Positive Solutions for Fractional Nonlocal Boundary Value Problems with Dependence on the First Order Derivative

Dehong Ji, Yitao Yang

***Abstract*—This research work is dedicated to an investigation of the existence results for a class of fractional nonlocal boundary value problems of the type**

$$D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0,\quad 0<t<1,\quad 2<\alpha\leq 3,$$

$$u(0)=u'(0)=0,\qquad D_{0+}^{\beta}u(1)=\int_0^{\eta}a(t)D_{0+}^{\gamma}u(t)dt,$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. A full analysis of existence of positive solutions is proved by using the monotone iterative technique. The interesting point is the nonlinear term f is involved with the first order derivative explicitly. The case $f=f(t,u)$ existence results are proved via Schauder and a classical Krasnosel'skii fixed point theorems.

***Index Terms*—Positive solution; Boundary value problem; Fractional differential equation; Fixed point theorem.**

I. Introduction

In this paper, we consider the existence results of positive solutions to the fractional nonlocal boundary value problems

$$D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0,\quad 0<t<1,\quad 2<\alpha\leq 3, \tag{1}$$

$$u(0)=u'(0)=0,\quad D_{0+}^{\beta}u(1)=\int_0^{\eta}a(t)D_{0+}^{\gamma}u(t)dt, \tag{2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $0<\beta<1, 0\leq\gamma<\alpha-1, \eta\in(0,1), f\in C([0,1]\times R^{+}\times R, R^{+}), a(t)\in L^{1}[0,1]\cap C(0,1)$ is nonnegative.

The study of differentiation and integration to a fractional order has caught importance and popularity among researchers compared to classical differentiation and integration. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property [1-3]. Existence of solutions is the basis of the theory of fractional differential equation. Most of the previous literature deals with the existence of solutions for fractional differential equations boundary value problems by the use of techniques of nonlinear analysis, see [6-36] and the references therein.

For example, in [17], Henderson and Luca considered the existence of positive solutions for the following fractional differential equation boundary value problems

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad n-1<\alpha\leq n,$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,$$

$$D_{0+}^{p}u(1)=\sum_{i=1}^{m}\eta_i D_{0+}^{q}u(\xi_i),$$

where $p\in[1,n-2], q\in[0,p]$.

See also [33] where, the authors studied the following fractional differential equation with infinite-point boundary value conditions

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad n-1<\alpha\leq n,$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,$$

$$D_{0+}^{\beta}u(1)=\sum_{i=1}^{\infty}\eta_i D_{0+}^{\beta}u(\xi_i).$$

By using the fixed point index theory in cones, Wang et al. [34] established the existence and multiplicity results of positive solutions for the following fractional boundary value problems

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad n-1<\alpha\leq n,$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ \ u(1)=\int_0^1 u(t)dv(t).$$

When $1\leq\beta<\alpha-1$, Zhang and Zhong [35] investigated the existence of triple positive solutions for the fractional boundary value problem

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad n-1<\alpha\leq n,$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,$$

$$D_{0+}^{\beta}u(1)=\int_0^{\eta}a(t)D_{0+}^{\gamma}u(t)dt,$$

by using the Leggett-Williams and Krasnosel'skii fixed point theorems.

Recently, [32] presented the existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad n-1<\alpha\leq n,$$

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,$$

$$D_{0+}^{\beta}u(1)=\int_0^{\eta}a(t)D_{0+}^{\gamma}u(t)dv(t).$$

All the above work was done under the assumption that f is allowed to depend just on u, while the first order derivative u' is not involved explicitly in the nonlinear term f. As we know, when the nonlinear term f is involved in the first-order derivative, difficulties arise immediately. In this work, we use the monotone iterative technique to overcome these difficulties. To the best knowledge of the authors, no work

This work is supported by the Natural Science Foundation of Tianjin (No.(19JCYBJC30700)).

Dehong Ji and Yitao Yang are with the College of Science, Tianjin University of Technology, Tianjin 300384, China (E-mail: jdh200298@163.com).

has been done for boundary value problem (1), (2) by use of the monotone iterative technique. The aim of this work is to fill the gap in the literature.

The paper is organized as follows. In section 2, we give some necessary concepts and results. Section 3 is devoted to study two existence results when $f = f(t, u)$. The first one uses the Schauder fixed point theorem, while in the second one, existence result is obtained via the classical Krasnosel'skii fixed point theorem. In section 4, the existence result when the nonlinearity f depends on the solution and its first derivative is established by using the monotone iterative technique.

In this paper, $E := C[0, 1]$ denotes the Banach space of all continuous functions on $[0, 1]$ with the norm $\|u\|_0 = \max\{|u(x)|, 0 \le x \le 1\}$ and $E^1 := C'[0, 1]$ will refer to the Banach space of continuously differentiable functions on $[0, 1]$ equipped with the norm $\|u\| = \max(\|u\|_0, \|u'\|_0)$.

II. The preliminary lemmas

To reformulate the problem (1), (2) into a fixed point theorem, we present some necessary definitions and lemmas from conformable fractional calculus theory in this section.

Definition 2.1 [32] The fractional integral of order $\alpha > 0$ for function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s) ds$$

provided that the right hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 [32] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of real number α, provided that the right hand side is point-wise defined on $(0, +\infty)$.

Lemma 2.1 *[32]* Let $\alpha > 0$, then the following equality holds for $u \in L(0, 1), D_{0+}^{\alpha} u \in L(0, 1)$

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

where $c_i \in R, \quad i = 1, 2, \cdots, n, n - 1 < \alpha \le n$.

Lemma 2.2 *[32]* Assume that $g \in L(0, 1)$ and $\alpha > \beta \ge 0$. Then

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s) ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s) ds.$$

Lemma 2.3 *[32]* Assume that $a \in L^1[0, 1] \cap C(0, 1)$,

$$\Delta := \Gamma(\alpha - \gamma) - \Gamma(\alpha - \beta) \int_0^{\eta} a(t) t^{\alpha-\gamma-1} dt \neq 0.$$

Then for any $y \in L[0, 1] \cap C(0, 1)$, the unique solution of the boundary value problem

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \qquad 0 < t < 1, \qquad 2 < \alpha \le 3, \tag{3}$$

$$u(0) = u'(0) = 0, \qquad D_{0+}^{\beta} u(1) = \int_0^{\eta} a(t) D_{0+}^{\gamma} u(t) dt, \tag{4}$$

is

$$u(t) = \int_0^1 G(t, s) y(s) ds, \tag{5}$$

where

$$G(t, s) = G_1(t, s) + h(s) t^{\alpha-1}, \tag{6}$$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1 - s)^{\alpha-\beta-1} & 0 \le t \le s \le 1, \\ t^{\alpha-1}(1 - s)^{\alpha-\beta-1} - (t - s)^{\alpha-1} & 0 \le s \le t \le 1. \end{cases} \tag{7}$$

$$G_2(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-\gamma-1}(1 - s)^{\alpha-\beta-1} & 0 \le t \le s \le 1, \\ t^{\alpha-\gamma-1}(1 - s)^{\alpha-\beta-1} - (t - s)^{\alpha-\gamma-1} & 0 \le s \le t \le 1. \end{cases} \tag{8}$$

$$h(s) = \frac{\Gamma(\alpha - \gamma)}{\Delta} \int_0^{\eta} a(t) G_2(t, s) dt. \tag{9}$$

We make the following assumptions throughout this paper:

(A_1) $a(t) \in L^1[0, 1] \cap C(0, 1)$;

(A_2) $\Delta := \Gamma(\alpha - \gamma) - \Gamma(\alpha - \beta) \int_0^{\eta} a(t) t^{\alpha-\gamma-1} dt \neq 0$ and $h(s) \ge 0$ for $s \in [0, 1]$.

Lemma 2.4 The function $G_1(t, s)$ defined by (7) satisfies the following properties.

1. $G_1(t, s) > 0$ for all $t, s \in (0, 1)$,
2. $\Gamma(\alpha) G_1(t, s) \le t^{\alpha-1}(1 - s)^{\alpha-\beta-1}$ for all $t, s \in [0, 1]$,
3. $\beta s(1 - s)^{\alpha-\beta-1} t^{\alpha-1} \le \Gamma(\alpha) G_1(t, s) \le s(1 - s)^{\alpha-\beta-1}$ for all $t, s \in [0, 1]$.

Lemma 2.5 The Green function $G(t, s)$ defined by (6) satisfies the following properties.

1. $G(t, s) > 0$ for all $t, s \in (0, 1)$,
2. $G(t, s) \le t^{\alpha-1} \Phi_1(s)$ for all $t, s \in [0, 1]$,
3. $\beta t^{\alpha-1} \Phi_2(s) \le G(t, s) \le \Phi_2(s)$ for all $t, s \in [0, 1]$,where

$$\Phi_1(s) = \frac{(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s),$$

$$\Phi_2(s) = \frac{s(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s).$$

Proof: It can be directly deduced from Lemma 2.4 and the definition of $G(t, s)$, so we omit the proof. ■

Remark 2.1 The function $u \in E^1$ is a solution of the boundary value problem (1), (2) if and only if it is a solution of the operator equation $u = Tu$, where

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s), u'(s)) ds. \tag{10}$$

III. The case $f = f(t, u)$

First, notice that the function $u \in E$ is a solution of the boundary value problem (1), (2) with $f = f(t, u(t))$ if and only if it is a solution of the operator equation $u = Lu$, where

$$Lu(t) = \int_0^1 G(t, s) f(s, u(s)) ds. \tag{11}$$

Using the Arzela-Ascoli theorem, it is easy to prove the following lemma.

Lemma 3.1 The operator $L : E \to E$ is completely continuous.

Theorem 3.2 [4] Let X be a Banach space and $C \subset X$ a bounded, closed convex subset of X. If $T : C \to C$ is a completely continuous operator, then T has a fixed point in C.

Theorem 3.3 [5] Let X be a Banach space, $K \subset X$ a cone and Ω_1, Ω_2 two bounded open subsets satisfying $0 \in$

$\Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Let $T: K\cap(\overline{\Omega_2}\backslash\Omega_1) \to K$ be a completely continuous operator, such that:
(a) either $\|Tv\| \le \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \ge \|v\|$ for $v \in K \cap \partial\Omega_2$,
(b) or $\|Tv\| \ge \|v\|$ for $v \in K \cap \partial\Omega_1$ and $\|Tv\| \le \|v\|$ for $v \in K \cap \partial\Omega_2$.
Then T has at least a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3.1 An existence result by the Schauder fixed point theorem

Theorem 3.4 Suppose that
(1) $f(t,\cdot)$ is nondecreasing on R^+, for all $t \in [0,1]$;
(2) Assume that there exists $R > 0$ such that

$$\int_0^1 f(t,R)dt \le \frac{R}{\dfrac{1}{\Gamma(\alpha)} + \dfrac{\Gamma(\alpha-\gamma)}{\Delta}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\eta a(t)dt}. \quad (12)$$

Then fractional boundary value problem (1), (2) has at least one nonnegative solution u such that $\|u\|_0 \le R$.

Proof: For $u \in B = \{u \in E: \ \|u\|_0 \le R\}$, following from Lemma 2.5, we have

$$\begin{aligned}
\Big|(Lu)(t)\Big| &\le \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\
&\le \int_0^1 G(t,s)\Big|f(s,u(s))\Big|ds \\
&\le \int_0^1 G(t,s)\Big|f(s,R)\Big|ds \\
&\le \int_0^1 t^{\alpha-1}\Big(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\Big)f(s,R)ds \\
&\le \Big[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big] \\
&\int_0^1 f(s,R)ds \\
&\le R.
\end{aligned}$$

So, $\|Lu\|_0 \le R$. Therefore, the operator L maps B into itself. Hence, by applying Theorem 3.2 and Lemma 3.1, L has a fixed point u in B. ■

3.2 An existence result by the classical Krasnosel'skii fixed point theorem

Construct the following cone

$$P = \{u \in E: \ u(t) \ge \beta\|u\|t^{\alpha-1}, \ t \in [0,1]\}. \quad (13)$$

Theorem 3.5 Assume that there exist $\rho \in (0,1), q_1, q_2 \in C([0,1], R^+)$, nondecreasing functions $\varphi_1, \varphi_2 \in C(R, R^+)$ and $r_0, R_0 > 0$ with $(r_0 \ne R_0)$ such that
(A) $0 \le f(t,u) \le q_1(t)\varphi_1(u)$ for all $t \in [0,1], \ 0 \le u \le r_0$ with

$$M\varphi_1(r_0) \le \frac{r_0}{\dfrac{1}{\Gamma(\alpha)} + \dfrac{\Gamma(\alpha-\gamma)}{\Delta}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^\eta a(t)dt}. \quad (14)$$

(B) $f(t,u) \ge q_2(t)\varphi_2(u)$ for all $t \in [\rho,1], \ \beta\rho^{\alpha-1}R_0 \le u \le R_0$ with

$$\beta\rho m\varphi_2(\beta R_0\rho^{\alpha-1})\frac{(1-\rho)^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha)} \ge R_0. \quad (15)$$

Then fractional boundary value problem (1), (2) has a positive solution satisfying

$$\min(r_0, R_0) \le \|u\|_0 \le \max(r_0, R_0). \quad (16)$$

Here $m = \min\{q_2(t), \ t \in [\rho,1]\}$ and $M = \max\{q_1(t), \ t \in [0,1]\}$.

Proof: (a) Let the open set $B_1 = \{u \in E: \ \|u\|_0 < r_0\}$ and $u \in P \cap \partial B_1$. Then for any $t \in [0,1]$, and since φ_1 is nondecreasing, we have

$$\begin{aligned}
\Big|(Lu)(t)\Big| &= \Big|\int_0^1 G(t,s)f(s,u(s))ds\Big| \\
&\le \int_0^1 G(t,s)q_1(s)\varphi_1(u)ds \\
&\le \int_0^1 G(t,s)q_1(s)\varphi_1(\|u\|_0)ds \\
&\le \int_0^1 G(t,s)q_1(s)\varphi_1(r_0)ds \\
&\le \int_0^1 G(t,s)M\varphi_1(r_0)ds \\
&\le \int_0^1 t^{\alpha-1}\Big(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\Big)M\varphi_1(r_0)ds \\
&\le \Big[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big]M\varphi_1(r_0) \\
&\le r_0 = \|u\|_0.
\end{aligned}$$

So, $\|Lu\|_0 \le \|u\|_0$, for all $u \in P \cap \partial B_1$.

(b) Let the open set $B_2 = \{u \in E: \ \|u\|_0 < R_0\}$ and $u \in P \cap \partial B_2$. So, Lemma 2.5 yields

$$u(t) \ge \beta R_0\rho^{\alpha-1}, \quad \forall t \in [\rho,1].$$

Then for any $t \in [0,1]$, and since φ_2 is nondecreasing, we get

$$\begin{aligned}
\|Lu\|_0 &\ge \max_{t\in[0,1]} \beta t^{\alpha-1}\int_\rho^1 \Phi_2(s)f(s,u(s))ds \\
&= \beta\int_\rho^1 \Phi_2(s)f(s,u(s))ds \\
&\ge \beta\int_\rho^1 \Phi_2(s)q_2(s)\varphi_2(u)ds \\
&\ge \beta\int_\rho^1 \Phi_2(s)m\varphi_2(\beta R_0\rho^{\alpha-1})ds \\
&= \beta m\varphi_2(\beta R_0\rho^{\alpha-1})\int_\rho^1 \Phi_2(s)ds \\
&\ge \beta m\varphi_2(\beta R_0\rho^{\alpha-1})\int_\rho^1 \frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}ds \\
&\ge \beta m\varphi_2(\beta R_0\rho^{\alpha-1})\rho\int_\rho^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}ds \\
&= \beta m\varphi_2(\beta R_0\rho^{\alpha-1})\rho\frac{(1-\rho)^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha)} \\
&\ge R_0 = \|u\|_0.
\end{aligned}$$

Then, $\|Lu\|_0 \ge \|u\|_0$, for all $u \in P \cap \partial B_2$. Moreover, from Lemma 2.5, we get $L(P) \subset P$. Then fractional boundary value problem (1), (2) has a positive solution satisfying

$$\min(r_0, R_0) \le \|u\|_0 \le \max(r_0, R_0).$$

■

IV. The case $f = f(t,u,v)$

In this sequel, we denote by K the positive cone of E^1 given by

$$K = \{u \in E^1 | u(t) \ge 0, \ t \in [0,1]\}. \quad (17)$$

Lemma 4.1 The Green function $G(t,s)$ defined by (6) satisfies

$$\frac{\partial G(t,s)}{\partial t} \le (\alpha-1)t^{\alpha-2}\left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\right]. \quad (18)$$

Proof: It follows from (6) that

$$\frac{\partial G(t,s)}{\partial t} = \frac{\partial G_1(t,s)}{\partial t} + (\alpha-1)h(s)t^{\alpha-2}.$$

(7) implies that

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)}\begin{cases} (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1} \\ \qquad 0 \le t \le s \le 1, \\ (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1} \\ -(\alpha-1)(t-s)^{\alpha-2} \\ \qquad 0 \le s \le t \le 1. \end{cases}$$

So, we have

$$\frac{\partial G(t,s)}{\partial t} \le (\alpha-1)t^{\alpha-2}\left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\right].$$

■

Lemma 4.2 The operator $T: K \to K$ is completely continuous.

Proof: First, using Lemma 2.5, we get $T(K) \subseteq K$, and each fixed point of T is a solution of problem (1),(2). We claim that $T: K \to K$ is completely continuous. The continuity of T is obvious since f is continuous. Now, we prove T is compact.

Let $\Omega \subset K$ be an bounded set. Then, there exists $R > 0$, such that $\Omega \subset \{u \in K | \|u\| \le R\}$. $f \in C[[0,1] \times R^+ \times R, R^+]$ implies there exists $\Psi_R(t) \in C[0,1]$ such that

$$f(t,u,v) \le \Psi_R(t),\ \forall t \in [0,1], u \in [0,R], v \in [-R,R].$$

For any $u \in \Omega$, we obtain

$$\begin{aligned} 0 &\le (Tu)(t) = \int_0^1 G(t,s)f(s,u(s),u'(s))ds \\ &\le \int_0^1 G(t,s)\Psi_R(s)ds \\ &\le \int_0^1 t^{\alpha-1}\left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\right] \\ &\Psi_R(s)ds \\ &\le \left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\right] \\ &\int_0^1 \Psi_R(s)ds \\ &=: \overline{M}. \end{aligned}$$

From the definition of T, we get

$$\|Tu\|_0 \le \overline{M}. \quad (19)$$

On the other hand, for all $u \in \Omega$, using Lemma 4.1, we find

$$\begin{aligned} (Tu)'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))ds \\ &\le \int_0^1 \frac{\partial G(t,s)}{\partial t}\Psi_R(s)ds \\ &\le \int_0^1 (\alpha-1)t^{\alpha-2}\left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + h(s)\right] \\ &\Psi_R(s)ds \\ &\le (\alpha-1)\left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\right] \\ &\int_0^1 \Psi_R(s)ds \\ &=: (\alpha-1)\overline{M}. \end{aligned}$$

Thus, we get

$$\|(Tu)'\|_0 \le (\alpha-1)\overline{M}. \quad (20)$$

In view of the above two equations (19),(20), we get $T\Omega$ is uniformly bounded.

It is clear that $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. This means for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [0,1],\ |t_1 - t_2| < \delta,\ s \in [0,1]$, one has

$$|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{\Psi_R(s)+1},$$

consequently,

$$\begin{aligned} &|(Tu)(t_2) - (Tu)(t_1)| \\ &\le \int_0^1 |G(t_2,s) - G(t_1,s)|f(s,u(s),u'(s))ds \\ &< \int_0^1 \frac{\varepsilon}{\Psi_R(s)+1}\Psi_R(s)ds < \varepsilon. \end{aligned}$$

Similarly, since $\frac{\partial G(t,s)}{\partial t}$ is uniformly continuous on $[0,1] \times [0,1]$, we can prove

$$|(Tu)'(t_2) - (Tu)'(t_1)| < \varepsilon.$$

This means that $T\Omega$ is equicontinuous. By the Arzela-Ascoli theorem, we know that $T: K \to K$ is completely continuous. ■

Theorem 4.3 Assume that there exists $a > 0$, such that

(C_1) $f(t,\ x_1,\ y_1) \le f(t,\ x_2,\ y_2)$, $for\ any$ $0 \le t \le 1,\quad 0 \le x_1 \le x_2 \le (\alpha-1)a,$ $0 \le |y_1| \le |y_2| \le (\alpha-1)a;$

(C_2) $\max\limits_{0\le t\le 1} f(t,(\alpha-1)a,(\alpha-1)a) \le \dfrac{a}{\left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\right]};$

(C_3) $f(t,\ 0,\ 0) \not\equiv 0$ for $0 \le t \le 1.$

Then the fractional boundary value problem *(1),(2)* has one positive solution $\omega^* \in K$ such that $0 < \omega^* \le (\alpha-1)a,\ 0 < |(\omega^*)'| \le (\alpha-1)a$ and $\lim\limits_{n\to\infty} T^n\omega_0 = \omega^*,\ \lim\limits_{n\to\infty}(T^n\omega_0)' = (\omega^*)'$where

$$\omega_0(t) = at^{\alpha-1},\quad 0 \le t \le 1.$$

Proof: We write

$$K_{(\alpha-1)a} = \{u \in K|\ \|u\| < (\alpha-1)a\},$$

and

$$\overline{K_{(\alpha-1)a}} = \{u \in K|\ \|u\| \le (\alpha-1)a\}.$$

We first claim $T:\ \overline{K_{(\alpha-1)a}} \to \overline{K_{(\alpha-1)a}}$. Let $u \in \overline{K_{(\alpha-1)a}}$, then

$$0 \le u(t) \le \max_{0\le t\le 1}|u(t)| \le \|u\| \le (\alpha-1)a, \quad (21)$$

$$|u'(t)| \le \max_{0\le t\le 1}|u'(t)| \le \|u\| \le a < (\alpha-1)a. \quad (22)$$

So, from assumptions (C_1) and (C_2), we get

$$\begin{aligned} 0 &\le f(t,u(t),u'(t)) \le f(t,(\alpha-1)a,(\alpha-1)a) \\ &\le \max_{0\le t\le 1} f(t,(\alpha-1)a,(\alpha-1)a) \\ &\le \frac{a}{\left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\right]},\ 0 \le t \le 1. \end{aligned} \quad (23)$$

Therefore, for $u \in \overline{K_{(\alpha-1)a}}$, according to Lemma 4.1, we get the following estimates

$$\begin{aligned}\Big|(Tu)(t)\Big| &= \Big|\int_0^1 G(t,s)f(s,u(s),u'(s))ds\Big| \\ &\le \int_0^1 G(t,s)\Big|f(s,u(s),u'(s))\Big|ds \\ &\le \int_0^1 G(t,s) \\ &\Big|f(s,(\alpha-1)a,(\alpha-1)a)\Big|ds \\ &\le \int_0^1 t^{\alpha-1}\Big(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s)\Big) \\ &f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le \Big[\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big] \\ &\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le a \le (\alpha-1)a.\end{aligned}$$

$$\begin{aligned}\Big|(Tu)'(t)\Big| &\le \int_0^1\Big|\frac{\partial G(t,s)}{\partial t}\Big|f(s,u(s),u'(s))ds \\ &\le \int_0^1\Big|\frac{\partial G(t,s)}{\partial t}\Big|f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le \int_0^1(\alpha-1)t^{\alpha-2}\Big[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s)\Big] \\ &f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le (\alpha-1)\Big[\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big] \\ &\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &=: (\alpha-1)a.\end{aligned}$$

Thus, we have

$$\|Tu\| \le (\alpha-1)a.$$

This means $T:\ \overline{K_{(\alpha-1)a}} \to \overline{K_{(\alpha-1)a}}$. Denote

$$\omega_0(t) = at^{\alpha-1}, \qquad 0 \le t \le 1,$$

Let $\omega_1 = T\omega_0$, then $\omega_1 \in \overline{K_{(\alpha-1)a}}$, we write

$$\omega_{n+1} = T\omega_n = T^{n+1}\omega_0, \quad (n = 0,1,2,\cdots). \tag{24}$$

Since $T:\ \overline{K_{(\alpha-1)a}} \to \overline{K_{(\alpha-1)a}}$, we have $\omega_n \in T\overline{K_{(\alpha-1)a}} \subseteq \overline{K_{(\alpha-1)a}}$, $n = 0,1,2,\cdots$. T is completely continuous implies $\{\omega_n\}_{n=0}^\infty$ is a sequentially compact set,

$$\begin{aligned}\omega_1(t) &= T\omega_0(t) = \int_0^1 G(t,s)f(s,\omega_0(s),\omega_0'(s))ds \\ &\le \int_0^1 G(t,s)\Big|f(s,(\alpha-1)a,(\alpha-1)a)\Big|ds \\ &\le \int_0^1 t^{\alpha-1}\Big(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s)\Big) \\ &f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le t^{\alpha-1}\Big[\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big] \\ &\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &= at^{\alpha-1} = \omega_0(t).\end{aligned}$$

$$\begin{aligned}\Big|\omega_1'(t)\Big| &= \Big|(T\omega_0)'(t)\Big| \\ &\le \int_0^1\Big|\frac{\partial G(t,s)}{\partial t}\Big|f(s,\omega_0(s),\omega_0'(s)))ds \\ &\le \int_0^1\Big|\frac{\partial G(t,s)}{\partial t}\Big|f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le \int_0^1(\alpha-1)t^{\alpha-2}\Big[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s)\Big] \\ &f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &\le t^{\alpha-2}(\alpha-1)\Big[\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(\alpha-\gamma)}{\Delta}\frac{1}{\Gamma(\alpha)}\int_0^\eta a(t)dt\Big] \\ &\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds \\ &=: a(\alpha-1)t^{\alpha-2} = \Big|\omega_0'(t)\Big|.\end{aligned}$$

then we have

$$\omega_1(t) \le \omega_0(t), \quad |\omega_1'(t)| \le |\omega_0'(t), \quad 0 \le t \le 1.$$

Thus,

$$\omega_2(t) = T\omega_1(t) \le T\omega_0(t) = \omega_1(t), \quad 0 \le t \le 1,$$

$$|\omega_2'(t)| = |T\omega_1)'(t)| \le |(T\omega_0)'(t)| = |\omega_1'(t), \qquad 0 \le t \le 1.$$

Hence by induction, we have

$$\omega_{n+1} \le \omega_n, \quad |\omega_{n+1}'(t)| \le |\omega_n'(t)|, \qquad 0 \le t \le 1, \quad n = 1,2,\cdots.$$

Thus, there exists $\omega^* \in \overline{K_{(\alpha-1)a}}$ such that $\omega_n \to \omega^*$. Letting $n \to \infty$ in (24), we obtain $T\omega^* = \omega^*$ since T is continuous.

If $f(t,\ 0,\ 0) \not\equiv 0,\ 0 \le t \le 1$, then the zero function is not the solution of (1),(2). Therefore, ω^* is a positive solution of (1),(2). ■

REFERENCES

[1] I. Podlubny, *Fractional Differential Equation*, Academic Press, New York, 1999.

[2] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, The Netherlands, 2006.

[3] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[4] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. I: Fixed Point Theorems, Noordhoff, Groningen, The Netherlands, 1964. Springer-Verlag, New York, 1986.

[5] M. A. Krasnosel'skii, *Nonlinear Positive Solutions of Operator Equations*, Noordhoof, Groningen, The Netherlands, 1964.

[6] Y. Cui, "Uniqueness of solution for boundary value problems for fractional differential equations," *Appl. Math. Lett.*, vol. 51, pp. 48-54, 2016.

[7] Y. Cui, W. Ma, Q. Sun, X. Su, "New uniqueness results for boundary value problem of fractional differentiale quation," *Nonlinear Anal. Model. Control*, vol. 23, pp. 31-39, 2018.

[8] X. Du, A. Mao, "Existence and multiplicity of nontrivial solutions for a class of semilinear fractional Schrodingerequations," *J. Funct. Spaces*, vol. 2017, ArticleID 3793872, 2017.

[9] Y. Guan, Z. Zhao, X. Lin, "On the existence of positive solutions and negative solutions of singular fractional differential equations via global bifurcation techniques," *Bound. Value Probl.*, vol. 2016, pp. 1-18, 2016.

[10] X. Hao, H. Wang, L. Liu, Y. Cui,"Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator," *Bound. Value Probl.*, vol. 2017, pp. 1-18, 2017.

[11] X. Hao, L. Zhang, L.Liu, "Positive solutions of higher order fractional integral boundary value problem with a parameter," *Nonlinear Anal. Model. Control*, vol. 24(2), pp. 210-223, 2019.

[12] X. Hao, L. Zhang, "Positive solutions of a fractional thermostatmodel with a parameter," *Symmetry*, vol. 11, pp. 1-9, 2019.

[13] X. Hao, H. Sun, L. Liu, "Existence results for fractional integral boundary value problem involving fractional derivatives on an infinite interval," *Math. Methods Appl. Sci.*, vol. 41, pp. 6984-6996, 2018.

[14] X. Hao, H. Wang, "Positive solutions of semipositone singular fractional differential systems with a parameter and integral boundary conditions," *Open Math.*, vol. 16, pp. 581-596, 2018.

[15] F. Yan, M. Zuo, X. Hao, "Positive solution for a fractional singular boundary value problem with p-Laplacian operator," *Bound. Value Probl.*, vol. 2018, pp. 1-10, 2018.

[16] X. Hao, L. Liu, Y. Wu, "Positive solutions for nonlinear nth-order singular eigenvalue problem with nonlocal conditions," *Nonlinear Anal.*, vol. 73(6), pp. 1653-1662, 2010.

[17] J. Henderson, R. Luca, "Existence of positive solutions for a singular fractional boundary value problem," *Nonlinear Anal. Model. Control*, vol. 22(1), pp. 99-114, 2017.

[18] Jingjing Cui, Guohua Peng, Quan Lu, Zhengge Huang, "A Class of Nonstationary Upper and Lower Triangular Splitting Iteration Methods for Ill-posed Inverse Problems," *IAENG International Journal of Computer Science*, vol. 47, no.1, pp. 118-129, 2020.

[19] J. Jiang, L. Liu, "Existence of solutions for a sequential fractional differential system with coupled boundary conditions," *Bound. Value Probl.*, vol. 2016, pp. 1-15, 2016.

[20] G. Wang, B. Ahmad, L. Zhang, "Existence results for nonlinear fractional differential equations with closed boundary conditions and impulses," *Adv. Differ. Equ.*, vol. 2012, pp. 1-13, 2012.

[21] Y. Wang, L. Liu, "Necessary and suffcient condition for the existence of positive solution to singular fractional differential equations," *Adv. Differ. Equ.*, vol. 2015, pp. 1-14, 2015.

[22] Y. Wang, L. Liu, Y. Wu, "Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity," *Nonlinear Anal.*, vol. 74, pp. 6434-6441, 2011.

[23] J. Wu, X. Zhang, L. Liu, "The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity," *Bound. Value Probl.*, vol. 2018, pp. 1-15, 2018.

[24] K. Zhang, "On sign-changing solution for some fractional differential equations," *Bound. Value Probl.*, vol. 2017, pp. 1-15, 2017.

[25] X. Zhang, L. Liu, Y. Wu, "Multiple positive solutions of a singular fractional differential equation with negatively perturbedterm," *Math. Comput. Model.*, vol. 55, pp. 1263-1274, 2012.

[26] X. Zhang, L. Liu, Y. Wu, "The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium," *Appl. Math. Lett.*, vol. 37, pp. 26-33, 2014.

[27] X. Zhang, L. Liu, Y. Wu, Y. Cui, "New result on the critical exponent for solution of an ordinary fractional differential problem," *J. Funct. Spaces*, vol. 2017, ArticleID3976469, 2017.

[28] X. Zhang, Z. Shao, Q. Zhong, "Positive solutions for semipositone (k,n-k) conjugate boundary value problems with singularities on space variables," *Appl. Math. Lett.*, vol. 72, pp. 50-57, 2017.

[29] Y. Zou, G. He, "On the uniqueness of solutions for a class of fractional differential equations," *Appl. Math. Lett.*, vol. 74, pp. 68-73, 2017.

[30] Mulin Li, Lifeng Wang, "Numerical Solution of Fractional Partial Differential Equation of Parabolic Type Using Chebyshev Wavelets Method," *Engineering Letters*, vol. 26, no.2, pp. 224-227, 2018.

[31] M. Zuo, X. Hao, L. Liu, Y. Cui, "Existence results for impulsive fractional integro-differential equation of mixed type with constant coeffcient and antiperiodic boundary conditions," *Bound. Value Probl.*, vol. 2017, pp. 1-15, 2017.

[32] Y. Wang, "Existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems," *Bound. Value Probl.*, vol. 2019, pp. 1-18, 2019.

[33] Y. Wang, L. Liu, "Positive solutions for a class of fractional infinite-point boundary value problems," *Bound. Value Probl.*, vol. 2018, pp. 1-14, 2018.

[34] Y. Wang, L. Liu, Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Anal.*, vol. 74, pp. 3599-3605, 2011.

[35] X. Zhang, Q. Zhong, "Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables," *Appl. Math. Lett.*, vol. 80, pp. 12-19, 2018.

[36] X. Tian, Z. Yang, "Adaptive Stabilization of Fractional-order Nonlinear System Considering Input Saturation Phenomenon," *Engineering Letters*, vol. 27, no.4, pp. 876-881, 2019.