# Data Transmission Feasibility Analysis from Fractional Covered Graph Prospect

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***Abstract*—By modeling the network structure with a graph, the corresponding data transmission problem can be transformed into a fractional flow problem on the graph. Under certain specific conditions, the transmission path of data is required to pass through certain special channels, thus the corresponding mathematical framework is transformed into the existence of fractional covered graphs under the special network graph structure. In this paper, by means of the necessary and sufficient conditions for fractional $(g, f, n', m)$-critical covered graphs, we obtain some sufficient conditions on graph parameters of fractional covered graphs. Furthermore, we point out that these results are tight in terms of some counterexamples, and the results have potential guiding value for network designing.**

***Index Terms*—network, graph, fractional factor, fractional covered graph.**

## I. Introduction

THE problem of data transmission has always been the focus of computer network research (see [1],[2]). From a mathematical perspective, the stations and channels in the network are represented by vertices and edges respectively, then the entire network is a graph [3], [4]. If some channels are restricted to transmit data only one way, the corresponding graph model is a directed graph; if all channels can transmit data in both directions, the corresponding graph model is an undirected graph [5], [6]. The data transmission problem on the network can be studied by the fractional flow on the graph, that is, the existence of the fractional factor [7], [8]. At the moment of resource scheduling, there may be some changes in the actual network. For example, some sites cannot be used due to congestion or failure; some channels cannot be used due to congestion or failure; some channels can only realize unidirectional resource transmission from a certain point to a certain point. These specific situations are actually considered in network resource scheduling. Model them from the perspective of graph theory and study the sufficient conditions for resource scheduling within a certain amount of scheduling resources under a specific network graph. More contents on networks and fractional factor can be referred to [9] and [10].

All graphs considered in this paper are finite, loopless, and without multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $n = |V(G)|$, i.e., the order of graph $G$. For a vertex $x \in V(G)$, the degree and the neighborhood of $x$ in $G$ are denoted by $d_G(x)$ and $N_G(x)$, respectively. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of $G$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$, and let $G - S = G[V(G)\backslash S]$. For two disjoint subsets $S$ and $T$ of $V(G)$, we use $e_G(S, T)$ to denote the number of edges with one end in $S$ and the other in $T$. The *binding number* $\text{bind}(G)$ of a graph $G$ is defined as follows:

$$\text{bind}(G) = \min\{\frac{|N_G(X)|}{|X|}|\emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

Let $\sigma_2(G) = \min\{d_G(u) + d_G(v)\}$ for each pair of non-adjacent vertices $u$ and $v$ of $G$.

Suppose that $g$ and $f$ are two integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A *fractional $(g, f)$-factor* is a function $h$ that assigns to each edge of a graph $G$ a number in [0,1] so that for each vertex $x$ we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of $x$ in $G$. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional $(g, f)$-factor is a fractional $f$-factor. Moreover, if $g(x) = f(x) = k$ ($k \geq 1$ is an integer) for all $x \in V(G)$, then a fractional $(g, f)$-factor is just a fractional $k$-factor. The recent results on fractional factor can be found in [11] and [12].

A graph $G$ is called a *fractional $(g, f, m)$-covered graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional $(g, f)$-factor $h$ such that $h(e) = 1$ for all $e \in H$. A graph $G$ is called a *fractional $(g, f, n')$-critical graph* if after delated any $n'$ vertices from $G$, the resulting graph still has a fractional $(g, f)$-factor.

The first author of this paper first introduced the concept of a fractional $(g, f, n', m)$-critical covered graph [13]. A graph $G$ is called a *fractional $(g, f, n', m)$-critical covered graph* if after deleting any $n'$ vertices from $G$, the resulting graph is still a fractional $(g, f, m)$-covered graph. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then fractional $(g, f, m)$-covered graph, fractional $(g, f, n')$-critical graph, and fractional $(g, f, n', m)$-critical covered graph are fractional $(a, b, m)$-covered graph, fractional $(a, b, n')$-critical graph, and fractional $(a, b, n', m)$-critical covered graph, respectively. If $g(x) = f(x)$ for all $x \in V(G)$, then fractional $(g, f, m)$-covered graph, fractional $(g, f, n')$-critical graph, and fractional $(g, f, n', m)$-critical covered graph are fractional $(f, m)$-covered graph, fractional $(f, n')$-critical graph, and fractional $(f, n', m)$-critical covered graph, respectively. Furthermore, if $g(x) = f(x) = k$ ($k \geq 1$ is an integer) for all $x \in V(G)$, then fractional $(g, f, m)$-covered graph, fractional $(g, f, n')$-critical graph, and fractional $(g, f, n', m)$-critical covered graph are just fractional $(k, m)$-covered graph, fractional $(k, n')$-critical graph, and fractional $(k, n', m)$-critical covered graph, respectively.

In particular, if $m = 1$, then fractional $(g, f, n', m)$-critical covered graph, fractional $(g, f, m)$-covered graph, fractional $(a, b, n', m)$-critical covered graph, fractional $(a, b, m)$-

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covered graph, fractional $(f, n', m)$-critical covered graph, fractional $(f, m)$-covered graph, fractional $(k, n', m)$-critical covered graph and fractional $(k, m)$-covered graph are fractional $(g, f, n')$-critical covered graph, fractional $(g, f)$-covered graph, fractional $(a, b, n')$-critical covered graph, fractional $(a, b)$-covered graph, fractional $(f, n')$-critical covered graph, fractional $f$-covered graph, fractional $(k, n')$-critical covered graph and fractional $k$-covered graph respectively. Several related contributions can be referred to [14], [15], [16] and [17].

The fractional covered graph describes the feasibility of transmission when data transmission is required to pass through certain specific channels. Gao and Wang [13] determined the Necessary and sufficient condition for a graph $G$ to be fractional $(g, f, n', m)$-critical covered.

*Lemma 1:* (Gao and Wang [13]) Let $G$ be a graph, $g, f$ be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let $n', m \in \mathbb{N} \cup \{0\}$. Then $G$ is a fractional $(g, f, n', m)$-critical covered graph if and only if for any $S \subseteq V(G)$ with $|S| \geq n'$,

$$\begin{aligned} & d_{G-S}(T) - g(T) + f(S) \quad (1) \\ \geq & \max_{U \subseteq S, |U|=n', H \subseteq E(G-U), |H|=m} \{f(U) + \sum_{x \in S} d_H(x) \\ & -e_H(T, S)\}, \end{aligned}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x) - 1\}$.

The sufficient condition can be stated as follows.

*Lemma 2:* (Gao and Wang [13]) Let $G$ be a graph, $g, f$ be two non-negative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let $n', m \in \mathbb{N} \cup \{0\}$. Then $G$ is a fractional $(g, f, n', m)$-critical covered graph if

$$\begin{aligned} & d_{G-S}(T) - g(T) + f(S) \quad (2) \\ \geq & \max_{U \subseteq S, |U|=n', H \subseteq E(G-U), |H|=m} \{f(U) + \sum_{x \in S} d_H(x) \\ & -e_H(T, S)\}, \end{aligned}$$

holds for any disjoint subsets $S$ and $T$ of $V(G)$ with $|S| \geq n'$.

The aim of this work is to study the network data transmission problem in lights of researching the fractional critical covered graphs, and several sufficient conditions given from the perspective of various graph parameters.

## II. Independent set conditions for fractional critical covered graphs

### A. Main results

Our first part results mainly focus on the independent set conditions for fractional $(g, f, n', m)$-critical covered graphs. The sharpness of the bounds will be presented in Section III, and the detailed proofs will be presented in the next subsection.

*Theorem 3:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $2 \leq a \leq b - \Delta$ and $n', m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$, $n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta} + n'$, and

$$\begin{aligned} & \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \\ \geq & \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

*Theorem 4:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $2 \leq a \leq b-\Delta$ and $n', m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$, and

$$\begin{aligned} & |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ \geq & \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

Set $n' = 0$ in Theorem 3 and Theorem 4, then the above two conclusions become the independent set conditions for fractional $(g, f, m)$-covered graphs.

*Corollary 5:* Let $G$ be a graph of order $n$. Let $a, b, m, \Delta$ be four integers with $i \geq 2$, $2 \leq a \leq b-\Delta$ and $m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + m$, $n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta}$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{(b-\Delta)n + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, m)$-covered graph.

*Corollary 6:* Let $G$ be a graph of order $n$. Let $a, b, m, \Delta$ be four integers with $i \geq 2$, $2 \leq a \leq b-\Delta$ and $m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta}$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{(b-\Delta)n + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, m)$-covered graph.

Furthermore, by setting $g(x) = f(x)$, the corresponding independent set conditions for fractional $(f, n', m)$-critical covered graphs are analyzed as follows.

*Corollary 7:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $2 \leq a \leq b - \Delta$ and $n', m, \Delta \geq 0$. Let $f$ be an integer-valued functions defined on $V(G)$ such that $a \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$, $n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta} + n'$, and

$$\begin{aligned} & \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \\ \geq & \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(f, n', m)$-critical covered graph.

*Corollary 8:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $2 \leq a \leq b - \Delta$ and $n', m, \Delta \geq 0$. Let $f$ be an integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) - \Delta \le b - \Delta$ for each $x \in V(G)$. If $\delta(G) \ge \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$, and

$$\begin{aligned} & |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ \ge \quad & \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(f, n', m)$-critical covered graph.

Moreover, the corresponding conclusions on $\Delta = 0$ are manifested in the following corollaries.

*Corollary 9:* Let $G$ be a graph of order $n$, and let $a, b, n', m$, and $i$ be non-negative integers such that $i \ge 2$, $2 \le a \le b$, $n > \frac{(a+b)(2m+ib-2)}{a} + n'$ and $\delta(G) \ge \frac{b^2(i-1)}{a} + m + n'$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $G$ satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{b(n+n') + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

*Corollary 10:* Let $G$ be a graph of order $n$. Let $a, b, n', m$ be four integers with $i \ge 2$, $2 \le a \le b$ and $n', m \ge 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\delta(G) \ge \frac{b^2(i-1)}{a} + m + n'$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a} + n'$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{b(n+n') + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

By setting $n' = 0$ in Corollary 15 and 16, we get the following corollaries on fractional $(g, f, m)$-covered graphs without parameter $\Delta$.

*Corollary 11:* Let $G$ be a graph of order $n$, and let $a, b, m$, and $i$ be non-negative integers such that $i \ge 2$, $2 \le a \le b$, $n > \frac{(a+b)(2m+ib-2)}{a}$ and $\delta(G) \ge \frac{b^2(i-1)}{a} + m$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $G$ satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{bn + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, m)$-covered graph.

*Corollary 12:* Let $G$ be a graph of order $n$. Let $a, b, m$ be four integers with $i \ge 2$, $2 \le a \le b$ and $m \ge 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\delta(G) \ge \frac{b^2(i-1)}{a} + m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{bn + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, m)$-covered graph.

Set $f(x) = g(x)$ for any $x \in V(G)$, then we obtain the results on fractional $(f, n', m)$-critical covered graphs without parameter $\Delta$ from Corollary 15 and 16.

*Corollary 13:* Let $G$ be a graph of order $n$, and let $a, b, n', m$, and $i$ be non-negative integers such that $i \ge 2$, $2 \le a \le b$, $n > \frac{(a+b)(2m+ib-2)}{a} + n'$ and $\delta(G) \ge \frac{b^2(i-1)}{a} + m + n'$. Let $f$ be an integer-valued functions defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. If $G$ satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{b(n+n') + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(f, n', m)$-critical covered graph.

*Corollary 14:* Let $G$ be a graph of order $n$. Let $a, b, n', m$ be four integers with $i \ge 2$, $2 \le a \le b$ and $n', m \ge 0$. Let $f$ be an integer-valued functions defined on $V(G)$ such that $a \le f(x) \le b$ for each $x \in V(G)$. If $\delta(G) \ge \frac{b^2(i-1)}{a} + m + n'$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a} + n'$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{b(n+n') + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(f, n', m)$-covered graph.

Set $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, we yield the following results on fractional $(a, b, n', m)$-critical covered graphs.

*Corollary 15:* Let $G$ be a graph of order $n$, and let $a, b, n', m$, and $i$ be non-negative integers such that $i \ge 2$, $2 \le a \le b$, $n > \frac{(a+b)(2m+ib-2)}{b} + n'$ and $\delta(G) \ge \frac{b^2(i-1)}{b} + m + n'$. If $G$ satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{an + bn' + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(a, b, n', m)$-critical covered graph.

*Corollary 16:* Let $G$ be a graph of order $n$. Let $a, b, n', m$ be four integers with $i \ge 2$, $2 \le a \le b$ and $n', m \ge 0$. If $\delta(G) \ge \frac{b^2(i-1)}{b} + m + n'$, $n > \frac{(a+b)(i(a+b)+2m-2)}{b} + n'$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{an + bn' + 2m}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(a, b, n', m)$-critical covered graph.

### B. *Proof of Theorem 3*

Assume that $G$ satisfies the conditions of the Theorem 3, but it's not a fractional $(g, f, n', m)$-critical covered graph. By Lemma 1 and $\sum_{x\in S} d_H(x) - e_H(T, S) \le 2m$, there exist $S \subseteq V(G)$ satisfying

$$\begin{aligned} & (a+\Delta)(|S| - n') + d_{G-S}(T) - (b-\Delta)|T| \\ \le \quad & f(S-U) + d_{G-S}(T) - g(T) \le 2m - 1, \end{aligned} \tag{3}$$

where $|S| \ge n' = |U|$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le g(x) - 1\}$. Obviously, $T \ne \emptyset$ and $d_{G-S}(x) \le g(x) - 1 \le b - \Delta - 1$ for any $x \in T$.

Let $d_1 = \min\{d_{G-S}(x) | x \in T\}$ and select $x_1 \in T$ with $d_{G-S}(x_1) = d_1$. If $z \ge 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \ne \emptyset$, let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and select $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ with $d_{G-S}(x_z) = d_z$. Thus, it generates a sequence with $0 \le d_1 \le d_2 \le \cdots \le d_\pi \le g(x) - 1 \le b - \Delta - 1$ and an independent set $\{x_1, x_2, \ldots, x_\pi\} \subseteq T$.

*Claim 1:* $|T| \ge (i-1)b + 1$.

**Proof.** Suppose $|T| \leq (i-1)b$. Then $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + m$. According to (3) and $0 \leq d_1 \leq b - \Delta - 1$, we get

$$\begin{aligned}
& 2m - 1 \\
\geq\ & (a+\Delta)(|S| - n') + d_{G-S}(T) - (b-\Delta)|T| \\
\geq\ & (a+\Delta)(|S| - n') + d_1|T| - (b-\Delta)|T| \\
=\ & (a+\Delta)(|S| - n') + (d_1 + \Delta - b)|T| \\
\geq\ & (a+\Delta)(\frac{b(b-\Delta)(i-1)}{a+\Delta} - d_1 + m) \\
& +(d_1 + \Delta - b)(i-1)b \\
\geq\ & 2m.
\end{aligned}$$

This produces a contradiction. □

Since $d_{G-S}(x) \leq b - \Delta - 1$ and $|T| \geq (i-1)b + 1$, we obtain $\pi \geq i$. Hence, we can select an independent set $\{x_1, x_2, \ldots, x_i\} \subseteq T$.

In light of independent set neighborhood union condition described in the theorem, we infer

$$\begin{aligned}
& \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} \\
\leq\ & \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \\
\leq\ & |S| + d_i
\end{aligned}$$

and

$$|S| \geq \frac{(b-\Delta)n + (a+\Delta)n' + 2m}{a+b} - d_i. \tag{4}$$

Since

$$|N_T[x_j]| - |N_T[x_j] \cap (\cup_{z=1}^{j-1} N_T[x_z])| \geq 1,$$

$$j = 2, 3, \ldots, i-1$$

and

$$\begin{aligned}
& |\cup_{z=1}^{j} N_T[x_z]| \leq \sum_{z=1}^{j} |N_T[x_z]| \\
\leq\ & \sum_{z=1}^{j} (d_{G-S}(x_z) + 1) \\
=\ & \sum_{z=1}^{j} (d_z + 1), j = 1, 2, \ldots, i,
\end{aligned}$$

we deduce

$$\begin{aligned}
& f(S-U) + d_{G-S}(T) - g(T) \\
\geq\ & (a+\Delta)(|S| - n') - (b-\Delta)|T| + d_1|N_T[x_1]| + \cdots \\
& +d_2(|N_T[x_2]| - |N_T[x_2] \cap N_T[x_1]|) \\
& +d_{i-1}(|N_T[x_{i-1}]| - |N_T[x_{i-1}] \cap (\cup_{j=1}^{i-2} N_T[x_j])|) \\
& +d_i(|T| - |\cup_{j=1}^{i-1} N_T[x_j])||) \\
\geq\ & (a+\Delta)(|S| - n') + (d_1 - d_i)|N_T[x_1]| + \sum_{j=2}^{i-1} d_j \\
& +(d_i + \Delta - b)|T| - d_i \sum_{j=2}^{i-1} |N_T[x_j]| \\
=\ & (a+\Delta)(|S| - n') + (d_1 - d_i)(d_1 + 1) + \sum_{j=2}^{i-1} d_j \\
& +(d_i + \Delta - b)|T| - d_i \sum_{j=2}^{i-1} (d_j + 1) \\
=\ & (a+\Delta)(|S| - n') + d_1^2 + \sum_{j=1}^{i-1} d_j \\
& +(d_i + \Delta - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1),
\end{aligned}$$

which implies

$$\begin{aligned}
& (n - |S| - |T|)(b - \Delta - d_i) \\
\geq\ & f(S-U) + d_{G-S}(T) - g(T) - 2m + 1 \\
\geq\ & (a+\Delta)(|S| - n') + d_1^2 + \sum_{j=1}^{i-1} d_j \\
& +(d_i + \Delta - b)|T| - d_i \sum_{j=1}^{i-1} (d_j + 1) - 2m + 1.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
0 \leq\ & n(b - \Delta - d_i) - (a + b - d_i)|S| + d_i \sum_{j=1}^{i-1} d_j \\
& -\sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 + (a+\Delta)n' \\
& +2m - 1.
\end{aligned} \tag{5}$$

By means of (4), (5), $d_1 \leq d_2 \leq \cdots \leq d_i \leq b - \Delta - 1$ and

$n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta} + n'$, we have

$$\begin{aligned} & 0 \\ \leq\ & n(b-\Delta-d_i) + d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j - (a+b \\ & -d_i)(\frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - d_i) \\ & +d_i(i-1) - d_1^2 + (a+\Delta)n' + 2m - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + (a+b)d_i - d_i^2 + d_i\sum_{j=1}^{i-1} d_j \\ & -\sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 \\ & +\frac{(a+\Delta)d_i}{a+b}n' + 2md_i - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + (d_id_1 - d_1 - d_1^2) \\ & +(d_i-1)\sum_{j=2}^{i-1} d_j + d_i(a+b+i-1) - d_i^2 \\ & +\frac{(a+\Delta)d_i}{a+b}n' + 2md_i - 1 \\ \leq\ & -\frac{(a+\Delta)n}{a+b}d_i + (d_i\frac{d_i-1}{2} - \frac{d_i-1}{2} - (\frac{d_i-1}{2})^2) \\ & +(d_i-1)\sum_{j=2}^{i-1} d_i + d_i(a+b+i-1) - d_i^2 \\ & +\frac{(a+\Delta)d_i}{a+b}n' + 2md_i - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{1}{2}) \\ & -\frac{3}{4} + \frac{(a+\Delta)d_i}{a+b}n' + 2md_i \\ <\ & 0 \end{aligned}$$

since $n > \frac{(a+b)(a+b+2m+(a+\Delta)n'-1+(i-2)(b-\Delta))}{a+\Delta}$, a contradiction.

Therefore, the desired theorem is proved. □

### *C. Proof of Theorem 4*

On the contrary, assume that $G$ satisfies the conditions of the Theorem 4, but it's not a fractional $(g,f,n',m)$-critical covered graph. By Lemma 1 and $\sum_{x\in S} d_H(x) - e_H(T,S) \leq 2m$, there exist disjoint subset $S \subseteq V(G)$ satisfying

$$\begin{aligned} & (a+\Delta)(|S|-n') + d_{G-S}(T) - (b-\Delta)|T| \\ \leq\ & f(S-U) + d_{G-S}(T) - g(T) \leq 2m-1, \end{aligned} \tag{6}$$

where $|S| \geq n'$ and $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) \leq g(x)-1\}$. Obviously, $T \neq \emptyset$ and $d_{G-S}(x) \leq g(x)-1 \leq b-\Delta-1$ for any $x \in T$.

Let $d_1 = \min\{d_{G-S}(x)|x \in T\}$ and select $x_1 \in T$ with $d_{G-S}(x_1) = d_1$. If $z \geq 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$, let

$$d_z = \min\{d_{G-S}(x)|x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and select $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ with $d_{G-S}(x_z) = d_z$. Thus, it generates a sequence with $0 \leq d_1 \leq d_2 \leq \cdots \leq d_\pi \leq g(x)-1 \leq b-\Delta-1$ and an independent set $\{x_1, x_2, \ldots, x_\pi\} \subseteq T$. Using the trick as depicted in the proofing of Theorem 3, we infer $|T| \geq (i-1)b+1$. Since $d_{G-S}(x) \leq b-\Delta-1$ and $|T| \geq (i-1)b+1$, we obtain $\pi \geq i$. Hence, we can select an independent set $\{x_1, x_2, \ldots, x_i\} \subseteq T$.

In light of independent set neighborhood union condition described in the theorem, we infer

$$\begin{aligned} & \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} \\ \leq\ & |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ \leq\ & |S| + \sum_{j=1}^{i} d_j \end{aligned}$$

and

$$|S| \geq \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - \sum_{j=1}^{i} d_j. \tag{7}$$

Using the same fashion, we have

$$\begin{aligned} 0 \leq\ & n(b-\Delta-d_i) - (a+b-d_i)|S| \\ & +d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i-1) \\ & -d_1^2 + (a+\Delta)n' + 2m - 1. \end{aligned} \tag{8}$$

By means of (7), (8), $d_1 \leq d_2 \leq \cdots \leq d_i \leq b-\Delta-1$ and $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$, we have

$$\begin{aligned} & 0 \\ \leq\ & n(b-\Delta-d_i) + d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j - (a+b \\ & -d_i)(\frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - \sum_{j=1}^{i} d_j) \\ & +d_i(i-1) - d_1^2 + (a+\Delta)n' + 2m - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + (a+b)\sum_{j=1}^{i} d_j - d_i\sum_{j=1}^{i} d_j \\ & +d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 \\ & +d_i\frac{(a+\Delta)n'}{a+b} + 2md_i - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + ((a+b-1)d_1 - d_1^2) \\ & +(a+b-1)\sum_{j=2}^{i-1} d_j + d_i(a+b+i-1) \\ & -d_i^2 + d_i\frac{(a+\Delta)n'}{a+b} + 2md_i - 1 \\ \leq\ & -\frac{(a+\Delta)n}{a+b}d_i + (a+b-1)d_i \\ & +(a+b-1)\sum_{j=2}^{i-1} d_i + d_i(a+b+i-1) \\ & -d_i^2 + d_i\frac{(a+\Delta)n'}{a+b} + 2md_i - 1 \\ =\ & -\frac{(a+\Delta)n}{a+b}d_i + i(a+b)d_i - d_i^2 \\ & +d_i\frac{(a+\Delta)n'}{a+b} + 2md_i - 1 \\ <\ & 0, \end{aligned}$$

since $n > \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta} + n'$, a contradiction.

Therefore, the desired theorem is proved. □

## III. SHARPNESS

The aim of this section is to present that the independent set results in Theorem 3-4 are tight when $m$ is small, i.e., $m < \frac{a+b}{2}$.

Theorem 3 and Theorem 4 are best possible, in some extent, on the conditions. Actually, we can construct some graphs such that the independent set degree condition in Theorem 3 can't be replaced by $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - 1$, and the independent set neighborhood union condition in Theorem 4 can't be replaced by $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - 1$.

Let $b = a + \Delta$, $G_1 = K_{iat+n'}$ be a complete graph, $G_2 = ibtK_1$ be a graph consisting of $ibt+1$ isolated vertices, and $G = G_1 \vee G_2$, where $t$ is sufficiently large. Then $n = |G_1| + |G_2| = i(a+b)t + n'$, and for any independent set $\{x_1, x_2, \ldots, x_i\} \subseteq V(G_2)$, we get

$$\begin{aligned} & \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} \\ > \;& \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \\ = \;& iat+n' > \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - 1, \end{aligned}$$

$$\begin{aligned} & \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} \\ > \;& |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ = \;& iat+n' > \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} - 1. \end{aligned}$$

Let $S = V(G_1)$, $T = V(G_2)$, $g(x) = a$ and $f(x) = b = a+\Delta$ for any $x \in V(G)$. Then $f(S\backslash U) - g(T) + d_{G-S}(T) - (\sum_{x\in S} d_H(x) - e_H(T,S)) = biat - aibt - 2m = -2m < 0$ for any $U \subseteq S$ and $|U| = n'$. Hence, $G$ is not a fractional $(g, f, n', m)$-critical covered graph according to Lemma 1.

### A. More remarks

If we allow $a = 1$ in theorems, it is found that the minimal degree condition should be strengthened in order to meet the same independent set conditions. Specifically, using the proofing processers, we can obtain the following result.

*Theorem 17:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $1 \leq a \leq b - \Delta$ and $n', m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + 2m$, $n > \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta} + n'$, and

$$\begin{aligned} & \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \\ \geq \;& \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

*Theorem 18:* Let $G$ be a graph of order $n$. Let $a, b, n', m, \Delta$ be five integers with $i \geq 2$, $1 \leq a \leq b - \Delta$ and $n', m, \Delta \geq 0$. Let $g, f$ be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b(b-\Delta)(i-1)}{a+\Delta} + n' + 2m$, $n > \frac{(a+b)(i(a+b)+2m-2)+(a+\Delta)n'}{a+\Delta}$, and

$$\begin{aligned} & |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ \geq \;& \frac{(b-\Delta)n+(a+\Delta)n'+2m}{a+b} \end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(g, f, n', m)$-critical covered graph.

Like discussed before, we can get various corollaries corresponding to different settings, and we didn't list here one by one. Again, using the counterexample presented in Subsection III, we can checked that the bound independent set conditions manifested above sharp as well.

## IV. OTHER DEGREE CONDITIONS FOR FRACTIONAL $(g, f, n', m)$-CRITICAL COVERED GRAPHS

In this section, we present several degree conditions for fractional $(g, f, n', m)$-critical covered graphs. The proof of these results are similar as these presented in Gao et al. [18], and thus we didn't give the detailed proofs.

*Theorem 19:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b - \Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a} + n'$. Functions $g, f$ are integer-valued on its vertex set and $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex $x$. Then $G$ is fractional $(g, f, n', m)$-critical covered if $\delta(G) \geq \frac{(b-\Delta)n+(\Delta+a)n'+2m}{b+a}$.

*Theorem 20:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq m + n' + \frac{b(b-\Delta)}{\Delta+a}$ and $n > \frac{(b+a+2m-1)(a+b)}{\Delta+a} + n'$. Functions $g, f$ as integer-valued on its vertex set and meet $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, n', m)$-critical covered if for any $xy \neq E(G)$, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n+(\Delta+a)n'+2m}{b+a}.$$

*Theorem 21:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m + n'$ and $n > \frac{(b+a+2m-2)(a+b)}{\Delta+a} + n'$. Functions $g, f$ are integer-valued defined on the vertex set so that $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, n', m)$-critical covered if $\sigma_2(G) \geq \frac{2(n(b-\Delta)+n'(\Delta+a))+2m}{b+a}$.

Set $f(x) = g(x)$ for all $x \in V(G)$, then from above three theorems, we get the following conclusions on fractional $(f, n', m)$-critical covered graphs.

*Corollary 22:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b - \Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a} + n'$. Functions $f$ is integer-valued on its vertex set and $b - \Delta \geq f(x) \geq a$ for every vertex $x$. Then $G$ is fractional $(f, n', m)$-critical covered if $\delta(G) \geq \frac{(b-\Delta)n+(\Delta+a)n'+2m}{b+a}$.

*Corollary 23:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq m + n' + \frac{b(b-\Delta)}{\Delta+a}$ and $n > \frac{(b+a+2m-1)(a+b)}{\Delta+a} + n'$. Functions $f$ is integer-valued on its vertex set and meet $b - \Delta \geq f(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(f, n', m)$-critical covered if for any $xy \neq E(G)$, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n+(\Delta+a)n'+2m}{b+a}.$$

*Corollary 24:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m + n'$ and $n > \frac{(b+a+2m-2)(a+b)}{\Delta+a} + n'$. Functions $f$ is integer-valued defined on the vertex set so that $b-\Delta \geq f(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(f, n', m)$-critical covered if $\sigma_2(G) \geq \frac{2(n(b-\Delta)+n'(\Delta+a))+2m}{b+a}$.

Also, the results for fractional $(g, f, m)$-covered graphs can be deduced by setting in above three theorems.

*Corollary 25:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a}$. Functions $g, f$ are integer-valued on its vertex set and $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex $x$. Then $G$ is fractional $(g, f, m)$-covered if $\delta(G) \geq \frac{(b-\Delta)n+2m}{b+a}$.

*Corollary 26:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq m + \frac{b(b-\Delta)}{\Delta+a}$ and $n > \frac{(b+a+2m-1)(a+b)}{\Delta+a}$. Functions $g, f$ as integer-valued on its vertex set and meet $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, m)$-covered if for any $xy \neq E(G)$, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n+2m}{b+a}.$$

*Corollary 27:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$ and $n > \frac{(b+a+2m-2)(a+b)}{\Delta+a}$. Functions $g, f$ are integer-valued defined on the vertex set so that $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, m)$-covered if $\sigma_2(G) \geq \frac{2n(b-\Delta)+4m}{b+a}$.

In particular, we have the following conclusion for a complete graph.

*Theorem 28:* Assume $G$ is a complete graph with $n$ vertices, and $b, \Delta, a, n', m$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{a+\Delta} + n'$. Functions $g, f$ are integer-valued on its vertex set satisfy $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, n', m)$-critical covered.

We have the following corollaries on fractional $(f, n', m)$-critical covered complete graph and fractional $(g, f, m)$-covered complete graph.

*Corollary 29:* Assume $G$ is a complete graph with $n$ vertices, and $b, \Delta, a, n', m$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{a+\Delta} + n'$. Functions $f$ is integer-valued on its vertex set satisfy $b-\Delta \geq f(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(f, n', m)$-critical covered.

*Corollary 30:* Assume $G$ is a complete graph with $n$ vertices, and $b, \Delta, a, m$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{a+\Delta}$. Functions $g, f$ are integer-valued on its vertex set satisfy $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex $x$ in $G$. Then $G$ is fractional $(g, f, m)$-covered.

### A. Specific case in setting $(g, f) = (a, b)$

We infer a likely conclusion for a graph without non-adjacent vertices.

*Theorem 31:* Assume $G$ is a complete graph having $n$ vertices, and $b, n', a, m, \Delta$ are non-negative integers meeting $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$ where $b-\Delta \geq a \geq 2$. Then $G$ is fractional $(a, b, n', m)$-critical covered.

We arrive the corollary below by setting $n' = 0$ in above theorem, which is a sufficient condition for a fractional $(a, b, m)$-covered complete graph.

*Corollary 32:* Assume complete graph $G$ having $n$ vertices, and $b, a, m, \Delta$ are non-negative integers meeting $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$ where $b-\Delta \geq a \geq 2$. Then $G$ is fractional $(a, b, m)$-covered.

*Theorem 33:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$. Then $G$ is fractional $(a, b, n', m)$-critical covered if $\delta(G) \geq \frac{(a+\Delta)n'+(b-\Delta)n+2m}{b+a}$.

*Theorem 34:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m + n'$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$. Then $G$ is fractional $(a, b, n', m)$-critical covered if for any $xy \neq E(G)$, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{(\Delta+a)n'+(b-\Delta)n+2m}{b+a}.$$

*Theorem 35:* Assume $G$ is a graph with $n$ vertices, and set $b, a, n', m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m + n'$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$. Then $G$ is fractional $(a, b, n', m)$-critical covered if $\sigma_2(G) \geq \frac{2(n'(\Delta+a)+n(b-\Delta))+4m}{b+a}$.

Setting $n' = 0$ and we get the following corollaries on fractional $(a, b, m)$-covered graphs from above three theorems.

*Corollary 36:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$. Then $G$ is fractional $(a, b, m)$-covered if $\delta(G) \geq \frac{(b-\Delta)n+2m}{b+a}$.

*Corollary 37:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$. Then $G$ is fractional $(a, b, m)$-covered if for any $xy \neq E(G)$, we have

$$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n+2m}{b+a}.$$

*Corollary 38:* Assume $G$ is a graph with $n$ vertices, and set $b, a, m$, and $\Delta$ as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$. Then $G$ is fractional $(a, b, m)$-covered if $\sigma_2(G) \geq \frac{2n(b-\Delta)+4m}{b+a}$.

It's not hard to check that all these degree bounds are tight if $m < \frac{a+b}{2}$.

## V. Results on fractional $(k, n', m)$-critical covered graphs

Fractional $k$-factor is a special case of fractional $(g, f)$-factor when $g(x) = f(x) = k$ for each $x \in V(G)$. Using the trick similar as presented above, we obtain the following conclusions.

*Theorem 39:* Let $k \geq 2$ and $n', m \geq 0$ be three integers, and let $G$ be a graph with $n \geq 8k + n' + 4m - 7$ and $\delta(G) \geq k + n' + m$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{n+n'}{2} + \frac{m}{k}$$

for each pair of non-adjacent vertices $x, y$ of $G$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

Set $n' = 0$ in Theorem 39, then it becomes the following necessary condition on fractional $(k, m)$-covered graph.

*Corollary 40:* Let $k \geq 2$ and $m \geq 0$ be two integers, and let $G$ be a graph with $n \geq 8k + 4m - 7$ and $\delta(G) \geq k + m$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2} + \frac{m}{k}$$

for each pair of non-adjacent vertices $x, y$ of $G$, then $G$ is a fractional $(k, m)$-covered graph.

*Theorem 41:* Let $G$ be a graph of order $n$. Let $k, i, n', m$ be four integers with $i \geq 2$, $k \geq 2$ and $n', m \geq 0$. If $\delta(G) \geq k(i-1) + n' + m$, $n > 2k(i-2) + 4k + n' + 4m - 2$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n+n'}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

*Theorem 42:* Let $G$ be a graph of order $n$. Let $k, i, n', m$ be four integers with $i \geq 2$, $k \geq 2$ and $n', m \geq 0$. If $\delta(G) \geq k(i-1) + n' + m$, $n > 4ki + n' + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n+n'}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

Set $n' = 0$ in Theorem 41 and 42, then it becomes the following necessary condition on fractional $(k, m)$-covered graph.

*Corollary 43:* Let $G$ be a graph of order $n$. Let $k, i, m$ be three integers with $i \geq 2$, $k \geq 2$ and $m \geq 0$. If $\delta(G) \geq k(i-1) + m$, $n > 2k(i-2) + 4k + 4m - 2$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, m)$-covered graph.

*Corollary 44:* Let $G$ be a graph of order $n$. Let $k, i, m$ be four integers with $i \geq 2$, $k \geq 2$ and $m \geq 0$. If $\delta(G) \geq k(i-1) + m$, $n > 4ki + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, m)$-covered graph.

If we allow $k = 1$ in Theorem 41 and 42, then the minimum degree condition should be more stronger. We state the revised results as follows.

*Theorem 45:* Let $G$ be a graph of order $n$. Let $k, i, n', m$ be four integers with $i \geq 2$, $k \geq 1$ and $n', m \geq 0$. If $\delta(G) \geq k(i-1) + n' + 2m$, $n > 2k(i-2) + 4k + n' + 4m - 2$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n+n'}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

*Theorem 46:* Let $G$ be a graph of order $n$. Let $k, i, n', m$ be four integers with $i \geq 2$, $k \geq 1$ and $n', m \geq 0$. If $\delta(G) \geq k(i-1) + n' + 2m$, $n > 4ki + n' + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n+n'}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

Set $n' = 0$ in Theorem 45 and 46, then it becomes the following necessary condition on fractional $(k, m)$-covered graph.

*Corollary 47:* Let $G$ be a graph of order $n$. Let $k, i, m$ be three integers with $i \geq 2$, $k \geq 1$ and $m \geq 0$. If $\delta(G) \geq k(i-1) + 2m$, $n > 2k(i-2) + 4k + 4m - 2$, and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, m)$-covered graph.

*Corollary 48:* Let $G$ be a graph of order $n$. Let $k, i, m$ be four integers with $i \geq 2$, $k \geq 1$ and $m \geq 0$. If $\delta(G) \geq k(i-1) + 2m$, $n > 4ki + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n}{2} + \frac{m}{k}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then $G$ is a fractional $(k, m)$-covered graph.

*Theorem 49:* Let $k \geq 2$, $n' \geq 0$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m + n' - 5$. If $\delta(G) \geq \frac{n+n'}{2} + \frac{m}{k}$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

*Theorem 50:* Let $k \geq 2$, $n' \geq 0$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m + n' - 3$, $\delta(G) \geq k + n' + m$. If

$$\max\{d_G(u), d_G(v)\} \geq \frac{n+n'}{2} + \frac{m}{k}$$

for each pair of non-adjacent vertices $u$ and $v$ in $G$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

*Theorem 51:* Let $k \geq 2$, $n' \geq 0$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m + n' - 5$, $\delta(G) \geq k + n' + m$. If $\sigma_2(G) \geq n + n' + \frac{2m}{k}$, then $G$ is a fractional $(k, n', m)$-critical covered graph.

Set $n' = 0$ in Theorem 49, 50 and 51, then it becomes the following necessary condition on fractional $(k, m)$-covered graph.

*Corollary 52:* Let $k \geq 2$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m - 5$. If $\delta(G) \geq \frac{n}{2} + \frac{m}{k}$, then $G$ is a fractional $(k, m)$-covered graph.

*Corollary 53:* Let $k \geq 2$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m - 3$, $\delta(G) \geq k + m$. If

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2} + \frac{m}{k}$$

for each pair of non-adjacent vertices $u$ and $v$ in $G$, then $G$ is a fractional $(k, m)$-covered graph.

*Corollary 54:* Let $k \geq 2$ and $m \geq 0$ be three integers. Let $G$ be a graph of order $n$ with $n \geq 4k + 4m - 5$, $\delta(G) \geq k + m$. If $\sigma_2(G) \geq n + \frac{2m}{k}$, then $G$ is a fractional $(k, m)$-covered graph.

It is valuable to point out that all these results are tight for $m < k$ via some counterexamples.

## VI. Conclusion and Discussion

At the same time, the scheduling of network resources is related to the structure of the network graph and the capacity of channel transmission resources in the network. Calculating the feasibility of the entire network resource scheduling at the same time from the perspective of modern graph theory and obtaining sufficient conditions for resource scheduling within a certain amount of scheduling resources can provide a theoretical basis for network design and actual scheduling algorithm design.

In this paper, we mainly obtain several sufficient conditions for fractional critical covered graphs in different settings in terms of the necessary and sufficient condition which are presented in Lemma 1. Furthermore, we show that these bounds are sharp show if $m$ is small enough. Unfortunately, we didn't know what is the tight bounds of these graph parameters for fractional critical covered graphs if $m$ is a large number. Hence, we raise the following open problem as the end of this article.

*Problem 1:* What are the sharp bounds of various degree conditions of fractional critical covered graphs for a general $m$?

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