Lattices of (Generalized) Fuzzy Ideals in Residuated Lattices

Kuanyun Zhu, Jingru Wang* and Yongwei Yang

Abstract—**In this paper, we discuss the relationships between fuzzy sets and residuated lattices. In fact, we consider a residuated lattice as a universal set and apply the concept of tip-extended pair of fuzzy sets for definition the (generalized) fuzzy ideals. And we explore some new operations of fuzzy sets in a residuated lattice. In particular, associated with the concept of tip-extended pair of fuzzy sets, it is proved, respectively, that the sets of all (generalized) fuzzy ideals form bounded distributive lattices.**

Index Terms—**Residuated lattice; (Generalized) fuzzy ideal; Bounded distributive lattice**

I. Introduction

It is well known that certain information processing, especially inferences based on certain information, is based on the classic logic (classic two-valued logic). Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. As a result, some various kinds of non-classical logic systems have been extensively introduced. In fact, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. Many-valued logic, as the extension and development of classical logic, has been a crucial direction in non-classical logic. Lattice-valued logic, an important many-valued logic, has two prominent roles: the one is to extend the chain-type truth-valued field of the current logics to some relatively general lattices, the other one is the incompletely comparable property of truth value characterized by the general lattice can more effectively reflect the uncertainty of human being's judging and decision. Thus, lattice-valued logic has been become a new research area, and it has been influenced the development of algebraic logic, computer science and artificial intelligent technology. In the meantime, various logical algebras have been proposed as the structures of truth degrees associated with logic systems, such as BL-algebras, MV-algebras, lattice implication algebras, MTL-algebras, NM-algebras and R_0-algebras and so on.

As far as know that, filters of logical algebras plays an important role in studying these logical algebras and the completeness of the corresponding non-classical logics. It is well known that, filters are very closed related to congruence relations with which one can associate quotient algebras [7]. From a logical point of view, various filters correspond to various sets of provable formulas. Moreover, a filter is also called a deductive system [26]. In particular, some types of filters in BL-algebras were investigated in [8], [13], [26], and it has been widely studied in residuated lattices [17], [30], [33].

So far, fuzzification ideals have been applied to some algebraic structures and fuzzy logical algebrass, for example, groups, rings, semirings and so on [22]. In addition, fuzzification ideals have also been applied to some fuzzy logical algebras. For example, Jun et al. [10] considered fuzzy filters of MTL-algebras. Liu and Li [16] discussed fuzzy filters of BL-algebras. Borzooei [2] investigated some types of filters in MTL-algebras. Gasse and Deschrijver [7] studied filters of residuated lattices and triangle algebra. Kondo [12] investigated filters on commutative residuated lattices. Liu et al. [18], [19] considered interval-valued I-fuzzy filters, interval-valued I-fuzzy congruences and interval-valued intuitionistic (T, S)-fuzzy filters theory on residuated lattices. In particular, in 2018, Gao et al. [6] introduced the notion of fuzzy extended filters on residusted lattices. Zhang and Li [30] studied intuitionistic fuzzy filters theory on residuated lattices. In 2019, Dong and Xin [4] introduced α-filters and prime α-filters in residuated lattices. Paad and Borzooei [23] discussed semi-maximal filters in BL-algebras. Rasouli and Davvaz [24] studied rough filters based on residuated lattices. Zhu et al. [32] introduced some stabilizers and studied related properties of them in residuated lattices. Then, they investigated the image and inverse image of a right and left stabilizer of a nonempty subset under a homomorphism. Besides, they discussed the relations between stabilizers and several special filters (ideals) in residuated lattices. They also characterized some special classes of residuated lattices, for example, Heyting algebras and linearly ordered Heyting algebras. Anusuya Ilamathi et al. [1] applied the notion of multisets to filters of residuated lattices and introduced the new concept of multiset filters. The relation between multiset filters and their n-level sets is showed and some principal characterizations of multiset filter are discussed. Furthermore, as an application of the proposed concept, they presented a decision making problem.

Form the point of view of uncertain information, the sets of provable formulas in corresponding inference systems can be described by fuzzy filters of those algebraic semantics. At the same time, ideal theory is a very effectively tool for investigating these various algebraic and logic systems. The concept of ideal has been introduced in many algebraic structure, for example, lattices, rings, MV-algebras and lattice implication algebras and so on. In these algebraic structure, the ideal is in the center position. In [14], [15], Lele and Nganou proposed the concept of (fuzzy) ideals in BL-algebras as a natural generalization of that of ideals in MV-algebras. Yang and Xin [28] introduced the notion of implicative ideals in BL-algebras. In 2016, Luo [20] first introduced the notion

Manuscript received December 3, 2019; revised April 25, 2020. This work was supported in part by Higher Education Key Scientific Research Program Funded by Henan Province (No. 20A110011) and Research and Cultivation Fund Project of Anyang Normal University (No. AYNUKP-2018-B26).

K. Zhu and J. Wang are with the School of Information and Mathematics, Yangtze University, Jingzhou, 434023, P.R. China e-mail: kyzhu@yangtzeu.edu.cn, wangjingru@hnu.edu.cn

Y. Yang is with the School of Mathematics and Statistics, Anyang Normal University, Anyang, 455000, P.R. China e-mail: yangyw@aynu.edu.cn

of ideals in residuated lattices as a natural generalization of the concept of ideals in BL-algebras. Then the author gave several equivalent characterizations of ideals in residuated lattices. In 2017, Liu et al. [17] introduced the notion of fuzzy ideals on residuated lattices. In 2018, Shoar [25] investigated some types of ideals in residuated lattices. In particular, in 2019, Kengne et al. [11] introduced the notion of L-fuzzy ideals of a residuated lattice and obtained their properties and equivalent characterizations. Also, they introduced the notion of prime fuzzy ideals, fuzzy prime ideals and fuzzy prime ideals of the second kind of a residuated lattice and established the relationships between these types of fuzzy ideals. At last, they studied the notions of fuzzy maximal ideal and maximal fuzzy ideal of a residuated lattice and presented some characterizations. Holdon [9] put forward the concepts of nodal and conodal ideals in a residuated lattice and studied some properties. At the same time, they gave a characterization of nodal ideals in terms of congruences and showed that if L is an MTL-algebra and I is a non-principal nodal ideal, then L/I is a chain. And, they proposed a characterization for Boolean residuated lattices (L is a Boolean residuated lattice if and only if L is an involution semi-G-agebra) and they discussed briefly the applications of their results in varieties of residuated lattices. Finally, they introduced the concept of a fuzzy (nodal) ideal of a residuated lattice, and gave some related results. In 2020, Yang et al. [29] presented a rough set model based on fuzzy ideals of distributive lattices, they studied the special properties of rough sets which can be constructed by means of the congruence relations determined by fuzzy ideals in distributive lattices. They also investigated the properties of the generalized rough sets with respect to fuzzy ideals in distributive lattices.

On the one hand, as for BL-algebras, MV-algebras, MTL-algebras and Heyting algebras and so on, we have observed that although they are essentially different algebras they are all particular types of residuated lattices. Therefore, it is meaningful to study the fuzzy ideals of residuated lattices for studying the common properties of fuzzy ideals in the above mentioned algebras and charactering some particular types of residuated lattices in terms of fuzzy ideals, for example, regular residuated lattices. On the other hand, as far as we know that in modern fuzzy logic theory, residuated lattices and some related algebraic systems play an extremely important role because they provide an algebraic frameworks to fuzzy logic and fuzzy reasoning. However, so far, mostly focus on filter and fuzzy filter in residuated lattices, there are few researches related to fuzzy ideals in residuated lattices, through Liu et al. [17] investigated the properties and equivalent characterizations of fuzzy ideals in residuated lattices. Therefore, as a supplement of this topic from the theoretical point of view, in this paper, we continue to study the properties of fuzzy ideals in residuated lattices. This is the motivation for us to investigate the (generalized) fuzzy ideals in residuated lattices.

This paper is organized as follows: In Section II, we present some preliminary concepts and results related to residuated lattices and fuzzy ideals. By means of the notion of tip-extended pair of fuzzy sets, in Sections III and IV, we show that the sets of all (generalized) fuzzy ideals form bounded distributive lattices. Finally, our researches are concluded in Section V.

II. Preliminaries

In this section, we recall some fundamental concepts and definitions which shall be needed in the sequel. At first, we give a brief reminder of the definitions of residuated lattices.

Definition 2.1: [3] A residuated lattice is an algebraic structure $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following axioms:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(2) $(L, \otimes, 1)$ is a commutative monid;

(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$.

In what follows, we denote by L a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, unless otherwise specified.

Definition 2.2: [33] For any $x \in L$, we define $x' = x \rightarrow 0$ and $x'' = (x')' = (x \rightarrow 0) \rightarrow 0$. L is called regular if for any $x \in L$, $x'' = x$.

In the following, we list some basic properties of (regular) residuated lattices.

Proposition 2.3: [3] For all $x, y, z, w \in L$, the following properties hold.

(1) $1 \rightarrow x = x$.

(2) $x \leq y$ if and only if $x \rightarrow y = 1$.

(3) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \geq x \rightarrow z$.

(4) If $x \leq z$ and $y \leq w$ then $x \otimes y \leq z \otimes w$.

(5) $x \rightarrow (y \rightarrow z) = x \otimes y \rightarrow z = y \rightarrow (x \rightarrow z)$.

(6) $x \leq x''$ and $x''' = x'$.

(7) $x \rightarrow y \leq y' \rightarrow x'$.

(8) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

(9) $x \otimes y = 0$ if and only if $x \leq y'$ if and only if $y \leq x'$.

Further, if L is regular, then we also have the following properties:

(10) $x' \rightarrow y' = y \rightarrow x, x' \rightarrow y = y' \rightarrow x$.

(11) $x \otimes y = (x \rightarrow y')'$.

(12) $x \rightarrow y = (x \otimes y')'$.

In order to introduce the notion of ideals in residuated lattices, in [20], Luo defined a binary operation $\oplus$ on L as follows: for any $x, y \in L$,

$$x \oplus y = x' \rightarrow y.$$

Proposition 2.4: [20] For any $x, y, z \in L$, the following statements hold.

(1) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $z \oplus x \leq z \oplus y$.

(2) $x \vee y \leq x \oplus y$.

(3) $0 \oplus x = x$ and $x \oplus 0 = x''$.

Remark 2.5: It follows from Example 1 [20] that in a general residuated lattice, the operation $\oplus$ is not associative and commutative. However, in [20], Luo showed that in any regular residuated lattice, the operation $\oplus$ is associative and commutative.

Next, we recall the notions of (fuzzy) ideals in residuated lattices.

Definition 2.6: [20] Let $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if for any $x, y \in F$,

(1) $x \leq y$ and $y \in I$, then $x \in I$,

(2) $x \oplus y \in I$.

An equivalent definition for an ideal in L as follows:

Definition 2.7: [20] Let $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if

(1) $0 \in I$,

(2) for any $x, y \in L$, if $x' \otimes y \in I$ and $x \in I$, then $y \in I$.

In what follows, we denote the set of all ideals in L by $I(L)$.

In 2017, Liu et al. [17] introduced the notion of fuzzy ideals of residuated lattices.

Definition 2.8: [17] A fuzzy set μ of L is called a fuzzy ideal if for any $x, y \in L$,

(1) $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$,

(2) $x \leq y$ implies $\mu(x) \geq \mu(y)$.

An equivalent definition for a fuzzy ideal in L as follows:

Definition 2.9: [17] A fuzzy set μ of L is called a fuzzy ideal if for any $x, y \in L$,

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(y) \geq \mu(x) \wedge \mu(x' \otimes y)$.

In what follows, we denote the set of all fuzzy ideals in L by $FI(L)$.

III. Lattices of fuzzy ideals in residuated lattices

In this section, by means of the notion of tip-extended pair of fuzzy sets, the lattices of fuzzy ideals are investigated. Firstly, we introduce the notion of tip-extended of fuzzy sets in residuated lattices.

Definition 3.1: Let μ and ν be two fuzzy sets of L. Then tip-extended pair of μ and ν of L can be defined as follows:

$$\mu^{\nu}(x) = \begin{cases} \mu(x), & \text{if } x \neq 0, \\ \mu(0) \vee \nu(0), & \text{if } x = 0, \end{cases}$$

and

$$\nu^{\mu}(x) = \begin{cases} \nu(x), & \text{if } x \neq 0, \\ \nu(0) \vee \mu(0), & \text{if } x = 0. \end{cases}$$

Example 3.2: Let $L = \{0, a, b, c, 1\}$ be a chain and operations $\otimes$ and $\rightarrow$ be defined as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
1	0	b	b	1	1
1	0	a	b	c	1

Then it is easy to verify that $L = \{0, a, b, c, 1\}$ is a residuated lattice. If we define

$$\mu = \frac{0.3}{0} + \frac{0.5}{a} + \frac{0.4}{b} + \frac{0.7}{c} + \frac{0.3}{1}$$

and

$$\nu = \frac{0.4}{0} + \frac{0.1}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.7}{1},$$

then

$$\mu^{\nu} = \frac{0.4}{0} + \frac{0.5}{a} + \frac{0.4}{b} + \frac{0.7}{c} + \frac{0.3}{1}$$

and

$$\nu^{\mu} = \frac{0.4}{0} + \frac{0.1}{a} + \frac{0.3}{b} + \frac{0.4}{c} + \frac{0.7}{1}.$$

By means of Definition 3.1, we can obtain the following result.

Theorem 3.3: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then

$$\mu^{t}(x) = \begin{cases} \mu(x), & \text{if } x \neq 0, \\ \mu(0) \vee t, & \text{if } x = 0, \end{cases}$$

is a fuzzy ideal of L.

Proof. Let $0 = x \leq y$. If $y = 0$, then $\mu^t(x) = \mu^t(y)$. If $y \neq 0$, then $\mu^t(x) = \mu^t(0) = \mu(0) \vee t$. Since μ is a fuzzy ideal of L, we have $\mu(0) \geq \mu(y)$. Thus $\mu^t(x) \geq \mu(0) \geq \mu(y) = \mu^t(y)$. If $0 \neq x \leq y$, then $\mu^t(x) = \mu(x) \geq \mu(y) = \mu^t(y)$.

Let $x, y \in L$. Then we have the following two cases.

Case I: $x \oplus y = 0$.

If one of x and y equals 0, say $x = 0$, then $y = 0 \oplus y = x \oplus y = 0$. In this subcase, it is obvious that $\mu^t(x) \wedge \mu^t(y) \leq \mu^t(x \oplus y)$.

If $x = 0, y \neq 0$ or $x \neq 0, y = 0$, it is a contradict.

If $x \neq 0, y \neq 0$, then, one has that

$$\begin{aligned} \mu^t(x) \wedge \mu^t(y) &= \mu(x) \wedge \mu(y) \\ &\leq \mu(x \oplus y) \\ &\leq \mu^t(x \oplus y). \end{aligned}$$

Case II: $x \oplus y \neq 0$.

In this case, it can not happen that $x = y = 0$.

If $x = 0, y \neq 0$, then, it obvious that $\mu^t(x) \wedge \mu^t(y) \leq \mu^t(x \oplus y)$.

If $x \neq 0, y \neq 0$, then we have

$$\begin{aligned} \mu^t(x) \wedge \mu^t(y) &= \mu(x) \wedge \mu(y) \\ &\leq \mu(x \oplus y) \\ &= \mu^t(x \oplus y). \end{aligned}$$

Therefore, μ^t is a fuzzy ideal of L. □

Definition 3.4: Let μ and ν be two fuzzy sets of L. Then the operation $\widetilde{\oplus}$ is defined as follows:

$$(\mu \widetilde{\oplus} \nu)(x) = \bigvee_{x \leq y \oplus z} (\mu(y) \wedge \nu(z))$$

for all $x \in L$.

Theorem 3.5: Let L be a regular residuated lattice and μ be a fuzzy set of L. Define a fuzzy set ν of L as follows:

$$\nu(x) = \bigvee_{x \leq x_1 \oplus \ldots \oplus x_n} (\mu(x_1) \wedge \mu(x_2) \wedge \ldots \wedge \mu(x_n))$$

for some $x_1, x_2, \ldots, x_n \in L$. Then ν is the smallest fuzzy ideal of L that contains μ.

Proof. Obviously, $\nu(0) \geq \nu(x)$ for any $x \in L$.

Next, assume that $x, y \in L$,

$$x \leq b_1 \oplus \ldots \oplus b_m,$$

for some $b_1, b_2, \ldots, b_m \in L$, and

$$x' \otimes y \leq a_1 \oplus \ldots \oplus a_n,$$

for some $a_1, a_2, \ldots, a_n \in L$.

Then

$$\begin{aligned} y &\leq x \oplus (x' \otimes y) \\ &\leq (b_1 \oplus \ldots \oplus b_m) \oplus (a_1 \oplus \ldots \oplus a_n), \end{aligned}$$

for some $b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n, \in L$.

It follows from the definition of ν that

$$\nu(y) \geq \mu(a_1) \wedge \mu(a_2) \wedge \ldots \wedge \mu(a_n) \wedge \mu(b_1) \wedge \ldots \wedge \mu(b_m).$$

Denote by

$$M = \{\mu(b_1) \wedge \mu(b_2) \wedge \ldots \wedge \mu(b_m) | x \leq b_1 \oplus \ldots \oplus b_m\},$$

for some $b_1, b_2, \ldots, b_m \in L$, and

$$N = \{\mu(a_1) \wedge \mu(a_2) \wedge \ldots \wedge \mu(a_n) | x' \otimes y \leq a_1 \ldots \oplus \ldots \oplus a_n\},$$

for some $a_1, a_2, \ldots, a_n \in L$.

Then we have

$$\begin{aligned}&\nu(x)\wedge\nu(x'\otimes y)\\&=(\bigvee M)\wedge(\bigvee N)\\&=\bigvee(M\wedge N)\\&=\bigvee\{\mu(a_1)\wedge ...\wedge\mu(a_n)\wedge\mu(b_1)\wedge ...\wedge\mu(b_m)|x'\otimes y\\&\qquad\leq a_1\oplus ...\oplus a_n, x\leq b_1\oplus ...\oplus b_m\},\end{aligned}$$

for some $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in L$.

Hence,

$$\nu(y)\geq\nu(x)\wedge\nu(x'\otimes y).$$

It follows from Definition 2.9 that ν is a fuzzy ideal of L. Further, since $x\leq x\oplus x$ for any $x\in L$, we have

$$\nu(x)\geq\mu(x)\wedge\mu(x)=\mu(x),$$

which implies that ν contains μ.
Last, we show that ν is the smallest fuzzy ideal of L.
Let η be a fuzzy ideal of L that contains μ, i.e., $\eta(x)\geq\mu(x)$ for any $x\in L$. It follows from Definition 2.8 that

$$\begin{aligned}\nu(x)&=\bigvee_{x\leq x_1\oplus...\oplus x_n}(\mu(x_1)\wedge\mu(x_2)\wedge ...\wedge\mu(x_n))\\&\leq\bigvee_{x\leq x_1\oplus...\oplus x_n}(\eta(x_1)\wedge\eta(x_2)\wedge ...\wedge\eta(x_n))\\&\leq\eta(x),\end{aligned}$$

for some $x_1, x_2, ..., x_n\in L$. Therefore, ν is the smallest fuzzy ideal of L that contains μ. □

Remark 3.6: (1) The smallest fuzzy ideal containing μ is said to be generated by μ. It is also the intersection of all fuzzy ideals of L that containing μ, denoted by $\langle\mu\rangle$.

(2) It follows from Theorem 3.5 that $\nu=\langle\mu\rangle$.

Theorem 3.7: Let L be a regular residuated lattice and μ and ν be two fuzzy ideals of L. Then

$$\mu^{\nu}\widetilde{\oplus}\nu^{\mu}=\langle\mu\vee\nu\rangle.$$

Proof. We first show that $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}$ is a fuzzy ideal.

$$\begin{aligned}&(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x\otimes y)\\&=\bigvee_{x\oplus y\leq u\oplus v}(\mu^{\nu}(u)\wedge\nu^{\mu}(v))\\&\geq\bigvee_{x\leq p\oplus q, y\leq r\oplus s}(\mu^{\nu}(p\oplus r)\wedge\nu^{\mu}(q\oplus s))\\&\geq\bigvee_{x\leq p\oplus q, y\leq r\oplus s}(\mu^{\nu}(p)\wedge\nu^{\mu}(q)\wedge\mu^{\nu}(r)\wedge\nu^{\mu}(s))\\&=\bigvee_{x\leq p\oplus q}(\mu^{\nu}(p)\wedge\nu^{\mu}(q))\wedge\bigvee_{y\leq r\oplus s}(\mu^{\nu}(r)\wedge\nu^{\mu}(s))\\&=(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x)\wedge(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(y).\end{aligned}$$

Let $x, y\in L$ and $x\leq y$. Then it is easy to see that $(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x)\geq(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(y)$.
Thus $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}$ is a fuzzy ideal of L.
Now, let $x\in L$. Then

$$\begin{aligned}(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x)&=\bigvee_{x\leq y\oplus z}(\mu^{\nu}(y)\wedge\nu^{\mu}(z))\\&\geq\mu^{\nu}(x)\wedge\nu^{\mu}(0)\\&\geq\mu(x)\wedge\mu(0)\\&=\mu(x).\end{aligned}$$

Thus $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}\geq\mu$.
In a similar way, we have $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}\geq\nu$. Hence, $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}\geq\mu\vee\nu$. Further, $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}\geq\langle\mu\vee\nu\rangle$.
Last, let $\lambda\in FI(L)$ and $\mu\vee\nu\leq\lambda$. Then we have the following two cases.

Case I: If $x=0$, then $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}(x)=\mu(0)\vee\nu(0)\leq\lambda(0)$.
Case II: If $x>0$, then

$$\begin{aligned}&(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x)\\&=\bigvee_{x\leq y\oplus z}(\mu^{\nu}(y)\wedge\nu^{\mu}(z))\\&=\bigvee_{x\leq y\oplus z, y\neq 0, z\neq 0}(\mu^{\nu}(y)\wedge\nu^{\mu}(z))\vee(\bigvee_{x\leq y}\mu(y))\vee(\bigvee_{x\leq z}\nu(z))\\&=\bigvee_{x\leq y\oplus z, y\neq 0, z\neq 0}(\mu(y)\wedge\nu(z))\vee(\bigvee_{x\leq y}\mu(y))\vee(\bigvee_{x\leq z}\nu(z))\\&\leq\bigvee_{x\leq y\oplus z, y\neq 0, z\neq 0}(\lambda(y)\wedge\lambda(z))\vee(\bigvee_{x\leq y}\lambda(y))\vee(\bigvee_{x\leq z}\lambda(z))\\&=\bigvee_{x\leq y\oplus z}(\lambda(y)\wedge\lambda(z))\\&\leq\lambda(x).\end{aligned}$$

Therefore, $\mu^{\nu}\widetilde{\oplus}\nu^{\mu}=\langle\mu\vee\nu\rangle$. □

Let L be a regular residuated lattice and $\mu,\nu\in FI(L)$, the operations on $FI(L)$ are defined by

$$\mu\sqcap\nu=\mu\wedge\nu$$

and

$$\mu\sqcup\nu=\mu^{\nu}\widetilde{\oplus}\nu^{\mu}.$$

Then we have the following theorem.

Theorem 3.8: Let L be a regular residuated lattice and μ and ν be two fuzzy ideals of L. Then $(FI(L),\sqcap,\sqcup,\emptyset, L)$ is a bounded distributive lattice.

Proof. It is easy to see that $(FI(L),\sqcap,\sqcup,\emptyset, L)$ is a bounded lattice.
Thus, we need only to prove the distributivity.
Let $\mu,\nu,\omega\in FI(L)$.
Obviously,

$$\omega\sqcap(\mu\sqcup\nu)\geq(\omega\sqcap\mu)\sqcup(\omega\sqcap\nu).$$

Thus we need only to show that

$$\omega\sqcap(\mu\sqcup\nu)\leq(\omega\sqcap\mu)\sqcup(\omega\sqcap\nu).$$

Let $x\in L$. Then we have the following two cases.
Case I: if $x=0$, then we have

$$\begin{aligned}&(\omega\sqcap(\mu\sqcup\nu))(0)\\&=\omega(0)\wedge(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(0)\\&=\omega(0)\wedge(\mu(0)\wedge\nu(0))\\&=(\omega(0)\wedge\mu(0))\wedge(\omega(0)\wedge\nu(0))\\&=(\omega\wedge\mu)(0))\wedge(\omega\wedge\nu)(0))\\&=((\omega\wedge\mu)^{\omega\wedge\nu}\widetilde{\oplus}(\omega\wedge\nu)^{\omega\wedge\mu})(0)\\&=(\omega\sqcap\mu)\sqcup(\omega\sqcap\nu)(0).\end{aligned}$$

Case II: if $x\neq 0$, then we have

$$\begin{aligned}&\omega(x)\wedge(\mu^{\nu}\widetilde{\oplus}\nu^{\mu})(x)\\&=\omega(x)\wedge\bigvee_{x\leq y\oplus z}(\mu^{\nu}(y)\wedge\nu^{\mu}(z))\\&=\bigvee_{x\leq y\oplus z}(\omega(x)\wedge\mu^{\nu}(y)\wedge\nu^{\mu}(z))\\&=\bigvee_{x\leq y\oplus z, y\neq 0, z\neq 0}(\omega(x)\wedge\mu(y)\wedge\omega(x)\wedge\nu(z))\\&\quad\vee(((\omega\wedge\mu)^{\omega\wedge\nu}(0)\wedge((\omega(x)\wedge\mu(x))\vee(\omega(x)\wedge\mu(x)))))\\&=\bigvee_{x\leq y\oplus z, y\neq 0, z\neq 0}((\omega\wedge\mu)^{\omega\wedge\nu}(y\vee x)\wedge(\omega\wedge\nu)^{\omega\wedge\mu}(z\vee x))\\&\quad\vee((\omega\wedge\mu)^{\omega\wedge\nu}(0)\wedge(\omega\wedge\mu)(y\vee x))((\omega\wedge\nu)^{\omega\wedge\mu}(0)\\&\qquad\wedge(\omega\wedge\nu)(z\vee x))\\&=\bigvee_{x\leq y\oplus z}((\omega\wedge\mu)^{\omega\wedge\nu}(y\vee x)\wedge(\omega\wedge\nu)^{\omega\wedge\mu}(z\vee x)).\end{aligned}$$

Let $y \vee x = y', z \vee x = z'$. Then we have $x \leq y' \oplus z'$. In addition,

$$\bigvee_{x \leq y \oplus z} ((\omega \wedge \mu)^{\omega \wedge \nu}(y \vee x) \wedge (\omega \wedge \nu)^{\omega \wedge \mu}(z \vee x))$$
$$= \bigvee_{x \leq y' \oplus z'} ((\omega \wedge \mu)^{\omega \wedge \nu}(y') \wedge (\omega \wedge \nu)^{\omega \wedge \mu}(z'))$$
$$= ((\omega \wedge \mu)^{\omega \wedge \nu} \widetilde{\oplus} (\omega \wedge \nu)^{\omega \wedge \mu})(x)$$
$$= (\omega \sqcap \mu) \sqcup (\omega \sqcap \nu).$$

Thus $\omega \sqcap (\mu \sqcup \nu) \leq (\omega \sqcap \mu) \sqcup (\omega \sqcap \nu)$. Therefore, $(FI(L), \sqcap, \sqcup, \emptyset, L)$ is a bounded distributive lattice. □

IV. Lattices of generalized fuzzy ideal in residuated lattices

In this section, using the notion of tip-extended pair of fuzzy sets in residuated lattices, the lattices of generalized fuzzy ideals are investigated. Firstly, we introduce the concept of generalized fuzzy ideals of residuated lattices.

Definition 4.1: Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of L is called a generalized fuzzy ideal if

(I1) $\mu(x \oplus y) \vee m \geq \mu(x) \wedge \mu(y) \wedge n$,

(I2) $x \leq y$ implies $\mu(x) \vee m \geq \mu(y) \wedge n$.

In the following parts of this paper, generalized fuzzy ideals means generalized fuzzy ideals with respect to a fixed pair (m, n). We denote the set of all generalized fuzzy ideals in L by $GFI(L)$.

Next, we present equivalent characterizations of generalized fuzzy ideals in residuated lattices.

Proposition 4.2: Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of L is called a generalized fuzzy ideal if

(I3) $\mu(0) \vee m \geq \mu(x) \wedge n$,

(I4) $\mu(y) \vee m \geq \mu(x) \wedge \mu(x' \otimes y) \wedge n$.

Proof. Let μ be a generalized fuzzy ideal of L.
Since $0 \leq x$ for any $x \in L$, it follows from Definition 4.1 that $\mu(0) \vee m \geq \mu(x) \wedge n$.
In addition, since $y \leq x \oplus (x' \otimes y)$, we have $\mu(y) \vee m \geq \mu(x \oplus (x' \otimes y)) \wedge n$.
Further,

$$\mu(y) \vee m \vee m$$
$$\geq (\mu(x \oplus (x' \otimes y)) \wedge n) \vee m$$
$$= (\mu(x \oplus (x' \otimes y) \vee m) \wedge (n \vee m)$$
$$\geq \mu(x) \wedge \mu(x' \otimes y) \wedge n \wedge n$$
$$= \mu(x) \wedge \mu(x' \otimes y) \wedge n.$$

Conversely, assume that (I3) and (I4) hold.
Let $x, y \in L$ and $x \leq y$. Then $y' \leq x'$ and $x \otimes y' \leq x \otimes x' = 0$. So $\mu(0) = \mu(y' \otimes x)$.
It follows from (I2) that $\mu(x) \vee m \geq \mu(y) \wedge \mu(y' \otimes x) \wedge n = \mu(y) \wedge \mu(0) \wedge n$.
Moreover,

$$\mu(x) \vee m \vee m$$
$$\geq ((\mu(y) \wedge n) \wedge \mu(0)) \vee m$$
$$= ((\mu(y) \wedge n) \vee m) \wedge (\mu(0) \vee m)$$
$$\geq ((\mu(y) \wedge n) \vee m) \wedge (\mu(y) \wedge n)$$
$$= \mu(y) \wedge n,$$

This implies that (I2) holds.
On the other hand, since $x' \otimes (x \oplus y) \leq y$, we have $\mu(x' \otimes (x \oplus y)) \vee m \geq \mu(y) \wedge n$.
It follows from (I2) that $\mu(x \oplus y) \vee m \geq \mu(x) \wedge \mu(x' \otimes (x \oplus y)) \wedge n$.
Further,

$$\mu(x \oplus y) \vee m \vee m$$
$$\geq (\mu(x) \wedge \mu(x' \otimes (x \oplus y)) \wedge n) \vee m$$
$$= ((\mu(x) \wedge n) \vee m) \wedge (\mu(x' \otimes (x \oplus y)) \vee m)$$
$$\geq ((\mu(x) \wedge n) \vee m) \wedge (\mu(y) \wedge n)$$
$$= (\mu(x) \wedge n) \wedge (\mu(y) \wedge n)$$
$$= \mu(x) \wedge \mu(y) \wedge n$$

This implies (I1) holds.
Therefore, μ is a generalized fuzzy ideal of L. □

Theorem 4.3: Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of L is called a generalized fuzzy ideal if and only if for all $x, y, z \in L$, $x \leq y \oplus z$ implies that $\mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n$.

Proof. Let μ be a generalized fuzzy ideal of L and $x \leq y \oplus z$. Then $\mu(x) \vee m \geq \mu(y \oplus z) \wedge n$.
Further, $\mu(x) \vee m \vee m \geq (\mu(y \oplus z) \wedge n) \vee m$, i.e.,

$$\mu(x) \vee m$$
$$\geq (\mu(y \oplus z) \vee m) \wedge (n \vee m)$$
$$\geq \mu(y) \wedge \mu(z) \wedge n \wedge n$$
$$= \mu(y) \wedge \mu(z) \wedge n.$$

Therefore, $\mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n$.
Conversely, since $x \oplus x \geq 0$, we have $\mu(0) \vee m \geq \mu(x) \wedge \mu(x) \wedge n$, i.e., $\mu(0) \vee m \geq \mu(x) \wedge n$.
On the other hand, since $y \leq x \oplus (x' \otimes y)$, we have $\mu(y) \vee m \geq \mu(x) \wedge \mu(x' \otimes y) \wedge n$.
Therefore, μ is a generalized fuzzy ideal of L. □

Corollary 4.4: Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of L is called a generalized fuzzy ideal if and only if for all $x, y_1, ...y_n \in L$, $x \leq y_1 \oplus y_2 \oplus ... \oplus y_n$ implies that $\mu(x) \vee m \geq \mu(y_1) \wedge ... \wedge \mu(y_n) \wedge n$.

Definition 4.5: Let μ be a fuzzy set of L, $m, n \in [0, 1]$ and $m < n$. Then the intersection of all generalized fuzzy ideals containing μ is called the generated generalized fuzzy ideals by μ, denoted by $\langle \mu \rangle^{(m,n)}$.

Theorem 4.6: Let L be a regular residuated lattice, μ be a fuzzy set of L, $m, n \in [0, 1]$ and $m < n$. Then

$$\langle \mu \rangle^{(m,n)}(x)$$
$$= m \vee (\bigvee_{x \leq a_1 \oplus ... \oplus a_n} (\mu(a_1) \wedge ... \wedge \mu(a_n) \wedge n)$$
$$= \bigvee_{x \leq a_1 \oplus ... \oplus a_n} ((\mu(a_1) \vee m) \wedge ... \wedge (\mu(a_n) \vee m) \wedge n)$$

for all $x \in L$.

Proof. Let $\theta(x) = m \vee (\bigvee_{x \leq a_1 \oplus ... \oplus a_n} (\mu(a_1) \wedge ... \wedge \mu(a_n) \wedge n))$.
Firstly, we show that θ is a generalized fuzzy ideal which contains μ.
Let $n = 1, a_1 = x$. Then $m \vee (n \wedge \mu(x)) \leq \theta(x)$. Hence $n \wedge \mu(x) \leq \theta(x) \vee m$.
Now it is easy to show that θ is a generalized fuzzy ideal.
Next we show that θ is the smallest generalized fuzzy ideal which contains μ.
Suppose that η is a generalized fuzzy ideal which contains

μ, i.e., $n \wedge \mu(x) \leq \eta(x) \vee m$. Then

$$\begin{aligned} & n \wedge \theta(x) \\ & = n \wedge (m \vee (\bigvee_{x \leq a_1 \oplus \ldots \oplus a_n} (\mu(a_1) \wedge \ldots \wedge \mu(a_n) \wedge n))) \\ & = m \vee (\bigvee_{x \leq a_1 \oplus \ldots \oplus a_n} (\mu(a_1) \wedge \ldots \wedge \mu(a_n) \wedge n)) \\ & \leq m \vee (\bigvee_{x \leq a_1 \oplus \ldots \oplus a_n} (\eta(x) \vee m) \wedge \ldots \wedge (\eta(x) \vee m) \wedge n)) \\ & \leq m \vee (\bigvee_{x \leq a_1 \oplus \ldots \oplus a_n} (\eta(x) \wedge \ldots \wedge \eta(x) \wedge n) \\ & \leq m \vee \eta(x). \end{aligned}$$

Therefore, θ is the smallest generalized fuzzy ideal which contains μ, i.e., $\langle \mu \rangle^{(m,n)} = \theta$. □

Definition 4.7: Let μ and ν be two fuzzy sets of L, $m, n \in [0,1]$ and $m < n$. Then the operation $\widetilde{\oplus}^{(m,n)}$ is defined by

$$(\mu \widetilde{\oplus}^{(m,n)} \nu)(x) = \bigvee_{x \leq y \oplus z} ((\mu(y) \vee m) \wedge (\nu(z) \vee m) \wedge n).$$

Remark 4.8: In Definition 4.7, the operation $\widetilde{\oplus}^{(m,n)}$ possesses the following properties:

(1) $\mu \widetilde{\oplus}^{(m,n)} \nu = m \vee \mu \widetilde{\oplus}^{(m,n)} \nu = n \wedge \mu \widetilde{\oplus}^{(m,n)} \nu$.

(2) If $x \leq y$, then $(\mu \widetilde{\oplus}^{(m,n)} \nu)(x) \geq (\mu \widetilde{\oplus}^{(m,n)} \nu)(y)$.

(3) $(\mu \widetilde{\oplus}^{(m,n)} \nu)(0) = (\mu(0) \vee \nu(0) \vee m) \wedge n$.

Theorem 4.9: Let μ be a generalized fuzzy ideal of L and $t \in [0,1]$. Then μ^t is also a generalized fuzzy ideal of L.

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 4.10: Let L be a regular residuated lattice, μ and ν be two fuzzy sets of L, $m, n \in [0,1]$ and $m < n$. Then

$$\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu} = \langle \mu \vee \nu \rangle^{(m,n)}.$$

Proof. First, we show that $\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu}$ is a generalized fuzzy ideal of L.

$$\begin{aligned} & m \vee (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x \otimes y) \\ & = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x \otimes y) \\ & = \bigvee_{x \oplus y \leq u \oplus v} ((\mu^{\nu}(u) \vee m) \wedge (\nu^{\mu}(v) \vee m) \wedge n) \\ & \geq \bigvee_{x \leq p \oplus q, y \leq r \oplus s} ((\mu^{\nu}(p \oplus r) \vee m) \\ & \qquad \wedge (\nu^{\mu}(q \oplus s) \vee m) \wedge n) \\ & \geq \bigvee_{x \leq p \oplus q, y \leq r \oplus s} ((\mu^{\nu}(p) \wedge \mu^{\nu}(p) \wedge n) \\ & \qquad \wedge (\nu^{\mu}(q) \wedge \nu^{\mu}(q)) \wedge n) \\ & = \bigvee_{x \leq p \oplus q} ((\mu^{\nu}(p) \vee m) \wedge (\nu^{\mu}(q) \vee m) \wedge n) \\ & \qquad \wedge \bigvee_{y \leq r \oplus s} ((\mu^{\nu}(r) \vee m) \wedge (\nu^{\mu}(s) \vee m) \wedge n) \\ & = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(y) \\ & = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(y) \wedge n. \end{aligned}$$

Let $x, y \in L$ and $x \leq y$. Then it is easy to see that $m \vee (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \geq n \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(y)$.

Thus, $\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu}$ is a generalized fuzzy ideal of L.

Now, let $x \in L$. Then

$$\begin{aligned} & m \vee (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \\ & = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \\ & = \bigvee_{x \leq y \oplus z} ((\mu^{\nu}(y) \vee m) \wedge (\nu^{\mu}(z) \vee m) \wedge n) \\ & \geq ((\mu^{\nu}(x) \vee m) \wedge (\nu^{\mu}(0) \vee m) \wedge n) \\ & = ((\mu^{\nu}(x) \vee m) \wedge (\mu^{\nu}(0) \vee m) \wedge n) \\ & = ((\mu^{\nu}(x) \wedge n) \vee m) \\ & \geq (\mu^{\nu}(x) \wedge n) \\ & \geq (\mu(x) \wedge n). \end{aligned}$$

Thus, $m \vee \mu^{\nu} \widetilde{\oplus} \nu^{\mu} \geq \mu \wedge n$.

In a similar way, we have $m \vee \mu^{\nu} \widetilde{\oplus} \nu^{\mu} \geq \nu \wedge n$. Hence, $\mu^{\nu} \widetilde{\oplus} \nu^{\mu} \geq \mu \vee \nu^{(m,n)}$.

Further, $\mu^{\nu} \widetilde{\oplus} \nu^{\mu} \geq \langle \mu \vee \nu \rangle^{(m,n)}$.

Last, we verify that $\mu^{\nu} \widetilde{\oplus} \nu^{\mu}$ is the smallest one. Let $\lambda \in GFI(L)$ and $n \wedge (\mu(x) \vee \nu(x)) \leq \lambda(x) \wedge n$. Then we have $n \wedge (m \vee \mu(x) \vee \nu(x)) \leq \lambda(x) \wedge n$. Consider the following two cases:

Case I: if $x = 0$, then $n \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(0) = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(0) = (\mu(0) \vee \nu(0) \vee m) \wedge n \leq \lambda(0) \vee m$.

Case II: if $x > 0$, then

$$\begin{aligned} & n \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \\ & = (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu}) \\ & = \bigvee_{x \leq y \oplus z} ((\mu^{\nu}(y) \vee m) \wedge (\nu^{\mu}(y) \vee m) \wedge n) \\ & = \bigvee_{x \leq y \oplus z, y \neq 0, z \neq 0} (\mu(y) \wedge \nu(z) \wedge n) \vee (\bigvee_{x \leq y} \mu(y) \wedge n) \\ & \qquad \vee (\bigvee_{x \leq z} \nu(z) \wedge n) \vee m \\ & \leq \bigvee_{x \leq y \oplus z, y \neq 0, z \neq 0} (\lambda(y) \wedge \lambda(z)) \vee (\bigvee_{x \leq y} \lambda(y)) \\ & \qquad \vee (\bigvee_{x \leq z} \lambda(z)) \vee m \\ & \leq m \vee \bigvee_{x \leq y \oplus z} (\lambda(y) \wedge \lambda(z) \wedge n) \\ & \leq m \vee \lambda(x). \end{aligned}$$

Thus $n \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \leq m \vee \lambda(x)$. Therefore, $\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu} = \langle \mu \vee \nu \rangle^{(m,n)}$ □

Theorem 4.11: Let L be a regular residuated lattice and μ and ν be two fuzzy ideals of L. Then $(GFI(L), \sqcap, \sqcup, \emptyset, L)$ is a bounded distributive lattice.

Proof. It is easy to see that $(GFI(L), \sqcap, \sqcup, \emptyset, L)$ is a bounded lattice. Thus we need only to prove the distributivity. Let $\mu, \nu, \omega \in GFI(L)$.

Obviously,

$$m \vee (\omega \sqcap (\mu \sqcup \nu)) \geq ((\omega \sqcap \mu) \sqcup (\omega \sqcap \nu)) \wedge n.$$

Thus, we only need to show

$$n \wedge (\omega \sqcap (\mu \sqcup \nu)) \leq ((\omega \sqcap \mu) \sqcup (\omega \sqcap \nu)) \vee m.$$

Let $x \in L$. Then we have the following two cases.

Case I: if $x = 0$, then we have

$$\begin{aligned} & n \wedge (\omega \sqcap (\mu \sqcup \nu))(0) \\ & = n \wedge (\omega(0) \wedge (\mu^{\nu} \widetilde{\oplus} \nu^{\mu})(0)) \\ & = n \wedge \omega(0) \wedge (\mu(0) \wedge \nu(0) \vee m) \wedge n \\ & = (\omega(0) \wedge \mu(0) \vee (\omega(0) \wedge \nu(0)) \vee (\omega(0) \wedge m) \wedge n \\ & \leq ((\omega \wedge \mu)(0) \vee (\omega \wedge \nu)(0) \vee m) \wedge n \\ & = ((\omega \wedge \mu)^{\omega \wedge \nu} \widetilde{\oplus}^{(m,n)} (\omega \wedge \nu)^{\omega \wedge \mu})(0) \\ & \leq (\omega \sqcap \mu) \sqcup (\omega \sqcap \nu)(0) \vee m. \end{aligned}$$

Case II: if $x \neq 0$, then we have

$$\begin{aligned}
& n \wedge \omega(x) \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \\
& = \omega(x) \wedge (\mu^{\nu} \widetilde{\oplus}^{(m,n)} \nu^{\mu})(x) \\
& = \omega(x) \wedge \bigvee_{x \leq y \oplus z} (((\mu^{\nu}(y) \vee m) \wedge (\nu^{\mu}(z) \vee m) \wedge n)) \\
& \leq \bigvee_{x \leq y \oplus z} ((((\mu^{\nu}(y) \wedge \omega(x)) \vee m) \\
& \qquad \wedge ((\nu^{\mu}(z) \wedge \omega(x)) \vee m) \wedge n)) \\
& = \bigvee_{x \leq y \oplus z, y \neq 0, z \neq 0} ((((\mu(y) \wedge \omega(x)) \vee m) \\
& \quad \wedge ((\nu(z) \wedge \omega(x)) \vee m) \wedge n)) \\
& \quad \wedge \bigvee_{x \leq y} (((\mu(y) \wedge \omega(x)) \vee m) \\
& \quad \wedge (\nu^{\mu}(0) \wedge \omega(x)) \vee m) \wedge n)) \\
& \quad \wedge \bigvee_{x \leq z} (((\nu^{\mu})(0) \wedge \omega(x)) \vee m) \\
& \quad \wedge (\nu(z) \wedge \omega(x)) \vee m) \wedge n)) \\
& \quad \bigvee_{x \leq y \oplus z, y \neq 0, z \neq 0} ((((\mu(y \vee x) \wedge \omega(y \vee x)) \vee m) \\
& \quad \wedge ((\nu(z \vee x) \wedge \omega(z \vee x)) \vee m) \wedge n)) \\
& \quad \wedge \bigvee_{x \leq y} (((\mu(y \vee x) \wedge \omega(y \vee x)) \vee m) \\
& \quad \wedge ((\nu \wedge \nu)^{\mu}(0) \vee m) \wedge n)) \\
& \quad \wedge \bigvee_{x \leq z} ((((\nu \wedge \mu)^{\mu \wedge \nu})(0) \\
& \quad \wedge \omega(x)) \vee m) \wedge (\nu(z \vee x) \wedge \omega(z \vee x)) \vee m) \wedge n)) \\
& \quad \bigvee_{x \leq y \oplus z} ((((\mu(y \vee x) \wedge \omega(y \vee x)) \vee m) \\
& \quad \wedge ((\nu(z \vee x) \wedge \omega(z \vee x)) \vee m) \wedge n)).
\end{aligned}$$

Let $y \vee x = y', z \vee x = z'$. Then we have $x \leq y' \oplus z'$. Further,

$$\begin{aligned}
& \bigvee_{x \leq y \oplus z} ((((\mu(y \vee x) \wedge \omega(y \vee x)) \vee m) \\
& \qquad \wedge ((\nu(z \vee x) \wedge \omega(z \vee x)) \vee m) \wedge n)) \\
& = \bigvee_{x \leq y' \oplus z'} (((\omega \wedge \mu)^{(\omega \wedge \nu}(y') \vee m) \\
& \qquad \wedge (\omega \wedge \nu)^{\omega \wedge \mu}(z') \vee m)) \wedge n \\
& = ((\omega \wedge \mu)^{\omega \wedge \nu} \widetilde{\oplus}^{(m,n)} (\omega \wedge \nu)^{\omega \wedge \mu})(x) \vee m \\
& = (\omega \sqcap \mu) \sqcup (\omega \sqcap \nu) \vee m.
\end{aligned}$$

Thus $n \wedge \omega \sqcap (\mu \sqcup \nu) \leq (\omega \sqcap \mu) \sqcup (\omega \sqcap \nu) \vee m$. Therefore, $(GFI(L), \sqcap, \sqcup, \emptyset, L)$ is a bounded distributive lattice. □

V. Conclusions

The research of (fuzzy) ideals in residuated lattices is an interesting topic. In this paper, by means of the notion of tip-extended pair of fuzzy sets in residuated lattices, it was proved that the sets of all (generalized) fuzzy ideals form bounded distributive lattices.

As an extension of this work, the following topics maybe considered:

(i) Investigating fuzzy prime ideals and maximal ideals in residuated lattices.

(ii) Studying MTL-L-ideals in residuated lattices.

(iii) Constructing (generalized) fuzzy ideals to other algebras, such as EQ-algebras, non-commutative residuated lattices and so on.

References

[1] V.S. Anusuya Ilamathi, J. Vimala, B. Davvaz, "Multiset filters of residuated lattices and its application in medical diagnosis", *J. Intell. Fuzzy Syst.,* vol. 36, no. 3, pp. 2297-2305, 2019.

[2] R.A. Borzooei, S.S. Khosravani, R. Ameri, "Some types of filters in MTL-algebras", *Fuzzy Sets Syst.,* vol. 187, pp. 92-102, 2012.

[3] R.P. Dilworth, M. Ward, "Residuated lattices", *Trans. Am. Math. Soc.,* vol. 45, pp. 335-354, 1939.

[4] Y.Y. Dong, X.L. Xin, "α-filters and prime α-filters in residuated lattices", *Soft Comput.,* vol. 23, no. 10, pp. 3207-3216, 2019.

[5] H. Farahani, O. Zahiri, "Algebraic view of MTL-filters", *Ann. Univ. Craiova. Math. Comput. Sci. Ser.,* vol. 40, pp. 34-44, 2013.

[6] N. Gao, Q. Li, X. Zhou, "Fuzzy extended filters on residusted lattices", *Soft Comput.,* vol. 22, pp. 2321-2328, 2018.

[7] B.V. Gasse, G. Deschrijver, C. Cornelis, E.E. Kerre, "Filters of residuated lattices and triangle algebra", *Inform. Sci.,* vol. 180, pp. 3006-3020, 2010.

[8] M. Haveshki, A.B. Saeid, E. Eslami, "Some types of filters in BL-algebras", *Soft Comput.,* vol. 10, pp. 657-664, 2006.

[9] L.C. Holdon, "On nodal and conodal ideals in residuated lattices", *J. Intell. Fuzzy Syst.,* vol. 37, no. 4, pp. 5249-5267, 2019.

[10] Y.B. Jun, Y. Xu, X.H. Zhang, "Fuzzy filters of MTL-algebras", *Inform. Sci.,* vol. 175, pp. 120-138, 2005.

[11] P.C. Kengne, B.B. Koguep, D. Akume, "L-fuzzy ideals of residuated lattices", *Discussiones Mathematicae-General Algebra and Applications,* vol. 39, no. 2, pp. 181-201, 2019.

[12] M. Kondo, "Filters on commutative residuated lattices", *Adv. Intell. Soft Comput.,* vol. 68, pp. 343-347, 2010.

[13] M. Kondo, W.A. Dudek, "Filters theory of BL-algebras", *Soft Comput.,* vol. 12, pp. 419-423, 2008.

[14] C. Lele, J.B. Nganou, "MV-algebra derived from ideals in BL-algebras", *Fuzzy Set Syst.,* vol. 218, pp. 103-113, 2013.

[15] C. Lele, J.B. Nganou, "Pseudo-addition and fuzzy ideals in BL-algebras", *Annals Fuzzy. Math. Inform.,* vol. 8, no. 2, pp. 193-207, 2014.

[16] L. Liu, K. Li, "Fuzzy filters of BL-algebras", *Inform. Sci.,* vol. 173, pp. 141-154, 2005.

[17] Y. Liu, Y. Qin, X.Y. Qin, Y. Xu, "Ideals and fuzzy ideals on residuated lattices", *Int. J. Mach. Learn. Cyber.,* vol. 8, pp. 239-253, 2017.

[18] Y. Liu, Y. Xu, X.Y. Qin, "Interval-valued I-fuzzy filters and interval-valued I-fuzzy congruences on residuated lattices", *J. Intell. Fuzzy Syst.,* vol. 26, pp. 2021-2033, 2014.

[19] Y. Liu, Y. Xu, X.Y. Qin, "Interval-valued intuitionistic (T,S)-fuzzy filters theory on residuated lattices", *Int. J. Mach. Learn. Cybern.,* vol. 5, pp. 683-696, 2014.

[20] Q. Luo, "Ideals in residuated lattices", *Quantitative Logic and Soft Comput.,* pp. 407-414, 2016.

[21] L. Liu, K. Li, "Boolean filters and positive implicative filters of residuated lattices", *Inform. Sci.,* vol. 177, pp. 5725-5738, 2007.

[22] J. Mordeson, "Fuzzy Commutative Algebra", *World. Sci., London,* 1998.

[23] A. Paad, R.A. Borzooei, "On semi-maximal filters in BL-algebras", *J. Alge. Syst.,* vol. 6, no. 2, pp. 101-116, 2019.

[24] S. Rasouli, B. Davvza, "Rough filters based on residuated lattices", *Knowl. Inf. Syst.* vol. 355, pp. 5647-5655, 2019.

[25] S.K. Shoar, "Some types of ideals in residuated lattices", *J. Intell. Fuzzy Syst.,* vol. 35, no. 5, pp. 5647-5655, 2018.

[26] E. Turunen, "Boolean deductive systems of BL-algebras", *Arch. Math. Logic,* vol. 40, pp. 467-473, 2001.

[27] Z.A. Xue, Y.H. Xiao, W.H. Liu, H.R. Cheng, Y.J. Li, "Intuitionistic fuzzy filter theory of BL-algebras", *Int. J. Mach. Learn. Cyber.,* vol. 4, pp. 659-669, 2013.

[28] Y. Yang, X. Xin, "On characterizations of BL-algebras via implicative ideals", *Ita. J. Pure Appl. Math.* vol. 37, pp. 493-506, 2017.

[29] Y. Yang, K. Zhu, X. Xin, "Rough sets based on fuzzy ideals in distributive lattices", *Open Math.* vol. 18, no. 1, pp. 122-137, 2020.

[30] X. Zhang, H. Zhou, X. Mao, "IMTL(MV)-filters of residuared lattices", *J. Intell. Fuzzy Syst.,* vol. 26, pp. 589-596, 2014.

[31] H. Zhang, Q. Li, "Intuitionistic fuzzy filters theory on residuated lattices", *Soft Comput.,* vol. 23, no. 16, pp. 6777-6783, 2019.

[32] K. Zhu, J. Wang, Y. Yang, "On two new classes of stabilizers in residuated lattices", *Soft Comput.,* vol. 23, no. 23, pp. 12209-12219, 2019.

[33] Y.Q. Zhu, Y. Xu, "On filter theory of residuated lattices", *Inform. Sci.,* vol. 180, pp. 3614-3632, 2010.