# Numerical Simulations to a One-dimensional Groundwater Pollution Measurement Model Through Heterogeneous Soil

Wasu Timpitak, and Nopparat Pochai

*Abstract*—**The problem of toxic contaminants in groundwater with groundwater pollution measurement model. The advection-diffusion equation is used to describe the concerned model. The theoretical solution of the advection-diffusion equation is limited only in ideal geometries. Applications of numerical solutions are influenced in several initial and boundary conditions when dealing with complex geometries. In this research, numerical simulations for one-dimensional groundwater pollution measurement around landfills model through heterogeneous soil are focused. The forward time center space and Saulyev finite difference techniques are used to approximate the solutions. The accuracy of proposed techniques are to examine by comparing the approximated solutions with the analytical solution. The purposed technique gives good agreement approximated solution.**

*Index Terms*—**advection, diffusion, semi-infinite domain, finite difference method, variable coefficient.**

## I. Introduction

THERE are several use of the advection-diffusion equation (ADE), including heat transfer, sediment transport and water pollutant concentration measurement. In [1], introduced a numerical technique for approximating ADE with constant coefficient. The developed scheme was based on a mathematical combine both Siemieniuch and Gradwell approximation for time and Dehghan's approximation for spatial variable. In [2], they have solved ADE with an explicit approach of finite difference method (EFDM) and variable coefficients in semi-infinite domain. This equation can analyze three dispersion problems: (i) solute dispersion along poised flow through inhomogeneous medium, (ii) temporarily dependent solute dispersion along uniform flow through homogeneous medium, and (iii) solute dispersion along temporarily dependent not poised flow through inhomogeneous medium. In [3], they have solved the one-dimensional (1D) ADE with variable coefficients in semi-infinite media by using EFDM for two dispersion problems: (i) temporarily dependent dispersion in uniform flow and (ii) spatially dependent dispersion in non-uniform flow, uniform pulse-type input condition and initial solute concentration, that decreasing function of distance were considered. In [4], the EFDM to obtain the dispersion through a heterogeneous horizontal semi-infinite medium. The heterogeneous nature of the medium was discoursed by a position dependency linear nonhomogeneous expression for velocity with not poised exponential variation with time. Velocity and dispersion was zero at the origin. In [5], an analytical solution to one-dimensions ADE with several point sources through arbitrary time-dependent discharging rate is proposed. They reported that the results had indicated, the proposed analytical solution could offer an accurate estimation of the contemplation. The limitations of the proposed solution were valid only for the constant-parameters condition, and was not computational performance for problems involving a high temporal or a high spatial resolution.

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The inhomogeneity of the medium causes variation in the flow velocity, [2], [6]. In [7], studied on the variation of the increasing nature. In this research, we will propose an explicit finite difference technique for an advection-diffusion equation with variable coefficient in a semi-infinite domain. Due to the low advection groundwater flow, the contaminated groundwater flow measurement need very long term transition time. The effective time of prediction will be larger than a year. The simulation of contaminated groundwater level in faraway point need to be introduced.

In [8], proposed Multiple BPNN and Genetic Algorithm (GA) to overcome the limitation of ARIMA/SARIMA, standalone BPNN and NARX for water level forecasting in a river are proposed. In [9], the application of PESN to a water quality evaluation problem are investigated. In [10], develop the ANN models in water resources engineering in Sarawak, in particular, in areas where precipitation data were absent. In [11], the knowledge from the digital communication theory to analyze a general watermarking problem are applied. In [12], used the theory of holomorphic functions, in particular, conformal mappings and the hodograph method to analytically find the shape of NAPL interfaces, whose very existence is puzzling.

In this research, the transient contaminated groundwater dispersion measurement around landfills model will be introduced. The inhomogeneous soil by the bottom topography will be also concerned in the proposed model. The finite difference technique to obtain the approximated solutions are purposed. The accuracy of the purposed numerical methods is tested by an analytical solution in an ideal case.

## II. Governing equation

### A. *Contaminated groundwater dispersion along unsteady flow through inhomogeneous soil*

The one-dimension advection-diffusion equation (ADE) is expressed as follows [6],

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C(x,t)}{\partial x}\right) - \frac{\partial u(x,t)\,C(x,t)}{\partial x}, \tag{1}$$

for all $(x,t) \in \Omega$ such that $\Omega = [0,L] \times [0,T]$ ,where are $C(x,t)$ the dispersing solute concentration, the longitudinal axis, and time, respectively.

### *B. Contaminated groundwater dispersion along unsteady flow through inhomogeneous soil*

If the values of $D$ and $u$ assumed as constants, then these values are named dispersion coefficient and uniform velocity of the flow field, respectively. In this study preferred to use Eq.(1) as follows:

$$\frac{\partial C(x,t)}{\partial t} = D_0 \frac{\partial}{\partial x}\left(f_1(x,t)\frac{\partial C(x,t)}{\partial x}\right) - \frac{u_0 \partial f_2(x,t) C(x,t)}{\partial x}, \tag{2}$$

for all $(x,t) \in [0,L] \times [0,T]$, where $D_0$ and $u_0$ are constant values whose dimensions depend upon the expressions $f_1(x,t)$ and $f_2(x,t)$ and $f_1(x,t)$ and $f_2(x,t)$ are given function. The analytical solutions of ADE for the previously mentioned two hydrodynamic dispersion problems were introduced by [6].

### *C. The initial and boundary conditions*

The initial condition is defined by an interpolation function of measured raw data. The boundary conditions can be classified into two cases.

*1) Potential contaminated groundwater:* An initially solute free condition is assumed for both of the problems in the semi-infinite domain. Meanwhile, a uniform distribution of nodes is applied at the origin of the domain. The initial condition, is assumed by

$$C(x,0) = f(x), \forall x \in [0,L], \tag{3}$$

where $f(x)$ is an initially pollutant concentration function.

*2) Contaminated groundwater at two monitoring points:* The boundary conditions, are also assumed by

$$C(0,t) = g_1(t), \forall t \in [0,T], \tag{4}$$

$$C(L,t) = g_2(t), \forall t \in [0,T], \tag{5}$$

where $g_1(t)$ and $g_2(t)$ boundary sources of pollutant concentration on the starting point and the end point of the radius of considered area, respectively.

*3) Contaminated groundwater at the single monitoring points:* The left boundary condition is assumed by the interpolation function of measured raw data at the considered landfill. The right boundary condition is assumed by the averaged rate of change of pollutant concentration around the right ended point. The boundary conditions, are also assumed by

$$C(0,t) = g_1(t), \forall t \in [0,T], \tag{6}$$

$$\frac{\partial C(L,t)}{\partial x} = \kappa, \forall t \in [0,T], \tag{7}$$

where $g_1(t)$ and $\kappa$ boundary sources of pollutant concentration on the starting point and the rate of change of pollutant concentration with respect to distance around the ended point on the considered area, respectively.

## III. Numerical Techniques

We now discretize the domain by dividing the interval $[0,L]$ into $M$ subintervals such that $M\Delta x = L$ and the time interval $[0,T]$ into $N$ subintervals such that $N\Delta t = T$. The grid points $(x_i, t_n)$ are defined by $x_i = i\Delta x$ for all $i = 0,1,2,\ \dots\ ,M$ and $t_n = n\Delta t$ for all $n = 0,1,2,\ \dots\ ,N$ in which $M$ and $N$ are positive integers. We can then approximate $C(x_i,t_n)$ by $C_i^n$ , value of the difference approximation of $C(x,t)$ at point $x = i\Delta x$ and $t = n\Delta t$, where $0 \le i \le M$ and $0 \le n \le N$. We will employ the forward time central space finite difference scheme (FTCS) into Eq.(2).

### *A. Forward Time Central Space Finite Difference Scheme*

$$C(x_i,t_n) \cong C_i^n, \tag{8}$$

$$\left.\frac{\partial C}{\partial t}\right|_{(x_i,t_n)} \cong \frac{C_i^{n+1} - C_i^n}{\Delta t}, \tag{9}$$

$$\left.\frac{\partial C}{\partial x}\right|_{(x_i,t_n)} \cong \frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}, \tag{10}$$

$$\left.\frac{\partial^2 C}{\partial x^2}\right|_{(x_i,t_n)} \cong \frac{C_{i+1}^n + C_{i-1}^n - 2C_i^n}{(\Delta x)^2}, \tag{11}$$

$$f_1(x_i,t_n) = f_{1_i}^n, \tag{12}$$

$$f_2(x_i,t_n) = f_{2_i}^n. \tag{13}$$

Substituting Eqs.(8)-(13) into Eq.(2), we get the finite difference equation,

$$\frac{C_i^{n+1} - C_i^n}{\Delta t} = \left(D_0 \cdot \left.\frac{\partial f_1}{\partial x}\right|_{(x_i,t_n)} - u_0 f_{2_i}^n\right) \cdot \left(\frac{C_{i+1}^n - C_{i-1}^n}{2\Delta x}\right) - u_0 \cdot \left.\frac{\partial f_2}{\partial x}\right|_{(x_i,t_n)} C_i^n + D_0 f_{1_i}^n \left(\frac{C_{i-1}^n - 2C_i^n + C_{i+1}^n}{(\Delta x)^2}\right), \tag{14}$$

for all $i = 1,2,3,\ \dots\ ,M$and $n = 0,1,2,\ \dots\ ,N-1$ Then the explicit finite difference equation becomes

$$C_i^{n+1} = \left[\lambda_i^n - \frac{1}{2}\gamma_i^n + \frac{1}{2}\beta_i^n\right] C_{i-1}^n + [1 - \Delta t\alpha_i^n - 2\lambda_i^n] C_i^n + \left[\lambda_i^n + \frac{1}{2}\gamma_i^n - \frac{1}{2}\beta_i^n\right] C_{i+1}^n, \tag{15}$$

where $\alpha_i^n = u_0 \cdot \left.\frac{\partial f_2}{\partial x}\right|_{(x_i,t_n)}$, $\gamma_i^n = \frac{\Delta t\, D_0}{\Delta x} \left.\frac{\partial f_1}{\partial x}\right|_{(x_i,t_n)}$, $\beta_i^n = \frac{\Delta t\, u_0 f_{2_i}^n}{\Delta x}$ and $\lambda_i^n = \frac{\Delta t\, D_0 f_{1_i}^n}{(\Delta x)^2}$, the explicit finite difference Eq.(15) can be written in a compact form as,

$$V_i^n = \lambda_i^n - \frac{1}{2}\gamma_i^n + \frac{1}{2}\beta_i^n, \tag{16}$$

$$G_i^n = 1 - \Delta t\alpha_i^n - 2\lambda_i^n, \tag{17}$$

$$P_i^n = \lambda_i^n + \frac{1}{2}\gamma_i^n - \frac{1}{2}\beta_i^n. \tag{18}$$

Then

$$C_i^{n+1} = V_i^n C_{i-1}^n + G_i^n C_i^n + P_i^n C_{i+1}^n. \tag{19}$$

According to the right boundary condition Eq.(7), if $i = M$, substituting the approximate unknown value of the right boundary [9], we can let $C_{M+1}^n = 2\kappa\Delta x + C_{M-1}^n$ and by rearranging, we obtain

$$C_M^{n+1} = \left(V_M^n + P_M^n\right) C_{M-1}^n + G_M^n C_M^n + 2\Delta x \cdot \kappa P_M^n. \quad (20)$$

The forward time central space scheme is conditionally stable subject to constraints in Eq.(15). The stability requirements for the scheme are [13], [14]

$$\lambda_i^n = \frac{\Delta t\, D_0 f_1\left(x_i, t_n\right)}{\left(\Delta x\right)^2} < \frac{1}{2},$$
$$\beta_i^n\,,\ \gamma_i^n\,,\ \alpha_i^n < 1,$$

where $\lambda_i^n$ is the diffusion number (dimensionless) and $\beta_i^n$ is the advection number (dimensionless). It can be obtained that the strictly stability requirements are the main disadvantage of this scheme.

The finite difference formula Eq.(19) has been derived in [15] that the truncation error for this method is $O\left\{(\Delta x)^2, \Delta t\right\}$

### B. Saulyev Explicit Finite Difference Scheme

The Saulyev scheme is unconditionally stable [16]. It is clear that the nonstrictly stability requirement of Saulyev scheme is the main of advantage and economical to use. Taking Saulyev technique [16] into Eq.(2), it can be obtained the following discretization:

$$C\left(x_i, t_n\right) \cong C_i^n, \quad (21)$$
$$\left.\frac{\partial C}{\partial t}\right|_{(x_i,t_n)} \cong \frac{C_i^{n+1} - C_i^n}{\Delta t}, \quad (22)$$
$$\left.\frac{\partial C}{\partial x}\right|_{(x_i,t_n)} \cong \frac{C_{i+1}^n - C_{i-1}^{n+1}}{2\Delta x}, \quad (23)$$
$$\left.\frac{\partial^2 C}{\partial x^2}\right|_{(x_i,t_n)} \cong \frac{C_{i+1}^n - C_i^n - C_i^{n+1} + C_{i-1}^{n+1}}{\left(\Delta x\right)^2}, \quad (24)$$
$$f_1\left(x_i, t_n\right) = f_{1\,i}^n, \quad (25)$$
$$f_2\left(x_i, t_n\right) = f_{2\,i}^n. \quad (26)$$

Substituting Eqs.(21)-(26) into Eq.(2), we get the finite difference equation,

$$\frac{C_i^{n+1} - C_i^n}{\Delta t} = \left(D_0 \cdot \left.\frac{\partial f_1}{\partial x}\right|_{(x_i,t_n)} - u_0 f_{2\,i}^n\right) \cdot \left(\frac{C_{i+1}^n - C_{i-1}^{n+1}}{2\Delta x}\right) - u_0 \cdot \left.\frac{\partial f_2}{\partial x}\right|_{(x_i,t_n)} C_i^n + D_0 f_{1\,i}^n \left(\frac{C_{i+1}^n - C_i^n - C_i^{n+1} + C_{i-1}^{n+1}}{\left(\Delta x\right)^2}\right), \quad (27)$$

for all $i = 1, 2, 3, \ \ldots\ , M$ and $n = 0, 1, 2, \ \ldots\ , N-1$. Then the explicit finite difference equation becomes

$$C_i^{n+1} = \frac{1}{1+\lambda_i^n}\left[\left(\lambda_i^n - \frac{1}{2}\gamma_i^n + \frac{1}{2}\beta_i^n\right) C_{i-1}^{n+1} + \left(1 - \Delta t\alpha_i^n - \lambda_i^n\right) C_i^n + \left(\lambda_i^n + \frac{1}{2}\gamma_i^n - \frac{1}{2}\beta_i^n\right) C_{i+1}^n\right] \quad (28)$$

where $\gamma_i^n = \frac{\Delta t\, D_0}{\Delta x}\left.\frac{\partial f_1}{\partial x}\right|_{(x_i,t_n)}$,$\alpha_i^n = u_0 \cdot \left.\frac{\partial f_2}{\partial x}\right|_{(x_i,t_n)}$, $\lambda_i^n = \frac{\Delta t\, D_0 f_{1\,i}^n}{(\Delta x)^2}$ and $\beta_i^n = \frac{\Delta t\, u_0 f_{2\,i}^n}{\Delta x}$, the explicit finite difference Eq.(28) can be written in a compact form as,

$$A_i^n = \frac{1}{1+\lambda_i^n}, \quad (29)$$
$$B_i^n = \lambda_i^n - \frac{1}{2}\gamma_i^n + \frac{1}{2}\beta_i^n, \quad (30)$$
$$Q_i^n = 1 - \Delta t\alpha_i^n - 2\lambda_i^n, \quad (31)$$
$$Z_i^n = \lambda_i^n + \frac{1}{2}\gamma_i^n - \frac{1}{2}\beta_i^n. \quad (32)$$

Then

$$C_i^{n+1} = A_i^n\left(B_i^n C_{i-1}^{n+1} + Q_i^n C_i^n + Z_i^n C_{i+1}^n\right). \quad (33)$$

According to the right boundary condition Eq.(7), if $i = M$, substituting the approximate unknown value of the right boundary [13], we can let $C_{M+1}^n = 2\kappa\Delta x + C_{M-1}^n$ and by rearranging, we obtain

$$C_M^{n+1} = A_M^n\left(B_M^n C_{M-1}^{n+1} + Q_M^n C_M^n + Z_M^n C_{M-1}^n + 2\Delta x \cdot \kappa Z_M^n\right). \quad (34)$$

Using Taylor series expansions on the approximation, [17] has shown that the truncation error is $O\left\{(\Delta x)^2 + (\Delta t)^2 + (\Delta t/\Delta x)^2\right\}$ or $O\left\{2, 2, (1/1)^2\right\}$

## IV. The accuracy of the proposed numerical technique

The variation in velocity is assumed as small order to insure that it can satisfy the essential conditions for velocity parameter in the ADE. Additionally, as the second assumption, dispersion parameter is considered proportional to the square of the velocity [18]. Thus, for Eq.(2), the expressions of $f_1\left(x, t\right)$ and $f_2\left(x, t\right)$ are assumed by [6]:

$$f_1\left(x, t\right) = \left(1 + ax\right)^2, \quad (35)$$
$$f_2\left(x, t\right) = 1 + ax, \quad (36)$$

where $a$ is a parameter that accounts for the inhomogeneity of the domain with dimension $(length)^{-1}$ In [6] , they have introduced an analytical solution that satisfies the specific $f_1\left(x, t\right)$ and $f_2\left(x, t\right)$ as in Eq.(35) and Eq.(36) ,

$$C = \frac{c_0}{2}\left[(1+ax)^{-1} erfc\left(\frac{\ln\left(1+ax\right)}{2a\sqrt{D_0 T}} - \beta\sqrt{t}\right) + (1+ax)^{\delta} erfc\left(\frac{\ln\left(1+ax\right)}{2a\sqrt{D_0 T}} + \beta\sqrt{t}\right)\right], \quad (37)$$

where $\beta = \sqrt{\frac{\omega_0^2}{4a^2 D_0} + au_0} = \frac{u_0 + aD_0}{2\sqrt{D_0}}$ ,$\delta = \frac{u_0}{aD_0}$ and $\omega_0 = \left(au_0 - a^2 D_0\right)$.

## V. Numerical experiments and results

*A. Simulation 1.1 : Two monitoring contaminated groundwater points; forward time central space finite difference scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1(x,t) = (1+x)^2$ and $f_2(x,t) = 1+x$. The boundary conditions and the initial condition are assumed by Eq.(37). By employing the FTCS finite difference technique Eq.(19), we get the approximated chemical concentrations along 1 km from the point source in 0.7 year. The results are shown in Tables I-III when time increment $(\Delta t)$ are varied and $\lambda = \Delta t \big/ (\Delta x)^2$ is divided by a half. The surface plot of approximated solutions is illustrated in Fig 1.

TABLE I
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.0005\,(year),\ \lambda = 0.2)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83862 | 0.70228 | 0.58759 | 0.49144 | 0.41106 |
| 0.5 | 0.88114 | 0.78073 | 0.69509 | 0.62143 | 0.55765 |
| 0.7 | 0.89079 | 0.79881 | 0.72039 | 0.65285 | 0.59416 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34398 | 0.28806 | 0.24146 | 0.20263 | 0.17027 |
| 0.5 | 0.50210 | 0.45347 | 0.41069 | 0.37291 | 0.33942 |
| 0.7 | 0.54281 | 0.49757 | 0.45749 | 0.42181 | 0.38990 |

TABLE II
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.00025\,(year),\ \lambda = 0.1)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83856 | 0.70217 | 0.58745 | 0.49130 | 0.41092 |
| 0.5 | 0.88113 | 0.78073 | 0.69508 | 0.62142 | 0.55764 |
| 0.7 | 0.89079 | 0.79881 | 0.72039 | 0.65284 | 0.59416 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34386 | 0.28796 | 0.24139 | 0.20260 | 0.17027 |
| 0.5 | 0.50209 | 0.45346 | 0.41068 | 0.37291 | 0.33942 |
| 0.7 | 0.54280 | 0.49756 | 0.45749 | 0.42181 | 0.38990 |

TABLE III
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83853 | 0.70212 | 0.58738 | 0.49122 | 0.41085 |
| 0.5 | 0.88113 | 0.78072 | 0.69507 | 0.62141 | 0.55763 |
| 0.7 | 0.89079 | 0.79881 | 0.72038 | 0.65284 | 0.59416 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34380 | 0.28791 | 0.24136 | 0.20258 | 0.17027 |
| 0.5 | 0.50209 | 0.45345 | 0.41068 | 0.37291 | 0.33942 |
| 0.7 | 0.54280 | 0.49756 | 0.45749 | 0.42181 | 0.38990 |

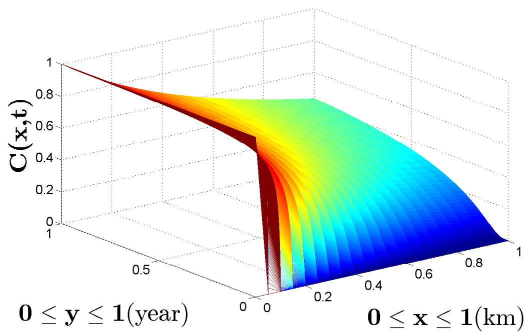


Fig. 1. The surface plot of computed chemical concentrations $\tilde{C}(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$.

*B. Simulation 1.2 : Two monitoring contaminated groundwater points; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1(x,t) = (1+x)^2$ and $f_2(x,t) = 1+x$. The boundary conditions and the initial condition are assumed by Eq.(37). By employing the Saulyev finite difference technique Eq.(33), we get the approximated chemical concentrations along 1 km from the point source in 0.7 year. The results are shown in Tables IV-VI when time increment $(\Delta t)$ are varied and $\lambda = \Delta t \big/ (\Delta x)^2$ is divided by a half. The surface plot of approximated solutions is illustrated in Fig 2.

TABLE IV
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.0005\,(year),\ \lambda = 0.2)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83801 | 0.70138 | 0.58663 | 0.49057 | 0.41034 |
| 0.5 | 0.88084 | 0.78024 | 0.69449 | 0.62080 | 0.55704 |
| 0.7 | 0.89063 | 0.79854 | 0.72006 | 0.65249 | 0.59382 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34344 | 0.28769 | 0.24124 | 0.20253 | 0.17027 |
| 0.5 | 0.50156 | 0.45303 | 0.41038 | 0.37275 | 0.33942 |
| 0.7 | 0.54250 | 0.49732 | 0.45731 | 0.42172 | 0.38990 |

TABLE V
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.00025\,(year),\ \lambda = 0.1)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83825 | 0.70172 | 0.58697 | 0.49086 | 0.41056 |
| 0.5 | 0.88098 | 0.78048 | 0.69478 | 0.62110 | 0.55734 |
| 0.7 | 0.89071 | 0.79868 | 0.72022 | 0.65266 | 0.59399 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34359 | 0.28777 | 0.24128 | 0.20255 | 0.17027 |
| 0.5 | 0.50182 | 0.45324 | 0.41053 | 0.37283 | 0.33942 |
| 0.7 | 0.54265 | 0.49744 | 0.45740 | 0.42176 | 0.38990 |

TABLE VI
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.83837 | 0.70189 | 0.58714 | 0.49100 | 0.41067 |
| 0.5 | 0.88106 | 0.78060 | 0.69492 | 0.62125 | 0.55748 |
| 0.7 | 0.89075 | 0.79874 | 0.72030 | 0.65275 | 0.59407 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.34366 | 0.28782 | 0.24130 | 0.20256 | 0.17027 |
| 0.5 | 0.50195 | 0.45334 | 0.41060 | 0.37287 | 0.33942 |
| 0.7 | 0.54272 | 0.49750 | 0.45744 | 0.42179 | 0.38990 |

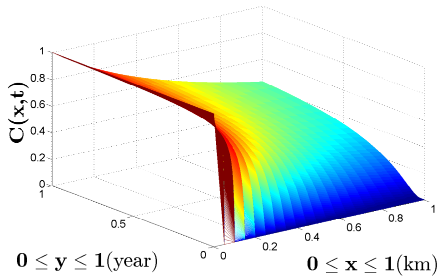


Fig. 2. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$ .

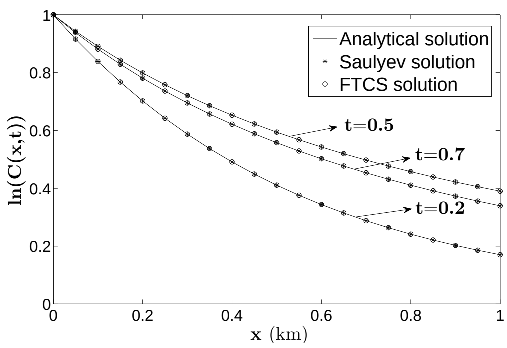


Fig. 3. The comparison of FTCS scheme and the Saulyev scheme and the analytical solution when $t = 0.2$ , $0.5$ and $0.7$

TABLE VII
THE COMPARISON THE MAXIMUM ROOT MEAN SQUARE ERROR OF FTCS SOLUTIONS AND SAULYEV SOLUTIONS

| $\lambda$ | $\Delta x$ | $\Delta t$ | FTCS $RSME_{\max}$ | Saulyev $RSME_{\max}$ |
|---|---|---|---|---|
| 0.025 | 0.050 | $6.2500\times10^{-5}$ | $1.2361\times10^{-2}$ | $1.5604\times10^{-2}$ |
| | 0.025 | $1.5625\times10^{-5}$ | $8.5507\times10^{-3}$ | $1.0739\times10^{-2}$ |
| 0.050 | 0.050 | $1.2500\times10^{-4}$ | $8.9331\times10^{-3}$ | $1.5631\times10^{-2}$ |
| | 0.025 | $3.1250\times10^{-5}$ | $6.2175\times10^{-3}$ | $1.0755\times10^{-2}$ |
| 0.100 | 0.050 | $2.5000\times10^{-4}$ | $2.5368\times10^{-3}$ | $1.5170\times10^{-2}$ |
| | 0.025 | $6.2500\times10^{-5}$ | $1.9303\times10^{-3}$ | $1.0466\times10^{-2}$ |
| 0.200 | 0.050 | $5.0000\times10^{-4}$ | $1.5331\times10^{-2}$ | $1.5476\times10^{-2}$ |
| | 0.025 | $1.2500\times10^{-4}$ | $1.0200\times10^{-2}$ | $1.0649\times10^{-2}$ |
| 0.400 | 0.050 | $1.0000\times10^{-3}$ | Unstable | $1.4332\times10^{-2}$ |
| | 0.025 | $2.5000\times10^{-4}$ | Unstable | $9.4084\times10^{-3}$ |
| 0.800 | 0.050 | $2.0000\times10^{-3}$ | Unstable | $2.4844\times10^{-2}$ |
| | 0.025 | $5.0000\times10^{-4}$ | Unstable | $1.5746\times10^{-2}$ |

### *C. Simulation 2 : The single monitoring contaminated groundwater when there is no rate of change of pollutant concentration around the ended-point; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1\,(x,t) = (1+x)^2$ and $f_2\,(x,t) = 1 + x$. The boundary conditions and the initial condition are assumed $C(0,t) = 1$, $\frac{\partial C(1,t)}{\partial x} = 0$, for all $t \in [0,1]$ and $C(x,0) = 0$, for all $x \in [0,1]$, respectively. By employing the Saulyev finite difference technique Eq.(33) and Eq.(34) , we get the approximated chemical concentrations along 1 km from the point source in 0.7 year. The results are shown in Table VIII, where $\lambda = \Delta t \big/ (\Delta x)^2$. The surface plot of approximated solutions is illustrated in Fig 4.

TABLE VIII
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.84921 | 0.72410 | 0.62234 | 0.54142 | 0.47886 |
| 0.5 | 0.92416 | 0.86342 | 0.81526 | 0.77767 | 0.74899 |
| 0.7 | 0.94083 | 0.89443 | 0.85824 | 0.83034 | 0.80926 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.43232 | 0.39967 | 0.37899 | 0.36860 | 0.36700 |
| 0.5 | 0.72786 | 0.71313 | 0.70384 | 0.69918 | 0.69846 |
| 0.7 | 0.79384 | 0.78314 | 0.77642 | 0.77305 | 0.77254 |

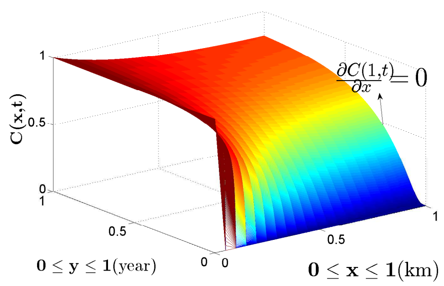


Fig. 4. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$ .

### *D. Simulation 3 : The single monitoring contaminated groundwater point when there is groundwater pollution flowing into the ended-point; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1\,(x,t) = (1+x)^2$ and $f_2\,(x,t) = 1+x$. The boundary conditions and the initial condition are assumed $C(0,t) = 1$,

$\frac{\partial C(1,t)}{\partial x} = 0.015$, for all $t \in [0,1]$ and $C(x,0) = 0$, for all $x \in [0,1]$, respectively. By employing the Saulyev finite difference technique Eq.(33) and Eq.(34) , we get the approximated chemical concentrations along 1 km from the point source in $0.7$ year. The results are shown in Table IX, where $\lambda = \Delta t \big/ (\Delta x)^2$. The surface plot of approximated solutions is illustrated in Fig 5.

TABLE IX
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.84922 | 0.72413 | 0.62238 | 0.54148 | 0.47894 |
| 0.5 | 0.92419 | 0.86347 | 0.81534 | 0.77777 | 0.74912 |
| 0.7 | 0.94087 | 0.89449 | 0.85832 | 0.83045 | 0.80940 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.43242 | 0.39978 | 0.37913 | 0.36875 | 0.36717 |
| 0.5 | 0.72801 | 0.71330 | 0.70403 | 0.69939 | 0.69869 |
| 0.7 | 0.79400 | 0.78332 | 0.77662 | 0.77328 | 0.77278 |

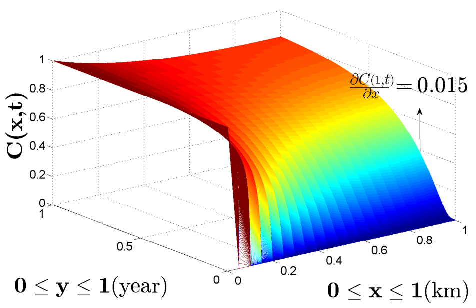


Fig. 5. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$ .

*E. Simulation 4 : The single monitoring contaminated groundwater point when there is groundwater pollution flowing outward at ended-point; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1\,(x,t) = (1+x)^2$ and $f_2\,(x,t) = 1+x$. The boundary conditions and the initial condition are assumed $C(0,t) = 1$, $\frac{\partial C(1,t)}{\partial x} = -0.015$, for all $t \in [0,1]$ and $C(x,0) = 0$, for all $x \in [0,1]$, respectively. By employing the Saulyev finite difference technique Eq.(33) and Eq.(34) , we get the approximated chemical concentrations along 1 km from the point source in $0.7$ year. The results are shown in Table X, where $\lambda = \Delta t \big/ (\Delta x)^2$. The surface plot of approximated solutions is illustrated in Fig 6.

TABLE X
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.84919 | 0.72407 | 0.62229 | 0.54136 | 0.47878 |
| 0.5 | 0.92413 | 0.86336 | 0.81518 | 0.77757 | 0.74886 |
| 0.7 | 0.94080 | 0.89437 | 0.85815 | 0.83023 | 0.80912 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.43223 | 0.39956 | 0.37886 | 0.36845 | 0.36683 |
| 0.5 | 0.72771 | 0.71296 | 0.70365 | 0.69897 | 0.69823 |
| 0.7 | 0.79368 | 0.78296 | 0.77621 | 0.77283 | 0.77229 |

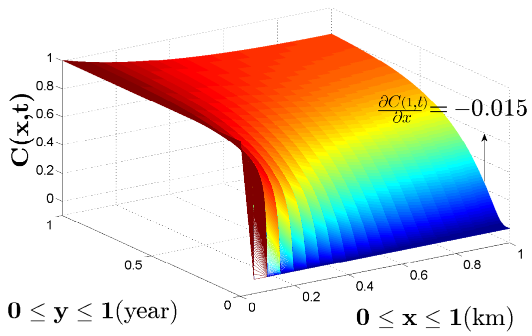


Fig. 6. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$ .

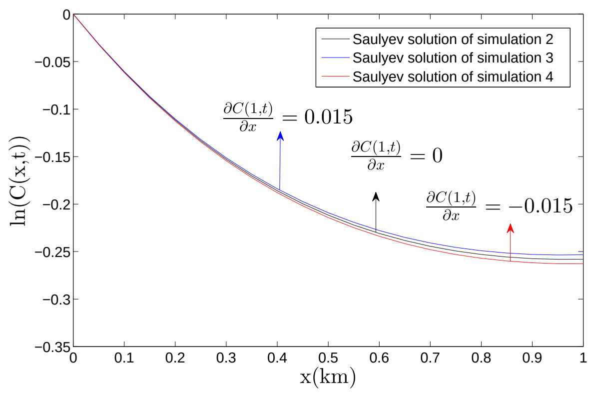


Fig. 7. The comparison of Saulyev solutions of simulation 2 , simulation 3 and simulation 4 when $t = 0.7$

*F. Simulation 5 : The single monitoring contaminated groundwater point when there is no rate of change of pollutant concentration around the ended-point in a high mixed soil topography; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement.*

Assuming that the chemical dispersion through high mixed inhomogeneous soil $f_1\,(x,t) = \sin(x)(1+x)^2$ , and $f_2\,(x,t) = \sin(x)\,(1+x)$ . The diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$). The boundary conditions and the initial condition are assumed $C(0,t) = 1$, $\frac{\partial C(1,t)}{\partial x} = 0$, for all $t \in [0,1]$ . and $C(x,0) = 0$, for all $x \in [0,1]$, respectively.

By employing the Saulyev finite difference technique Eq.(33) and Eq.(34), we get the approximated chemical concentrations along 1 km from the point source in 0.7 year. The results are shown in Table XI, where $\lambda = \Delta t \big/ (\Delta x)^2$. The surface plot of approximated solutions is illustrated in Fig 8.

TABLE XI
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.13435 | 0.03968 | 0.01457 | 0.00609 | 0.0028 |
| 0.5 | 0.23427 | 0.11088 | 0.06247 | 0.03896 | 0.026294 |
| 0.7 | 0.26724 | 0.14103 | 0.08805 | 0.06073 | 0.04527 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.00139 | 0.00075 | 0.00044 | 0.00031 | 0.00027 |
| 0.5 | 0.01909 | 0.01493 | 0.01261 | 0.01147 | 0.01115 |
| 0.7 | 0.03614 | 0.03072 | 0.02763 | 0.02610 | 0.02566 |

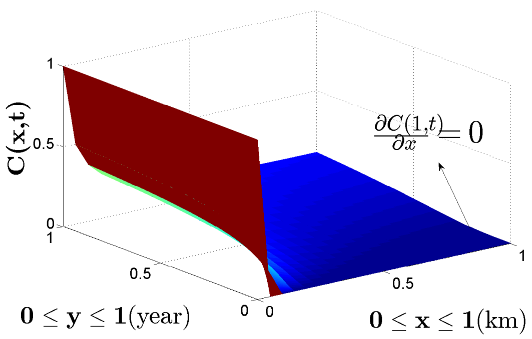


Fig. 8. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,1] \times [0,1]$ .

*G. Simulation 6 : The single monitoring contaminated groundwater point when there is no rate of change of pollutant concentration around the ended-point; Saulyev Explicit Finite Difference Scheme technique for an ideal contaminated groundwater dispersion measurement with long run simulation.*

Assuming that the chemical dispersion through inhomogeneous soil the diffusion coefficient and flow velocity field are averaged to be $D_0 = 0.71$ (km$^2$/year) and $u_0 = 0.6$ (km/year), respectively. The parameter that accounts for the inhomogeneity of the soil is assumed by $a = 1.0$ (km$^{-1}$) and $f_1\,(x,t) = (1+x)^2$ , and $f_2\,(x,t) = (1+x)$ . The boundary conditions and the initial condition are assumed $C(0,t) = 1$, $\frac{\partial C(3,t)}{\partial x} = 0$, for all $t \in [0, 16.67]$ . and $C(x,0) = 0$, for all $x \in [0,3]$ , respectively. By employing the Saulyev finite difference technique Eq.(33) and Eq.(34), we get the approximated chemical concentrations along 1 km from the point source in 0.7 year. The results are shown in Table XII, where $\lambda = \Delta t \big/ (\Delta x)^2$. The surface plot of approximated solutions is illustrated in Fig 9.

TABLE XII
THE APPROXIMATED CHEMICAL CONCENTRATION IN A HETEROGENEOUS SOIL

$(\Delta x = 0.05\,(km)\,,\ \Delta t = 0.000125\,(year),\ \lambda = 0.05)$

| t/x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.2 | 0.76944 | 0.62509 | 0.52613 | 0.45404 | 0.39931 |
| 0.5 | 0.76952 | 0.62536 | 0.52672 | 0.45508 | 0.40091 |
| 0.7 | 0.76952 | 0.62536 | 0.52673 | 0.45508 | 0.40091 |
| t/x | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.2 | 0.35679 | 0.32373 | 0.29912 | 0.28337 | 0.27849 |
| 0.5 | 0.35900 | 0.3266 | 0.30250 | 0.28713 | 0.28238 |
| 0.7 | 0.35900 | 0.32657 | 0.30250 | 0.28714 | 0.28238 |

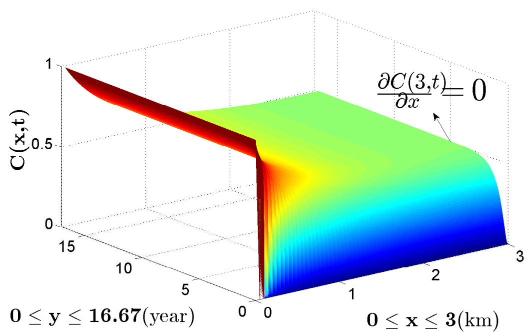


Fig. 9. The surface plot of computed chemical concentrations $\tilde{C}\,(x,t)$ for all $(x,t) \in [0,3] \times [0,16.67]$ .

## VI. DISCUSSION

In simulation 1.1, the approximated FTCS solutions are good agreement when $\lambda$ is divided by a half in 3 cases as show in Tables I-III. In simulation 1.2, if the Saulyev method is employ to approximate the groundwater pollutant concentration, it turn out that the solutions are closed to the FTCS solutions as show in Tables IV-VI. The comparison of both approximation techniques are illustrated in Fig 3. Both method give accurately approximated groundwater pollutant concentration but the FTCS gives unstable solutions when greater than 0.4 as show in Table VII. Although, The proposed Saulyev method still gives accurately groundwater pollutant concentration that is free from illustrated grid spacing. In simulations 2-4, the problems of the single monitoring contaminated groundwater point when there is no rate of change, there is positive rate of change and there is negative rate of change of pollutant concentration a the ended point are simulated as show in Tables VIII-X and Fig 4-6 we can see that the groundwater pollutant concentration in simulation 3 is higher than simulation 2 and 4, respectively as show Fig 7. In simulation 5, the realistic heterogeneous soil function are experimented such as $f_1$ and $f_2$, we can get the approximated groundwater pollutant concentration by using the proposed Saulyev method as show in Table XI and Fig 8. In the last simulation, the long-term situation is experimented. The considered domain is large as the realistic area, 3 km. The simulation time is long as prediction requirement, 16.67 year.

## VII. Conclusion

The numerical simulations to a one-dimensional groundwater pollution measurement model through heterogeneous soil are simulated. The numerical solutions for approximating the chemical concentration in heterogeneous medium are proposed. The forward time center space method and Saulyev finite difference technique are used to approximate the solution of the several simulations. The proposed finite difference technique gives good agreement approximated solutions under different conditions. We can see that the computed solutions are applicable to the real-world problems.

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