_1 r_1 r_2 - a_2 k_1 m_2 r_1 - k_2 m_2 r_1 r_2 \\ &\quad + E b_1 b_2 k_1 k_2 m_1 - E b_2 k_1 m_1 r_1 \\ &\quad + E b_1 k_2 m_1 r_2 + E q r_2, \\ A_3 &= -E (m_1 r_1 - q)(b_2 k_1 k_2 + a_2 k_1 + r_2 k_2). \end{aligned}$$

Under the assumption of Lemma 2.1, $A_1 > 0$ and $A_3 < 0$, hence, (2.4) admits a unique positive solution x^*, Consequently, system (1.1) admits a unique positive equilibrium $E(x^*, y^*)$, where

$$y^* = \frac{r_2\ (x^* + k_2)}{b_2 k_2 + b_2 x^* + a_2}. \tag{2.5}$$

This ends the proof of Lemma 2.1.

Lemma 2.2([19])*Let* $f(u) = u \exp(\alpha - \beta u)$*, where* α *and* β *are positive constants, then* $f(u)$ *is nondecreasing for* $u \in (0, \frac{1}{\beta}]$.

Lemma 2.3([19]) *Assume that sequence* $\{u(k)\}$ *satisfies*

$$u(k+1) = u(k) \exp(\alpha - \beta u(k)),\ k = 1, 2.....$$

where α *and* β *are positive constants and* $u(0) > 0$*. Then*
(i) If $\alpha < 2$*, then* $\lim\limits_{k \to +\infty} u(k) = \frac{\alpha}{\beta}$.
(ii) If $\alpha \leq 1$*, then* $u(k) \leq \frac{1}{\beta}, k = 2, 3.....$

Lemma 2.4([36]) *Suppose that functions* $f, g : Z_+ \times [0, \infty) \to [0, \infty)$ *satisfy* $f(k, x) \leq g(k, x) (f(k, x) \geq g(k, x))$ *for* $k \in Z_+$ *and* $x \in [0, \infty)$ *and* $g(k, x)$ *is nondecreasing with respect to* x*. If* $\{x(k)\}$ *and* $\{u(k)\}$ *are the nonnegative solutions of the following difference equations:*

$$x(k+1) = f(k, x(k)),\ u(k+1) = g(k, u(k)).$$

respectively, and $x(0) \leq u(0) (x(0) \geq u(0))$*, then*

$x(k) \leq u(k) (x(k) \geq u(k))$ *for all* $k \geq 0$.

Lemma 2.5. ([15]) *Assume that* $\{x(k)\}$ *satisfies* $x(k) > 0$ *and*

$$x(k+1) \leq x(k) \exp\Big\{a(k) - b(k) x(k)\Big\}$$

for $k \in N$, where $a(k)$ and $b(k)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b^l}\exp(a^u - 1).$$

Lemma 2.6.([15]) *Assume that $\{x(k)\}$ satisfies*

$$x(k+1) \ge x(k)\exp\Big\{a(k) - b(k)x(k)\Big\},\ \ k \ge N_0,$$

$\limsup_{k\to+\infty} x(k) \le x^*$ and $x(N_0) > 0$, where $a(k)$ and $b(k)$ are nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{k\to+\infty} x(k) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

III. Permanence

This section we will establish a set of sufficient conditions which ensure the permanence of the system (1.1).

Theorem 3.1. *Assume that*

(B_1) $r_1 > \dfrac{q}{m_1}$

holds, then system (1.1) is permanent.

Proof. From the first equation of system (1.1), we have

$$x(k+1) \le x(k)\exp\{r_1 - b_1 x(k)\}. \tag{3.1}$$

Applying Lemma 2.5 to (3.1) leads to

$$\limsup_{k\to+\infty} x(k) \le \frac{1}{b_1}\exp\{r_1 - 1\} \stackrel{\text{def}}{=} M_1. \tag{3.2}$$

From the second equation of system (1.1), we have

$$y(k+1) \le y(k)\exp\Big\{r_2 - b_2 y(k)\Big\}, \tag{3.3}$$

Applying Lemma 2.5 to (3.3) leads to

$$\limsup_{k\to+\infty} y(k) \le \frac{1}{b_2}\exp\{r_2 - 1\} \stackrel{\text{def}}{=} M_2. \tag{3.4}$$

From the first equation of system (1.1), we also have

$$\begin{aligned} & x(k+1) \\ \ge\ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & \qquad - \frac{Eq}{m_1 E + m_2 x(k)}\Big\} \\ \ge\ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{k_1} - \frac{Eq}{m_1 E}\Big\}. \end{aligned} \tag{3.5}$$

Applying Lemma 2.6 to (3.5) leads to

$$\liminf_{k\to+\infty} x(k) \ge \min\left\{\Delta_2\exp\{\Delta_1\}, \Delta_2\right\}, \tag{3.6}$$

where

$$\begin{aligned} \Delta_1 &= r_1 - \frac{q}{m_1} - (b_1 + \frac{a_1}{k_1})M_1 \\ \Delta_2 &= \frac{r_1 - \dfrac{q}{m_1}}{b_1 + \dfrac{a_1}{k_1}}. \end{aligned} \tag{3.7}$$

From the second equation of system (1.1), we also have

$$\begin{aligned} & y(k+1) \\ =\ & y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{x(k)+k_2}\Big\} \\ \ge\ & y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{k_2}\Big\}. \end{aligned} \tag{3.8}$$

Applying Lemma 2.6 to (3.8) leads to

$$\liminf_{k\to+\infty} y(k) \ge \min\left\{\Delta_3\exp\{\Delta_4\}, \Delta_3\right\}. \tag{3.9}$$

where

$$\begin{aligned} \Delta_3 &= \frac{r_2}{b_2 + \dfrac{a_2}{k_2}}, \\ \Delta_4 &= r_2 - (b_2 + \frac{a_2}{k_2})M_2. \end{aligned}$$

(3.2), (3.4), (3.6) and (3.9) show that under the assumption (B_1) holds, system (1.1) is permanent.

Remark 3.1. Noting that q represents the catchablity coefficient, hence, condition (B_1) shows that if catchablity coefficient is enough small, then the system is permanent. i. e., limited harvesting has no influence to the persistent property of the system.

IV. Extinction of the first species

Concerned with the extinction of the first species, we have the following result.

Theorem 4.1.*Assume that*

$$r_1 < \frac{qE}{m_1 E + m_2 F}, \tag{4.1}$$

where

$$F = \frac{1}{b_1 + \dfrac{a_1}{M_2 + k_1}}\exp\{r_1 - 1\},$$

and M_2 is defined by (3.4). Then the first species will be driven to extinction, i.e.,

$$\lim_{k\to+\infty} x(k) = 0.$$

Proof. Condition (4.1) implies that for enough small positive constant $\varepsilon > 0$,

$$r_1 < \frac{qE}{m_1 E + m_2 F(\varepsilon)} \tag{4.2}$$

holds, where

$$F(\varepsilon) = \frac{1}{b_1 + \dfrac{a_1}{M_2 + \varepsilon + k_1}}\exp\{r_1 - 1\} + \varepsilon.$$

Already, in the proof of Theorem 3.1, we had showed in (3.4) that

$$\limsup_{k\to+\infty} y(k) \le \frac{1}{b_2}\exp\{r_2 - 1\} \stackrel{\text{def}}{=} M_2. \tag{4.3}$$

Therefore, for $\varepsilon > 0$ small enough which satisfies (4.2), there exists a $N_1 > 0$ such that

$$y(k) < M_2 + \varepsilon \ \text{ for all } \ k \ge N_1. \tag{4.4}$$

For $k > N_1$, from (4.4) and the first equation of system (1.1), we have

$$\begin{aligned} & x(k+1) \\ = \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & -\frac{Eq}{m_1E + m_2 x(k)}\Big\} \\ \le \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{M_2+\varepsilon+k_1}\Big\}. \end{aligned} \tag{4.5}$$

Applying Lemma 2.5 to (4.5) leads to

$$\lim_{k\to+\infty} x(k) \le \frac{1}{b_1 + \dfrac{a_1}{M_2+\varepsilon+k_1}}\exp\{r_1-1\}, \tag{4.6}$$

Hence, there exists a $N_2 > N_1$ such that

$$\begin{aligned} x(k) \quad & < \quad \frac{1}{b_1 + \dfrac{a_1}{M_2+\varepsilon+k_1}}\exp\{r_1-1\}+\varepsilon \\ & \overset{def}{=} \quad F(\varepsilon) \ \text{ for all } \ k > N_2. \end{aligned} \tag{4.7}$$

For $k > N_2$, again, from the first equation of system (1.1), we have

$$\begin{aligned} & x(k+1) \\ = \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & -\frac{Eq}{m_1E + m_2 x(k)}\Big\} \\ < \ & x(k)\exp\Big\{r_1 - \frac{Eq}{m_1E+m_2x(k)}\Big\} \\ \le \ & x(k)\exp\Big\{r_1 - \frac{Eq}{m_1E+m_2F(\varepsilon)}\Big\}. \end{aligned} \tag{4.8}$$

Hence,

$$x(k) \le x(N_2)\exp\Big\{\Big(r_1 - \frac{Eq}{m_1E+m_2F(\varepsilon)}\Big)(k-N_2)\Big\}. \tag{4.9}$$

It then immediately follows from (4.2) that

$$\lim_{k\to+\infty} x(k) = 0.$$

This ends the proof of Theorem 4.1.

Remark 4.1. Condition (4.1) seems a little complicated, the reason is that we here try to considering the influence of the second species, however, if we use the estimate of upper bound (3.2) in Theorem 3.1, we could establish the following more stronger but concise result.

Corollary 4.1. *Assume that*

$$r_1 < \frac{qE}{m_1E+m_2M_1}. \tag{4.10}$$

Then the first species will be driven to extinction, i.e.,

$$\lim_{k\to+\infty} x(k) = 0.$$

V. Global attractivity

In section 3, we had showed that under very simple assumption, system (1.1) is permanent, also , in section 2, we obtain a set of sufficient conditions which ensure the existence of the positive equilibrium. It is nature to ask: what would ensure the global attractivity of the positive equilibrium? Concerned with this topic, we have the following result.

Theorem 5.1. *Assume that*

(A_2) $r_i,\ b_i, a_i, E, q,\ i = 1,\ 2$ are all positive constants, and

(A_3) $\dfrac{q}{m_1} < r_1 \le \min\{1, b_1k_2\},\ r_2 \le 1$

hold, then system (1.1) admits a unique positive equilibrium $E(x^*, y^*)$, which is globally attractive.

Proof Let $\big(x(k), y(k)\big)$ be arbitrary solution of system (1.1) with $x(0) > 0$ and $y(0) > 0$. Denote

$$U_1 = \limsup_{k\to+\infty} x(k), \qquad V_1 = \liminf_{k\to+\infty} x(k).$$

$$U_2 = \limsup_{k\to+\infty} y(k), \qquad V_2 = \liminf_{k\to+\infty} y(k).$$

We claim that $U_1 = V_1 = x^*$ and $U_2 = V_2 = y^*$.

From the first equation of system (1.1), we obtain

$$\begin{aligned} & x(k+1) \\ = \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & -\frac{Eq}{m_1E + m_2 x(k)}\Big\} \\ \le \ & x(k)\exp\Big\{r_1 - b_1 x\Big\}. \end{aligned} \tag{5.1}$$

Considering the auxiliary equation as follows:

$$u(k+1) = u(k)\exp\big\{r_1 - b_1u(k)\big\}, \tag{5.2}$$

From (A_3) we have $0 < r_1 \le 1$, according to (ii) of Lemma 2.3, we can obtain $u(k) \le \frac{1}{b_1}$ for all $k \ge 2$, where $u(k)$ is arbitrary positive solution of (5.2) with initial value $u(0) > 0$. From Lemma 2.2, $f(u) = u\exp(r_1 - b_1u)$ is nondecreasing for $u \in (0, \frac{1}{b_1}]$. According to Lemma 2.4 we can obtain $x(k) \le u(k)$ for all $k \ge 2$, where $u(k)$ is the solution of (5.2) with the initial value $u(2) = x(2)$. According to (i) of Lemma 2.3, we can obtain

$$U_1 = \limsup_{k\to+\infty} x(k) \le \lim_{k\to+\infty} u(k) = \frac{r_1}{b_1}. \tag{5.3}$$

From the second equation of system (1.1), we obtain

$$y(k+1) \le y(k)\exp\Big\{r_2 - b_2y(k)\Big\}.$$

Similar to the analysis of (5.1)-(5.3), we have

$$U_2 = \limsup_{k\to+\infty} y(k) \le \frac{r_2}{b_2}. \tag{5.4}$$

Then, for sufficiently small constant $\varepsilon > 0$, without loss of generality, we may assume that

$$\varepsilon < \frac{1}{2}\min\left\{\frac{r_2}{b_2+\dfrac{a_2}{k_2}}, \frac{r_1 - \dfrac{q}{m_1}}{b_1+\dfrac{a_1}{k_1}}\right\},$$

it follows from (5.3) and (5.4) that there is an integer $k_1 > 2$ such that

$$\begin{aligned} x(k) &< \frac{r_1}{b_1} + \varepsilon \stackrel{\text{def}}{=} M_1^x, \\ y(k) &< \frac{r_2}{b_2} + \varepsilon \stackrel{\text{def}}{=} M_1^y \ \text{ for all } k > k_1. \end{aligned} \tag{5.5}$$

(5.5) combine with the first equation of system (1.1) leads to

$$\begin{aligned} & x(k+1) \\ = \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & -\frac{Eq}{m_1 E + m_2 x(k)}\Big\} \\ \leq \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{M_1^y + k_1} \\ & -\frac{Eq}{m_1 E + m_2 M_1^x}\Big\}. \end{aligned} \tag{5.6}$$

Considering the auxiliary equation as follows:

$$\begin{aligned} u(k+1) = \ & u(k)\exp\Big\{r_1 - b_1 u(k) - \frac{a_1 u(k)}{M_1^y + k_1} \\ & -\frac{Eq}{m_1 E + m_2 M_1^x}\Big\}. \end{aligned} \tag{5.7}$$

Because of $0 < r_1 - \frac{q}{m_1} \leq 1$, obviously, $\frac{q}{m_1} > \frac{Eq}{m_1 E + m_2 M_1^x}$, and so, $0 < r_1 - \frac{q}{m_1} < r_1 - \frac{Eq}{m_1 E + m_2 M_1^x} < r_1 \leq 1$, according to (ii) of Lemma 2.3, we can obtain

$$u(k) \leq \frac{1}{b_1 + \frac{a_1}{M_1^y + k_1}}$$

for all $k \geq k_1$, where $u(k)$ is arbitrary positive solution of (5.7) with initial value $u(k_1) > 0$. From Lemma 2.2,

$$f(u) = u\exp\left(r_1 - b_1 u - \frac{a_1 u}{M_1^y + k_1} - \frac{Eq}{m_1 E + m_2 M_1^x}\right)$$

is nondecreasing for

$$u \in \left(0, \frac{1}{b_1 + \frac{a_1}{M_1^y + k_1}}\right].$$

According to Lemma 2.4 we can obtain $x(k) \leq u(k)$ for all $k \geq k_1+1$, where $u(k)$ is the solution of (5.7) with the initial value $u(k_1+1) = x(k_1+1)$. According to (i) of Lemma 2.3, we can obtain

$$U_1 = \limsup_{k\to+\infty} x(k) \leq \lim_{k\to+\infty} u(k) = \frac{r_1 - \frac{Eq}{m_1 E + m_2 M_1^x}}{b_1 + \frac{a_1}{M_1^y + k_1}}. \tag{5.8}$$

(5.5) combine with the second equation of system (1.1) leads to

$$\begin{aligned} & y(k+1) \\ = \ & y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{x(k)+k_2}\Big\} \\ \leq \ & y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{M_1^x + k_2}\Big\}, \end{aligned} \tag{5.9}$$

for all $k > k_1$. Similar to the analysis of (5.6)-(5.8), we can obtain

$$U_2 = \limsup_{k\to+\infty} y(k) \leq \frac{r_2}{b_2 + \frac{a_2}{M_1^x + k_2}}. \tag{5.10}$$

Then, for above $\varepsilon > 0$, it follows from (5.8) and (5.10) that there is an integer $k_2 > k_1$ such that for all $k > k_2$,

$$\begin{aligned} x(k) &< \frac{r_1 - \frac{Eq}{m_1 E + m_2 M_1^x}}{b_1 + \frac{a_1}{M_1^y + k_1}} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^x, \\ y(k) &< \frac{r_2}{b_2 + \frac{a_2}{M_1^x + k_2}} + \frac{\varepsilon}{2} \stackrel{\text{def}}{=} M_2^y. \end{aligned} \tag{5.11}$$

Noting that

$$\begin{aligned} & r_1 - \frac{Eq}{m_1 E + m_2 M_1^x} < r_1, \\ & \frac{a_1}{M_1^y + k_1} > 0, \\ & \frac{a_2}{M_1^x + k_2} > 0, \end{aligned} \tag{5.12}$$

it immediately follows that

$$\begin{aligned} M_2^x &= \frac{r_1 - \frac{Eq}{m_1 E + m_2 M_1^x}}{b_1 + \frac{a_1}{M_1^y + k_1}} + \frac{\varepsilon}{2} \\ &< \frac{r_1}{b_1} + \varepsilon = M_1^x; \\ M_2^y &= \frac{r_2}{b_2 + \frac{a_2}{M_1^x + k_2}} + \frac{\varepsilon}{2} \\ &< \frac{r_2}{b_2} + \varepsilon = M_1^y. \end{aligned} \tag{5.13}$$

According to the first equation of system (1.1) and the positivity of $x(k), y(k)$, we can obtain

$$\begin{aligned} & x(k+1) \\ = \ & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & -\frac{Eq}{m_1 E + m_2 x(k)}\Big\} \\ \geq \ & x(k)\exp\Big\{r_1 - \frac{q}{m_1} - b_1 x(k) - \frac{a_1 x(k)}{k_1}\Big\}. \end{aligned} \tag{5.14}$$

Considering the auxiliary equation as follows:

$$u(k+1) = u(k)\exp\Big\{r_1 - \frac{q}{m_1} - b_1 u(k) - \frac{a_1 u(k)}{k_1}\Big\}. \tag{5.15}$$

Since $0 < r_1 - \frac{q}{m_1} < r_1 \leq 1$, according to (ii) of Lemma 2.3, we can obtain $u(k) \leq \frac{1}{b_1 + \frac{a_1}{k_1}}$ for all $k \geq k_2$, where $u(k)$ is arbitrary positive solution of (5.15) with initial value $u(k_2) > 0$. From Lemma 2.2, $f(u) = u\exp\Big\{r_1 - \frac{q}{m_1} - b_1 u - \frac{a_1 u}{k_1}\Big\}$ is nondecreasing for $u \in \Big(0, \frac{1}{b_1 + \frac{a_1}{k_1}}\Big]$. According to Lemma 2.4 we can obtain $x(k) \geq u(k)$ for all $k \geq k_2$, where $u(k)$ is the solution of (5.15) with the initial value $u(k_2) = x(k_2)$. According to (i) of Lemma 2.3, we have

$$V_1 = \liminf_{k\to+\infty} x(k) \geq \lim_{k\to+\infty} u(k) = \frac{r_1 - \frac{q}{m_1}}{b_1 + \frac{a_1}{k_1}}. \qquad (5.16)$$

From the second equation of system (1.1) and the positivity of $x(k)$, we can obtain

$$y(k+1) \geq y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{k_2}\Big\}. \qquad (5.17)$$

Similar to the analysis of (5.14)-(5.16), we have

$$V_2 = \liminf_{k\to+\infty} x_2(k) \geq \frac{r_2}{b_2 + \frac{a_2}{k_2}}.$$

Then, for above $\varepsilon > 0$, there is an integer $k_3 > k_2$ such that for all $k > k_3$,

$$\begin{aligned} x(k) &> \frac{r_1 - \frac{q}{m_1}}{b_1 + \frac{a_1}{k_1}} - \varepsilon \overset{\text{def}}{=} m_1^x, \\ y(k) &> \frac{r_2}{b_2 + \frac{a_2}{k_2}} - \varepsilon \overset{\text{def}}{=} m_1^y. \end{aligned} \qquad (5.18)$$

(5.18) combine with the first equation of system (1.1) leads to

$$\begin{aligned} & x(k+1) \\ = \; & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{y(k)+k_1} \\ & \quad - \frac{Eq}{m_1 E + m_2 x(k)}\Big\} \\ \geq \; & x(k)\exp\Big\{r_1 - b_1 x(k) - \frac{a_1 x(k)}{m_1^y + k_1} \\ & \quad - \frac{Eq}{m_1 E + m_2 m_1^x}\Big\}, \; k > k_3. \end{aligned} \qquad (5.19)$$

Similar to the analysis of (5.14)-(5.16), we have

$$V_1 = \liminf_{k\to+\infty} x(k) \geq \frac{r_1 - \frac{Eq}{m_1 E + m_2 m_1^x}}{b_1 + \frac{a_1}{m_1^y + k_1}}. \qquad (5.20)$$

(5.18) combined with the second equation of system (1.1) leads to

$$y(k+1) \geq y(k)\exp\Big\{r_2 - b_2 y(k) - \frac{a_2 y(k)}{m_1^x + k_2}\Big\} \qquad (5.21)$$

for all $k > k_3$. Similar to the analysis of (5.14)-(5.16), we can obtain

$$V_2 = \liminf_{k\to+\infty} y(k) \geq \frac{r_2}{b_2 + \frac{a_2}{m_1^x + k_2}}. \qquad (5.22)$$

Then, for above $\varepsilon > 0$, it follows from (5.20) and (5.22) that there is an integer $k_4 > k_3$ such that for all $k > k_4$,

$$\begin{aligned} x(k) &> \frac{r_1 - \frac{Eq}{m_1 E + m_2 m_1^x}}{b_1 + \frac{a_1}{m_1^y + k_1}} - \frac{\varepsilon}{2} \overset{\text{def}}{=} m_2^x, \\ y(k) &> \frac{r_2}{b_2 + \frac{a_2}{m_1^x + k_2}} - \frac{\varepsilon}{2} \overset{\text{def}}{=} m_2^y. \end{aligned} \qquad (5.23)$$

Also, since $m_1^x > 0, m_1^y > 0$, it follows that

$$\begin{aligned} \frac{Eq}{m_1 E + m_2 m_1^x} &< \frac{q}{m_1}, \\ \frac{a_1}{m_1^y + k_1} &< \frac{a_1}{k_1}, \\ \frac{a_2}{m_1^x + k_2} &< \frac{a_2}{k_2}, \end{aligned} \qquad (5.24)$$

and so

$$\begin{aligned} m_2^x &= \frac{r_1 - \frac{Eq}{m_1 E + m_2 m_1^x}}{b_1 + \frac{a_1}{m_1^y + k_1}} - \frac{\varepsilon}{2} \\ &> \frac{r_1 - \frac{q}{m_1}}{b_1 + \frac{a_1}{k_1}} - \varepsilon = m_1^x; \\ m_2^y &= \frac{r_2}{b_2 + \frac{a_2}{m_1^x + k_2}} - \frac{\varepsilon}{2} \\ &> \frac{r_2}{b_2 + \frac{a_2}{k_2}} - \varepsilon = m_1^y. \end{aligned} \qquad (5.25)$$

Continuing the above steps, we can get four sequences $\{M_k^x\}$, $\{M_k^y\}$, $\{m_k^x\}$ and $\{m_k^y\}$ such that

$$\begin{aligned} M_k^x &= \frac{r_1 - \frac{Eq}{m_1 E + m_2 M_{k-1}^x}}{b_1 + \frac{a_1}{M_{k-1}^y + k_1}} + \frac{\varepsilon}{k}, \\ M_k^y &= \frac{r_2}{b_2 + \frac{a_2}{M_{k-1}^x + k_2}} + \frac{\varepsilon}{k}; \end{aligned} \qquad (5.26)$$

and

$$\begin{aligned} m_k^x &= \frac{r_1 - \frac{Eq}{m_1 E + m_2 m_{k-1}^x}}{b_1 + \frac{a_1}{m_{k-1}^y + k_1}} - \frac{\varepsilon}{k}, \\ m_k^y &= \frac{r_2}{b_2 + \frac{a_2}{m_{k-1}^x + k_2}} - \frac{\varepsilon}{k}. \end{aligned} \qquad (5.27)$$

Clearly, we have

$$m_k^x \leq V_1 \leq U_1 \leq M_k^x, \; m_k^y \leq V_2 \leq U_2 \leq M_k^y, \; k = 0, 1, 2.... \qquad (5.28)$$

Now, we will prove $\{M_k^x\}$, $\{M_k^y\}$ is monotonically decreasing, $\{m_k^x\}$, $\{m_k^y\}$ is monotonically increasing by means of inductive method. First of all, from (5.13) and (5.25) it is clear that $M_2^x < M_1^x$, $M_2^y < M_1^y, m_2^x > m_1^x \; m_2^y > m_1^y$. Now we assume that $M_k^x < M_{k-1}^x, M_k^y < M_{k-1}^y$ and $m_k^x > m_{k-1}^x, m_k^y > m_{k-1}^y$ hold, then

$$\begin{aligned} \frac{Eq}{m_1 E + m_2 M_k^x} &> \frac{Eq}{m_1 E + m_2 M_{k-1}^x}, \\ b_1 + \frac{a_1}{M_k^y + k_1} &> b_1 + \frac{a_1}{M_{k-1}^y + k_1}, \\ b_2 + \frac{a_2}{M_k^x + k_2} &> b_2 + \frac{a_2}{M_{k-1}^x + k_2}. \end{aligned} \qquad (5.29)$$

From (5.29) and the expression of M_k^x, M_k^y, it immediately follows that

$$\begin{aligned} M_{k+1}^x &= \frac{r_1 - \dfrac{Eq}{m_1E + m_2M_k^x}}{b_1 + \dfrac{a_1}{M_k^y + k_1}} + \frac{\varepsilon}{k+1} \\ &< \frac{r_1 - \dfrac{Eq}{m_1E + m_2M_{k-1}^x}}{b_1 + \dfrac{a_1}{M_{k-1}^y + k_1}} + \frac{\varepsilon}{i} = M_k^x; \\ M_{k+1}^y &= \frac{r_2}{b_2 + \dfrac{a_2}{M_k^x + k_2}} + \frac{\varepsilon}{k+1} \\ &< \frac{r_2}{b_2 + \dfrac{a_2}{M_{k-1}^x + k_2}} + \frac{\varepsilon}{i} = M_k^y. \end{aligned} \tag{5.30}$$

Also, it follows from $m_k^x > m_{k-1}^x, m_k^y > m_{k-1}^y$ that

$$\begin{aligned} &\frac{Eq}{m_1E + m_2m_k^x} < \frac{Eq}{m_1E + m_2m_{k-1}^x}, \\ &b_1 + \frac{a_1}{m_k^y + k_1} < b_1 + \frac{a_1}{m_{k-1}^y + k_1}, \\ &b_2 + \frac{a_2}{m_k^x + k_2} < b_2 + \frac{a_2}{m_{k-1}^x + k_2}. \end{aligned} \tag{5.31}$$

From (5.31) and the expression of m_k^x, m_k^y, it immediately follows that

$$\begin{aligned} m_{k+1}^x &= \frac{r_1 - \dfrac{Eq}{m_1E + m_2m_k^x}}{b_1 + \dfrac{a_1}{m_k^y + k_1}} - \frac{\varepsilon}{k+1} \\ &> \frac{r_1 - \dfrac{Eq}{m_1E + m_2m_{k-1}^x}}{b_1 + \dfrac{a_1}{m_{k-1}^y + k_1}} - \frac{\varepsilon}{k} = m_k^x, \\ m_{k+1}^y &= \frac{r_2}{b_2 + \dfrac{a_2}{m_k^x + k_2}} - \frac{\varepsilon}{k+1} \\ &> \frac{r_2}{b_2 + \dfrac{a_2}{m_{k-1}^x + k_2}} - \frac{\varepsilon}{k} = m_k^y. \end{aligned} \tag{5.32}$$

(5.29)-(5.32) show that $\{M_k^x\}$ and $\{M_k^y\}$ are monotonically decreasing, $\{m_k^x\}$ and $\{m_k^y\}$ are monotonically increasing. Consequently, $\lim\limits_{k\to+\infty}\{M_k^x\}$, $\lim\limits_{k\to+\infty}\{M_k^y\}$ and $\lim\limits_{k\to+\infty}\{m_k^x\}$ $\lim\limits_{k\to+\infty}\{m_k^y\}$ both exist. Let

$$\lim_{k\to+\infty} M_k^x = \overline{X}, \ \lim_{k\to+\infty} m_k^x = \underline{X}. \tag{5.33}$$

$$\lim_{k\to+\infty} M_k^y = \overline{Y}, \ \lim_{k\to+\infty} m_k^x = \underline{Y}. \tag{5.34}$$

From (5.26) and (5.27) we have

$$\overline{X} = \frac{r_1 - \dfrac{Eq}{m_1E + m_2\overline{X}}}{b_1 + \dfrac{a_1}{\overline{Y} + k_1}}, \ \ \overline{Y} = \frac{r_2}{b_2 + \dfrac{a_2}{\overline{X} + k_2}}; \tag{5.35}$$

$$\underline{X} = \frac{r_1 - \dfrac{Eq}{m_1E + m_2\underline{X}}}{b_1 + \dfrac{a_1}{\underline{Y} + k_1}}, \ \ \underline{Y} = \frac{r_2}{b_2 + \dfrac{a_2}{\underline{X} + k_2}}. \tag{5.36}$$

(5.35) and (5.36) are equivalent to

$$\begin{aligned} b_1\overline{X} + \frac{a_1\overline{X}}{\overline{Y} + k_1} &= r_1 - \frac{Eq}{m_1E + m_2\overline{X}}, \\ b_2\overline{Y} + \frac{a_2\overline{Y}}{\overline{X} + k_2} &= r_2. \end{aligned} \tag{5.37}$$

$$\begin{aligned} b_1\underline{X} + \frac{a_1\underline{X}}{\underline{Y} + k_1} &= r_1 - \frac{Eq}{m_1E + m_2\underline{X}}, \\ b_2\underline{Y} + \frac{a_2\underline{Y}}{\underline{X} + k_2} &= r_2. \end{aligned} \tag{5.38}$$

(5.37) and (5.38) show that $(\overline{X}, \overline{Y})$ and $(\underline{X}, \underline{Y})$ are all solutions of system (2.2). however, under the assumption of Theorem 5.1, system (2.2) has unique positive solution (x^*, y^*). Therefore

$$\begin{aligned} U_1 = V_1 = \lim_{k\to+\infty} x(k) = x^*, \\ U_2 = V_2 = \lim_{k\to+\infty} y(k) = y^*. \end{aligned} \tag{5.39}$$

That is, $E_+(x^*, y^*)$ is globally attractive. This ends the proof of Theorem 5.1.

VI. Examples

In this section, we shall give three examples to illustrate the feasibility of main result.

Example 6.1. Considering the following system:

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{1.5 - \frac{1}{1 + x(k)} \\ &\qquad -0.3x(k) - \frac{0.1x(k)}{y(k)+1}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{1.5 - 0.2y(k) \\ &\qquad -\frac{0.1y(k)}{x(k)+0.2}\Big\}. \end{aligned} \tag{6.1}$$

Corresponding to system (1.1), we have $r_1 = 1.5; r_2 = 1.5; b_1 = 0.3; b_2 = 0.2; a_1 = 0.1; a_2 = 0.1; E = 1; q = 1; k_1 = 1; k_2 = 0.2$; $m_1 = 1; m_2 = 1$; One could easily see that $r_1 = 1.5 > 1 = \dfrac{q}{m_1}$, thus the coefficients of system (6.1) satisfies condition (A_1) in Theorem 3.1. From Theorem 3.1, system (6.1) is permanent. Numeric simulations also support our finding(see Fig. 1 and 2).

Example 6.2. Considering the following system:

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{1 - \frac{2}{1 + 0.9\times x(k)} \\ &\qquad -0.3x(k) - \frac{0.1x(k)}{y(k)+1}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{1.5 - 0.2y(k) \\ &\qquad -\frac{0.1y(k)}{x(k)+0.2}\Big\}. \end{aligned} \tag{6.2}$$

Here all the other coefficients are as that of Example 6.1, only change $r_1 = 1$, $q_1 = 2$, $m_2 = 0.9$. By calculating, we have

$$M_1 = \frac{1}{b_1}\exp\{r_1 - 1\} = 1,$$

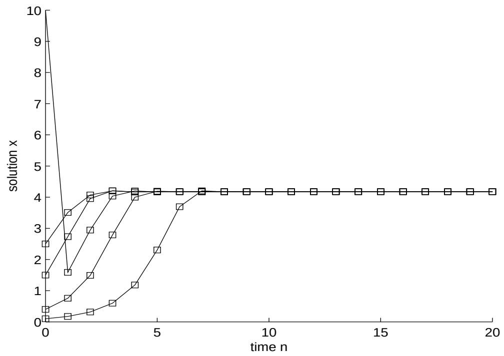

Fig. 1. Dynamic behaviors of the first component of the solution $(x(k), y(k))$ of system (6.1) with the initial conditions $(x(0), y(0))$= $(0.1, 3), (1.5, 2)$ $(2.5, 1)$, $(10, 3)$ and $(0.4, 0.5)$, respectively.

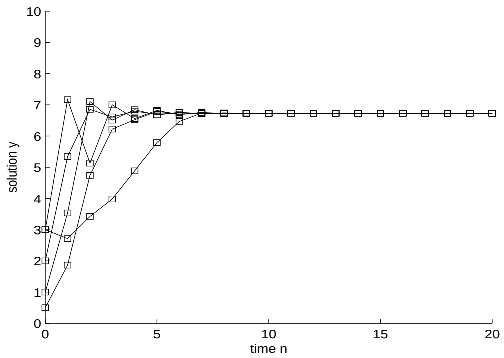

Fig. 2. Dynamic behaviors of the second component of the solution $(x(k), y(k))$ of system (6.1) with the initial conditions $(x(0), y(0))$= $(0.1, 3), (1.5, 2)$ $(2.5, 1)$, $(10, 3)$ and $(0.4, 0.5)$, respectively.

consequently

$$r_1 = 1 < \frac{2}{1.9} = \frac{qE}{m_1 E + m_2 M_1}. \tag{6.3}$$

Hence, it follows from Corollary 4.1 that the first species will be driven to extinction. numeric simulation (see Fig. 3) supports this assertion.

Example 6.3. Considering the following system:

$$\begin{aligned} x(k+1) &= x(k)\exp\Big\{0.5 - \frac{0.1\times 1}{1\times 1 + 1\times x(k)} \\ &\qquad -0.3x(k) - \frac{0.1x(k)}{y(k)+1}\Big\}, \\ y(k+1) &= y(k)\exp\Big\{0.5 - 0.2y(k) \\ &\qquad -\frac{0.1y(k)}{x(k)+0.2}\Big\}, \end{aligned} \tag{6.4}$$

Corresponding to system (1.1), we have $r_1 = 0.5; r_2 = 0.5; b_1 = 0.3; b_2 = 0.2; a_1 = 0.1; a_2 = 0.1; E = 1; q = 0.1; k_1 = 1; k_2 = 0.2; m_1 = 1; m_2 = 1$; One could easily see that $b_1 k_2 = 0.6 \geq r_1 = 0.5 > 0.1 = \frac{q}{m_1}, 1 \geq r_2$, thus the coefficients of system (6.4) satisfy (A_2) and (A_3) in Theorem

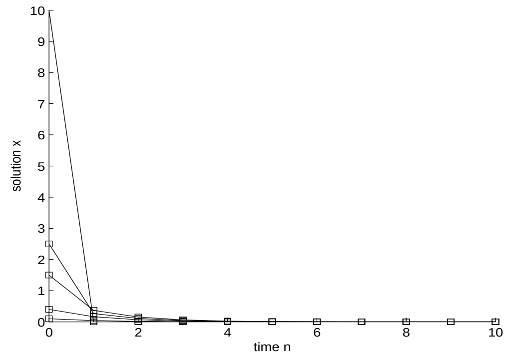

Fig. 3. Dynamic behaviors of the first component of the solution $(x(k), y(k))$ of system (6.2) with the initial conditions $(x(0), y(0))$= $(0.1, 3), (1.5, 2)$ $(2.5, 1)$ and $(0.4, 0.5)$, respectively.

5.1. From Theorem 5.1, system (6.4) admits a unique positive equilibrium, which is globally stable. Numeric simulations also support our finding(see Fig. 4 and 5).

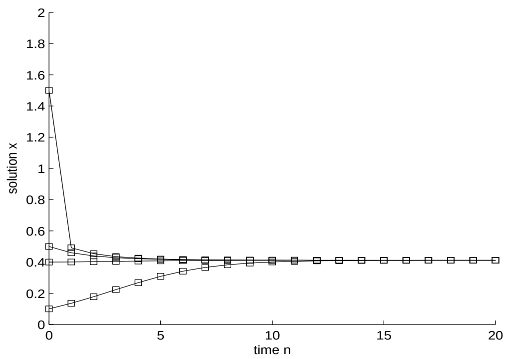

Fig. 4. Dynamic behaviors of the first component of the solution $(x(k), y(k))$ of system (6.4) with the initial conditions $(x(0), y(0))$= $(0.1, 3), (1.5, 2)$ $(2.5, 1)$ and $(0.4, 0.5)$, respectively.

VII. Discussion

Stimulated by recently works of [3]-[5], [31]-[35], we propose a discrete May cooperative system incorporating Michaelis-Menten type harvesting. To the best of authors knowledge, this is the first time that the discrete population model incorporating Michaelis-Menten type harvesting is proposed and studied.

With the help of two Lemmas (Lemma 2.5 and 2.6), we are able to establish a set of sufficient conditions which ensure the permanence of the system (Theorem 3.1), the condition is very simple and easily verified. Numeric simulations (Fig.1 and 2) also support our findings.

We also focus our attention to the extinction property of the system, since with the develop of the modern so ociety, more and more species becomes endangered due to the overfishing. Our result (Theorem 4.1) also confirm this phenomena. If the catchablity coefficient q is enough large, then inequality (4.1) holds, and it follows from Theorem 4.1 that the first species will be driven to extinction, despite the

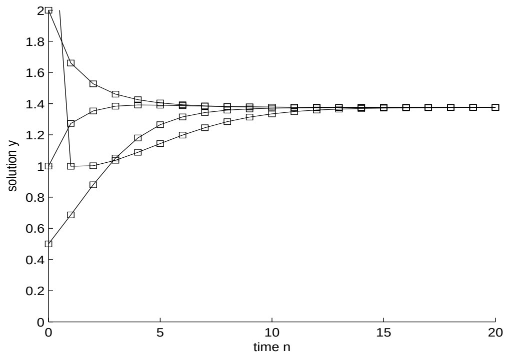

Fig. 5. Dynamic behaviors of the second component of the solution $(x(k), y(k))$ of system (6.4) with the initial conditions $(x(0), y(0))$= $(0.1, 3)$, $(1.5, 2)$ $(2.5, 1)$ and $(0.4, 0.5)$, respectively.

cooperation of the two species.

We finally focus our attention to the stability property of the positive equilibrium, by using the iterative method, a set of very simple sufficient conditions which ensure the global stability property of the positive equilibrium is established. Numeric simulations (Fig. 4 and 5) also shows the feasibility of this result.

At the end of the paper, we would like to point out that numeric simulations (Fig. 1 and 2) show that system (6.1) also admits a unique positive equilibrium which is globally attractive, however, since in system (6.1), $r_1 = 1.5 > 1$, we could not testify the stability property of the positive equilibrium by using Theorem 5.1. That is, there are still have room to improve for our Theorem 6.1, however, with the restriction of our method, we could not give more insight to this issue at present. We would like to leave this for future investigation.

VIII. Declarations

Competing interests

The authors declare that there is no conflict of interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

[1] R. M. May, “Theoretical Ecology, ” Principles and Applications, Sounders, Philadelphia,1976.
[2] J.A. Roberts, N.G. Joharjee, “Stability analysis of a continuous model of mutualism with delay dynamics, ” International Mathematical Forum, vol. 11, no. 10, pp. 463-473, 2016.
[3] F. Y. Wei, C. Y. Li, “Permanence and globally asymptotic stability of cooperative system incorporating harvesting, ” Advances in Pure Mathematics, vol. 3, no. 3, pp. 627-632, 2013.
[4] X. D. Xie, F. D. Chen, Y. L. Xue, “Note on the stability property of a cooperative system incorporating harvesting, ” Discrete Dynamics in Nature and Society Volume 2014, Article ID 327823, 5 pages. 2014.
[5] F. D. Chen, H. L. Wu, X. D. Xie, “Global attractivity of a discrete cooperative system incorporating harvesting, ” Advances in Difference Equations, Volume 2016, Article ID 268, 2016.
[6] K. Yang, Z. S. Miao, F. D. Chen, X. D. Xie, “Influence of single feedback control variable on an autonomous Holling-II type cooperative system, ” Journal of Mathematical Analysis and Applications, vol. 435, no. 1, pp.874-888, 2016.
[7] Y. Xue, X. Xie, F. Chen, et al, “Almost periodic solution of a discrete commensalism system, ” Discrete Dynamics in Nature and Society Volume 2015, Article ID 295483, 11 pages, 2015.
[8] X. D. Xie, F. D. Chen, K. Yang, et al, “Global attractivity of an integrodifferential model of mutualism, ” Abstract and Applied Analsis, Volume 2014, Article ID 928726, 6 pages, 2014.
[9] K. Yang, X. D. Xie, F. D. Chen, “Global stability of a discrete mutualism model, ” Abstract and Applied Analysis, Volume 2014, Article ID 709124, 7 pages, 2014.
[10] F. D. Chen, X. D. Xie, X. F. Chen, “Dynamic behaviors of a stage-structured cooperation model, ” Communications in Mathematical Biology and Neuroscience, Vol 2015, Article ID 4, 2015.
[11] W. S. Yang, X. P. Li, “Permanence of a discrete nonlinear N-species cooperation system with time delays and feedback controls, ” Applied Mathematics and Computation, vol. 218, no. 3, pp.3581- 3586, 2011.
[12] L. J. Chen, L. J. Chen, Z. Li, “Permanence of a delayed discrete mutualism model with feedback controls, ” Mathematical and Computer Modelling, vol. 50, no. 3, pp.1083-1089, 2009.
[13] L. J. Chen, X. D. Xie, “Permanence of an n-species cooperation system with continuous time delays and feedback controls, ” Nonlinear Analysis-Real World Applications, vol. 12, no. 1, pp. 34-38, 2011.
[14] R. Han, X. Xie, F. Chen, “Permanence and global attractivity of a discrete pollination mutualism in plant-pollinator system with feedback controls, ” Advances in Difference Equations, Volume 2016, Article ID 199, 2016.
[15] L. J. Chen, X. D. Xie, “Feedback control variables have no influence on the permanence of a discrete N-species cooperation system, ” Discrete Dynamics in Nature and Society, Volume 2009, Article ID 306425, 10 pages, 2009.
[16] R. Han, F. Chen, X. Xie, et al, “Global stability of May cooperative system with feedback controls, ” Advances in Difference Equations, Volume 2015, Article ID 360, 2015.
[17] F. D. Chen, J. H. Yang, L. J. Chen, X. D. Xie, “On a mutualism model with feedback controls, ” Applied Mathematics and Computation, vol. 214, no. 2, 581-587, 2009.
[18] F. Chen, Y. Xue, Q. Lin, et al, “Dynamic behaviors of a Lotka-Volterra commensal symbiosis model with density dependent birth rate, ” Advances in Difference Equations, Volume 2018, Article ID 296, 2018.
[19] G. Y. Chen, Z. D. Teng, “On the stability in a discrete two-species competition system, ” Journal of Applied Mathematics and Computing, vol 38, no.1, pp.25-36, 2012.
[20] L. Yang, X. Xie, F. Chen, “Dynamic behaviors of a discrete periodic predator-prey-mutualist system, ” Discrete Dynamics in Nature and Society, Volume 2015, Article ID 247269, 11 pages, 2015.
[21] L. Yang, X. D. Xie, F. Chen, Y. Xue, “Permanence of the periodic predator-prey-mutualist system, ” Advances in Difference Equations, Volume 2015, Article ID 331, 2015.
[22] R. Wu, L. Lin, “Dynamic behaviors of a commensal symbiosis model with ratio-dependent functional response and one party can not survive independently, ” Journal of Mathematics and Computer Science, vol. 16, no. 2, pp. 495-506, 2016.
[23] R. X. Wu, L. Li, Q. F. Lin, “A Holling type commensal symbiosis model involving Allee effect, ” Communications in Mathematical Biology and Neuroscience, Volume 2018, Article ID 6, 2018.
[24] A. Muhammadhaji, Z. D. Teng, “Global attractivity of a periodic delayed n-species model of facultative mutualism, ” Discrete Dynamics in Nature and Society, Volume 2013, Article ID 580185, 11 pages, 2013.
[25] T. T. Li, F. D. Chen, J. H. Chen, et al, “Stability of a stage-structured plant-pollinator mutualism model with the Beddington-DeAngelis functional response, ” Journal of Nonlinear Functional Analysis, Volume 2017, Article ID 50, pp. 1-18, 2017.
[26] Q. Lin, “ Allee effect increasing the final density of the species subject to the Allee effect in a Lotka-Volterra commensal symbiosis model, ” Advances in Difference Equations, Volume 2018, Article ID 196, 2018.
[27] Q. F. Lin, “Dynamic behaviors of a commensal symbiosis model with non-monotonic functional response and non-selective harvesting

in a partial closure, ” Communications in Mathematical Biology and Neuroscience, Volume 2018, Article ID 4, 2018.
[28] H. Deng, X. Huang, “ The influence of partial closure for the populations to a harvesting Lotka-Volterra commensalism model, ” Communications in Mathematical Biology and Neuroscience, Volume 2018, Article ID 10, 2018.
[29] C. Lei, “Dynamic behaviors of a non-selective harvesting May cooperative system incorporating partial closure for the populations, ” Communications in Mathematical Biology and Neuroscience, Volume 2018, Article ID 12, 2018.
[30] C. Lei, “Dynamic behaviors of a stage-structured commensalism system, ” Advances in Difference Equations, Volume 2018, Article ID 301, 2018.
[31] B. G. Chen, “Dynamic behaviors of a commensal symbiosis model involving Allee effect and one party can not survive independently, ” Advances in Difference Equations, Volume 2018, Article ID 212, 2018.
[32] B. G. Chen, “The influence of commensalism to a Lotka-Volterra commensal symbiosis model with Michaelis-Menten type harvesting, ” Advances in Difference Equations, Volume 2019, Article ID 43, 2019.
[33] M. A. Idlangoa, J. J. Shepherdb, J. A. Gear, “Logistic growth with a slowly varying Holling type II harvesting term, ” Communications in Nonlinear Science and Numerical Simulation, vol.49, no.1, pp. 81-92, 2017.
[34] D. P. Hu, H. J. Cao, “ Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvesting, ” Nonlinear Analysis: Real World Applications, vol. 33, pp.58-82, 2017.
[35] Q. Lin, X. Xie, F. Chen, et al, “Dynamical analysis of a logistic model with impulsive Holling type-II harvesting, ” Advances in Difference Equations, Volume 2018, Article ID 112, 2018.
[36] L. Wang, M. Q. Wang, “Ordinary Difference Equation, ” Xinjing University Press, Urmuqi, 1989.
[37] Z. W. Xiao, Z. Li, “Stability and bifurcation in a stage-structured predator-prey model with Allee effect and time delay, ” IAENG International Journal of Applied Mathematics, vol. 49, no.1, pp. 6-13, 2019.
[38] Q. Yue, “Permanence of a delayed biological system with stage structure and density-dependent juvenile birth rate, ” Engineering Letters, vol.27, no. 2, pp.263-268, 2019.
[39] S. B. Yu, “Effect of predator mutual interference on an autonomous Leslie-Gower predator-prey model, ” IAENG International Journal of Applied Mathematics, vol. 49, no. 2, pp. 229-233, 2019.
[40] B. G. Chen, “The influence of density dependent birth rate to a commensal symbiosis model with Holling type functional response, ” Engineering Letters, vol. 27, no. 2, pp. 295-302, 2019.