The Ergodicity and Extinction of Stochastic HIV Model with Saturated Incidence

Hui Miao, Chengjun Kang* and Meiyan Jiao

***Abstract*—In this paper, stochastic perturbations into HIV models with saturated incidence is proposed and discussed. Some sufficient conditions on the boundedness, extinction, persistence, and stability in distribution are established. Asymptotic behaviors around equilibria for infection-free and endemic of deterministic system are obtained. Finally, numerical simulations are carried out to validate our analytical results.**

***Index Terms*—Stochastic HIV model; Brownian motion; Extinction; Persistence; Stability in distribution.**

I. Introduction

HUMAN immunodeficiency virus (HIV) is one of the very dangerous viruses that infect the human and cause acquired immune deficiency syndrome (AIDS). There have been serious attempts from mathematicians and biologist to understand the dynamical behaviors of HIV in the human body by using mathematical models [1-10,16-17,19-20,24,27]. Mathematical models can be a useful tool for designing antiviral treatment strategies. It is well known that when HIV infects the CD4$^+$ T cell, the body's immune system is impaired and eventually loses its ability to fight other diseases. Therefore, the treatment of infected patients by HIV is of a great importance.

It is well known that HIV has the long incubation and infectious periods. In the absence of anti-retroviral therapy, the average time of progression from HIV infection to AIDS is 9-10 years, and the average survival time after developing AIDS is just 9.2 months. Moreover, the rate of clinical disease progression alters widely between individuals from two weeks up to 20 years. There are many factors which affect the rate of progression. These factors include the body's ability to defend against HIV, poor access to health care and existence of coexisting infections.

Upon infection with viruses, there are two main classes of anti-HIV drugs: (i) the reverse transcriptase inhibitors (RTIs) drugs, which prevent HIV from infecting the target cells; and (ii) the protease inhibitors (PIs) drugs, which prevent the infected cells from producing new infectious viruses. With antiretroviral therapy, treated people can live longer free of HIV-related symptoms. There is currently no cure for HIV. The only known methods of prevention are based on avoiding contact with the virus.

Manuscript received Aug 3, 2019; revised Oct 21, 2019. This work was supported by the National Natural Science Foundation of China (Grant nos. 11901363 and 11771373), the Youth Research Fund for the Shanxi basic research project (Grant nos. 201801D221002 and 2015021025), Scientific and Technological Innovation Project of Universities in Shanxi Province (Grant no. 2019L0489) and the Youth Research Fund for the Shanxi University of Finance and Economics (Grant no. QN-2018007).

Corresponding author: Chengjun Kang is with Faculty of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi, P.C. 030006 China e-mail: miaohuixju@163.com.

Hui Miao, Chengjun Kang and Meiyan Jiao are with Faculty of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi, P.C. 030006 China e-mail: chengjun0102@126.com.

Following above assumptions, we divide the host population into the uninfected CD4$^+$ T cells $x_1(t)$, latently infected CD4$^+$ T cells $x_2(t)$, infected CD4$^+$ T cells $x_3(t)$ and free viruses $x_4(t)$. To investigate effects among viral endemic infections, the viral model with the saturated incidence rate $\frac{\beta x_1 x_4}{1+mx_4}$ is described by

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= \lambda - \frac{\beta x_1(t)x_4(t)}{1+mx_4(t)} - \delta x_1(t),\\
\frac{dx_2(t)}{dt} &= \frac{k\beta x_1(t)x_4(t)}{1+mx_4(t)} - (\mu+\alpha)x_2(t),\\
\frac{dx_3(t)}{dt} &= \frac{(1-k)\beta x_1(t)x_4(t)}{1+mx_4(t)} + \alpha x_2(t) - ax_3(t),\\
\frac{dx_4(t)}{dt} &= cx_3(t) - \gamma x_4(t) - \frac{\eta\beta x_1(t)x_4(t)}{1+mx_4(t)},
\end{aligned} \tag{1}$$

where λ is the birth rate, β is the transmission coefficient between uninfected cells and infective virus particles, δ, μ, a and γ are the natural death rate of the uninfected CD4$^+$ T cells, latently infected CD4$^+$ T cells, infected CD4$^+$ T cells and free viruses, respectively. α is the rate of avtivation of latent cells. k is the fraction of infections leading to latency. c is the rate of virion emission by infected CD4$^+$ T cells. η is the fraction of infection rate per uninfected CD4$^+$ T cells. Throughout this paper, we assume that the parameters are all positive.

Obviously, model (1) always has the infection-free equilibrium $E_0 = (\frac{\lambda}{\delta}, 0, 0, 0)$. Define the basic reproduction number for viral infection $R_0 = \frac{\beta\lambda c(\mu+\alpha-k\mu)}{a(\mu+\alpha)(\beta\lambda\eta+\delta\gamma)}$. It is showed if $R_0 < 1$, then E_0 is globally asymptotically stable. If $R_0 > 1$, then E_0 is unstable, and there is an endemic equilibrium $E^* = (x_1^*, x_2^*, x_3^*, x_4^*)$, where $x_1^* = \frac{\lambda m(c(\mu+\alpha-k\mu)-a\eta(\mu+\alpha))+ak\delta\gamma(\mu+\alpha)}{(\beta+m\delta)(c(\mu+\alpha-k\mu)-a\eta(\mu+\alpha))}$, $x_2^* = \frac{k(\beta\lambda c(\mu+\alpha-k\mu)-a(\mu+\alpha)(\beta\lambda\eta+\delta\gamma))}{(c(\mu+\alpha-k\mu)-a\eta(\mu+\alpha))(\beta+m\delta)(\mu+\alpha)}$, $x_3^* = \frac{(\mu+\alpha-k\mu)x_2^*}{ak}$ and $x_4^* = \frac{k(\beta\lambda c(\mu+\alpha-k\mu)-a(\mu+\alpha)(\beta\lambda\eta+\delta\gamma))}{(\beta+m\delta)ak\gamma(\mu+\alpha)}$, which is globally asymptotically stable.

In fact, epidemic models are inevitably affected by environmental white noise which is an important component, because it can provide an additional degree of realism in compared to their deterministic counterparts. Therefore, many stochastic models for the epidemic populations have been developed. Here, we mainly mention three approaches. The first one is through time Markov chain model to consider environment noise [19,20]. The second is with parameters perturbation [15,18,21-23,26-29,31,32,34]. The last important issue to model stochastic epidemic model is to robust the positive equilibria of deterministic models. In this situation, it is mainly to investigate whether the stochastic model preserves the asymptotic stability properties of the positive equilibria of deterministic models [25,30,33].

In this paper, we introduce randomness into the model (1) by replacing the parameters δ, μ, a and γ by $\delta \to \delta +$

$\sigma_1 dB_1(t)$, $\mu \to \mu + \sigma_2 dB_2(t)$, $a \to a + \sigma_3 dB_3(t)$ and $\gamma \to \gamma + \sigma_4 dB_4(t)$ with the second approaches as [11]. Therefore, in this paper we consider the following stochastic HIV model with saturated incidence rate

$$\begin{aligned} dx_1(t) = & (\lambda - \frac{\beta x_1(t)x_4(t)}{1+mx_4(t)} - \delta x_1(t))dt \\ & +\sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = & (\frac{k\beta x_1(t)x_4(t)}{1+mx_4(t)} - (\mu+\alpha)x_2(t))dt \\ & +\sigma_2 x_2(t)dB_2(t), \\ dx_3(t) = & (\frac{(1-k)\beta x_1(t)x_4(t)}{1+mx_4(t)} + \alpha x_2(t) - a x_3(t))dt \\ & +\sigma_3 x_3(t)dB_3(t), \\ dx_4(t) = & (cx_3(t) - \gamma x_4(t) - \frac{\eta\beta x_1(t)x_4(t)}{1+mx_4(t)})dt \\ & +\sigma_4 x_4(t)dB_4(t), \end{aligned} \tag{2}$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$ and $B_4(t)$ are independent Brownian motions, σ_1, σ_2, σ_3 and σ_4 are their intensities.

The organization of this paper is as follows. In Section 2, we prove the positivity of the solution for model (2) which is essential in stochastic viral model. In Section 3, we drive the conditions which lead the disease to die out. In Section 4, we investigate that the solution of model (2) is stable in distribution under the condition $R_0 < 1$. In Section 5, we discuss the almost sure persistence of CD4$^+$ T cells. In the next two Sections, we obtain asymptotic behavior around the disease-free equilibrium E_0 and the endemic equilibrium E^* of the deterministic model (1), respectively. In Section 8, numerical simulations are carried out to illustrate the main theoretical results. A brief discussion is given in the end to conclude this work.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all Prob-null sets). Let $B(t)$ be the one-dimensional Brownian motion defined on this probability space. Also, let $R^4_+ = \{x \in R^4 \mid x_i(t) > 0 \text{ for all } 1 \leq i \leq 4\}$ and $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$.

II. Existence of unique global positive solution

To investigate the dynamical behavior of model (2), the most important concern is whether the solution is of global existence. The following Theorem shows that the solution of model (2) is global and positive.

Theorem 1 *For any initial value $x_0 \in R^4_+$, then model (2) has a unique solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ defined on $t \in [0, \infty)$ and the solution will remain in R^4_+ with probability one.*

Proof: Since the coefficients of model (2) are locally Lipschitz continuous, then for any given initial value $(x_1(0), x_2(0), x_3(0), x_4(0)) \in R^4_+$, model (2) has a unique local solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ defined on $t \in [0, \tau_e)$, where τ_e is the explosion time [12]. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 \geq 0$ be sufficiently large so that $(x_1(0), x_2(0), x_3(0), x_4(0))$ all lie within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\begin{aligned} \tau_m = & \inf\{t \in [0, \tau_e) \mid \min\{x_1(t), x_2(t), x_3(t), x_4(t)\} \\ & \leq \frac{1}{m} \text{ or } \max\{x_1(t), x_2(t), x_3(t), x_4(t)\} \geq m\}, \end{aligned}$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x_1(t), x_2(t), x_3(t), x_4(t)) \in R^4_+$ a.s. for all $t \geq 0$. In other words, to complete the proof, we need to show that $\tau_\infty = \infty$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \epsilon. \tag{3}$$

Hence there is an integer $m_1 \geq m_0$ such that $P\{\tau_m \leq T\} \geq \epsilon$ for all $m \geq m_1$.

Define a C^2-function $V(t) : R^4_+ \to R_+$ by $V(t) = \sum_{i=1}^4 (x_i + 1 - \log x_i)$.
The non-negativity of this function can be seen from $u + 1 - \log u \geq 0$, $\forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV(x(t)) = & \{\lambda - \delta x_1(t) - \frac{\lambda}{x_1(t)} + \delta + \frac{\beta x_4(t)}{1+mx_4(t)} \\ & -\mu x_2(t) - \frac{k\beta x_4(t)}{(1+mx_4(t))x_2(t)} + \mu + \alpha \\ & -a x_3(t) - \frac{(1-k)\beta x_1(t)x_4(t)}{(1+mx_4(t))x_3(t)} - \frac{\alpha x_2(t)}{x_3(t)} \\ & +a + cx_3(t) - \gamma x_4(t) - \frac{\eta\beta x_1(t)x_4(t)}{1+mx_4(t)} \\ & -\frac{cx_3(t)}{x_4(t)} + \gamma + \frac{\eta\beta x_1(t)}{1+mx_4(t)} + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 \\ & +\sigma_3^2 + \sigma_4^2)\}dt + \sigma_1(x_1(t) - 1)dB_1(t) \\ & +\sigma_2(x_2(t) - 1)dB_2(t) + \sigma_3(x_3(t) \\ & -1)dB_3(t) + \sigma_4(x_4(t) - 1)dB_4(t). \end{aligned}$$

Hence

$$\begin{aligned} dV(x(t)) \leq & [\lambda + \delta + \mu + \alpha + a + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \\ & +\sigma_4^2) + \eta\beta x_1(t) + \alpha x_2(t) + cx_3(t) \\ & +\beta x_4(t)]dt + \sigma_1(x_1(t) - 1)dB_1(t) \\ & +\sigma_2(x_2(t) - 1)dB_2(t) + \sigma_3(x_3(t) \\ & -1)dB_3(t) + \sigma_4(x_4(t) - 1)dB_4(t). \end{aligned}$$

Let $c_1 = \lambda + \delta + \mu + \alpha + a + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)$ and $c_2 = 2\eta\beta + 2\alpha + 2c + 2\beta$.

Since $x_i \leq 2(x_i + 1 - \log x_i)$, we have

$$\eta\beta x_1(t) + \alpha x_2(t) + cx_3(t) + \beta x_4(t) \leq c_2 V(x(t)).$$

Hence,

$$\begin{aligned} dV(x(t)) \leq & c_3(1 + V(x(t)))dt + \sigma_1(x_1(t) - 1)dB_1(t) \\ & +\sigma_2(x_2(t) - 1)dB_2(t) + \sigma_3(x_3(t) \\ & -1)dB_3(t) + \sigma_4(x_4(t) - 1)dB_4(t), \end{aligned} \tag{4}$$

where $c_3 = \max\{c_1, c_2\}$.

Integrating both sides of (4) from 0 to $\tau_m \wedge T$ and then

taking the expectations

$$\begin{aligned} & E[V(x(\tau_m \wedge T))] \\ \leq\ & E\int_0^{\tau_m \wedge T} (c_3(1+V(x(t)))) \, \mathrm{d}t + V(x_0) \\ \leq\ & V(x_0) + c_3 T + c_3 \int_0^T (EV(x(\tau_m \wedge T))) \, \mathrm{d}t. \end{aligned} \tag{5}$$

By the Gronwall inequality, we have $E[V(x(\tau_m \wedge T))] \leq c_4$, where $c_4 = (V(x_0) + c_3 T)e^{c_3 T}$. Set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$, and by (3), $P(\Omega_m) \geq \epsilon$. Note that for every $\omega \in \Omega_m$, there is some $i (1 \leq i \leq 4)$ such that $x_i\ (\tau_m, \omega)$ equals either m or $\frac{1}{m}$, then

$$V(x(\tau_m, \omega)) \geq (\tfrac{1}{m} + 1 - \log \tfrac{1}{m}) \wedge (m + 1 - \log m). \tag{6}$$

It then follows from (5) and (6) that

$$\begin{aligned} c_4 \geq\ & E[1_{\Omega_m}(\omega) V(x(\tau_m, \omega))] \\ \geq\ & \varepsilon([\tfrac{1}{m} + 1 - \log \tfrac{1}{m}] \wedge [m + 1 - \log m]), \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$, which leads to the contradiction $\infty > c_4 = \infty$. So, we have $\tau_\infty = \infty$ a.s. This completes the proof.

III. Extinction

In this section, we investigate the sufficient condition for the extinction of the disease.

Considering the matrix

$$A = \begin{pmatrix} M & P & -(\gamma+\mu)\eta \\ P & Q & N \\ -(\gamma+\mu)\eta & N & -2\gamma - \sigma_4^2 \end{pmatrix},$$

where

$$M = -(\sigma_2^2 + 2\mu)\eta^2,\ N = c - (a+\gamma)\eta,$$
$$P = c\eta - (a+\mu)\eta^2,\ Q = 2\eta c - (2a + \sigma_3^2)\eta^2.$$

Theorem 2 *Let $(x_1(t), x_2(t), x_3(t), x_4(t))$ be the solution of model (2) with initial value $x_0(t) \in R_+^4$. If*

- *(a) $(c\eta - (a+\mu)\eta^2)^2 < (2\mu + \sigma_2^2)\eta^2[(2a+\sigma_3^2)\eta^2 - 2c\eta]$;*
- *(b) $|A| < 0$,*

then $x_2(t)$, $x_3(t)$ and $x_4(t)$ tend to zero exponentially with probability one.

Proof: From model (2), we have

$$\begin{aligned} & d(\eta(x_2(t) + x_3(t)) + x_4(t)) \\ =\ & (-\mu\eta x_2(t) + (c - a\eta)x_3(t) - \gamma x_4(t))dt \\ & + \sigma_2 \eta x_2(t) dB_2(t) + \sigma_3 \eta x_3(t) dB_3(t) \\ & + \sigma_4 x_4(t) dB_4(t). \end{aligned}$$

Let $V(x) = \log(\eta(x_2(t) + x_3(t)) + x_4(t))$, using Itô's formula, we have

$$\begin{aligned} & dV(x) \\ =\ & \frac{1}{2(\eta(x_2(t)+x_3(t)) + x_4(t))^2}[2(\eta(x_2(t)+x_3(t)) \\ & + x_4(t))(-\mu\eta x_2(t) + (c - a\eta)x_3(t) - \gamma x_4(t)) \\ & - \sigma_2^2\eta^2 x_2^2(t) - \sigma_3^2 \eta^2 x_3^2(t) - \sigma_4^2 x_4^2(t)]dt \\ & + \frac{1}{\eta(x_2(t)+x_3(t)) + x_4(t)}(\sigma_2 \eta x_2(t) dB_2(t) \\ & + \sigma_3 \eta x_3(t) dB_3(t) + \sigma_4 x_4(t) dB_4(t)) \\ =\ & \frac{1}{2(\eta(x_2(t)+x_3(t)) + x_4(t))^2} \\ & \times \begin{pmatrix} x_2(t) & x_3(t) & x_4(t) \end{pmatrix} A \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} dt \\ & + \frac{1}{\eta(x_2(t)+x_3(t)) + x_4(t)}(\sigma_2 \eta x_2(t) dB_2(t) \\ & + \sigma_3 \eta x_3(t) dB_3(t) + \sigma_4 x_4(t) dB_4(t)). \end{aligned}$$

Under assumption (a) and (b), we obtain the matrix A negative-definite. Assume that λ_{max} is the largest eigenvalue, then

$$\begin{aligned} dV(x) \leq\ & -\mid \lambda_{max} \mid \frac{1}{2(\eta(x_2(t)+x_3(t)) + x_4(t))^2}(x_2^2(t) \\ & + x_3^2(t) + x_4^2(t))dt \\ & + \frac{1}{\eta(x_2(t)+x_3(t)) + x_4(t)}(\sigma_2 \eta x_2(t) dB_2(t) \\ & + \sigma_3 \eta x_3(t) dB_3(t) + \sigma_4 x_4(t) dB_4(t)). \end{aligned}$$

Since $(x_2(t) + x_3(t) + x_4(t))^2 \leq 2(x_2^2(t) + x_3^2(t) + x_4^2(t))$, we get

$$\begin{aligned} & d(\log(\eta(x_2(t) + x_3(t)) + x_4(t))) \\ \leq\ & -\frac{1}{4} \mid \lambda_{max} \mid dt + \frac{\sigma_2 \eta x_2(t)}{\eta(x_2(t)+x_3(t)) + x_4(t)} dB_2(t) \\ & + \frac{\sigma_3 \eta x_3(t)}{\eta(x_2(t)+x_3(t)) + x_4(t)} dB_3(t) \\ & + \frac{\sigma_4 x_4(t)}{\eta(x_2(t)+x_3(t)) + x_4(t)} dB_4(t). \end{aligned}$$

Integrating the above inequality and using $\limsup_{t\to\infty} \frac{1}{t} \mid B_i(t) \mid = 0$, for $i = 2, 3, 4$. So, $\limsup_{t\to\infty} \frac{1}{t} \log(\eta(x_2(t) + x_3(t)) + x_4(t)) \leq -\frac{1}{4}|\lambda_{max}| < 0$ *a.s.*. Hence, $x_2(t) \to 0, x_3(t) \to 0$ and $x_4(t) \to 0$ a.s. as $t \to \infty$. This completes the proof.

IV. Stability in distribution

In this section, we now concentrate on $x_1(t)$. We shall eventually show that $x_1(t)$ is stable in distribution in the sense that it stabilities around the mean value $\frac{\lambda}{\delta}$. To do this we introduce a new stochastic process $Z(t)$ which is defined by its initial condition $Z(0) = x_1(0)$ and the stochastic differential equation

$$dZ(t) = (\lambda - \delta Z(t))dt + \sigma_1 Z(t) dB_1(t). \tag{7}$$

Lemma 1 *([13]) Consider a diffusion $dX(t) = BX(t)dt + \sigma(X(t))dB(t)$, where B is a $k \times k$ matrix, $\sigma(\cdot)$ is a Lipschitzian $(k \times l)$-matrix-valued function on R^k and $\{W(t) : t \geq 0\}$ is a standard l-dimensional Brownian motion. Let $a(x, y) = (\sigma(x) - \sigma(y))(\sigma(x) - \sigma(y))'$. If there exists a symmetric positive definite matrix C and a positive constant γ such that*

$$\begin{aligned} & 2C(x-y)B(x-y) - \frac{2C(x-y)a(x,y)C(x-y)}{(x-y)C(x-y)} \\ & + tr(a(x,y)C) \\ \leq\ & -\gamma \mid x - y \mid^2, x \neq y, \end{aligned}$$

then the diffusion is a stable in distribution.

Theorem 3 *Under the conditions of Theorem 1, then* $\lim_{t\to\infty}(Z(t) - x_1(t)) = 0$ *in probability, a.e., $Z(t) \to^w \upsilon$, as $t \to \infty$,*

where $\overset{w}{\to}$ *means the convergence in distribution and* υ *is a probability measure in* R_+^1 *such that* $\int_0^\infty x\upsilon(dx) = \frac{\lambda}{\delta}$. *In particularly,* υ *has density* $(A\sigma_1^2x^2p(x))^{-1}$, *where A is a normal constant,* $p(x) = e^{(-\frac{2\lambda}{\delta})}x^{\frac{2\delta}{\sigma_1^2}}e^{\frac{2\lambda}{\sigma_1^2 x}}, x > 0$.

Proof: By comparison theorem, we see that $x_1(t) \le Z(t)$, where $Z(t)$ is the global solution of equation (7) with initial value $Z(0) = x_1(0)$.

Firstly, we show (7) is stable in distribution and ergodic. Let $Y(t) = Z(t) - \frac{\lambda}{\delta}$, then $Y(t)$ satisfies

$$dY(t) = -\delta Y(t)dt + \sigma_1(Y(t) + \frac{\lambda}{\delta})dB_1(t). \tag{8}$$

Since Lemma 1, when $C = 1$ implies that the diffusion process $Y(t)$ is stable in distribution as $t \to \infty$, so does $Z(t)$.

To prove the ergodicity of $Z(t)$, we define $p(z) = \exp(-2\int_1^z \frac{\lambda-\delta y}{\sigma_1^2 y^2}\,\mathrm{d}y)$. By computation, $p(z) = \exp(-\frac{2\lambda}{\sigma_1^2})z^{\frac{2\delta}{\sigma_1^2}}\exp(\frac{2\lambda}{\sigma_1^2 z})$, and it is noted that for each integer $n \ge 1$, there exist positive constants $C_1(n)$, $C_2(n)$ and $M(n)$ such that

$$\begin{cases} p(z) \ge C_1(n)z^{\frac{2\delta}{\sigma_1^2}-n}, & \text{as } 0 < z < \frac{1}{M(n)}, \\ p(z) \ge C_2(n)z^{\frac{2\delta}{\sigma_1^2}}, & \text{as } z > M(n). \end{cases} \tag{9}$$

Therefore, with (9) we see

$$\int_1^\infty p(z)\,\mathrm{d}z = \infty,\ \int_0^1 p(z)\,\mathrm{d}z = \infty,$$
$$\int_0^\infty \frac{1}{\sigma_1^2 p(z)z^2}\,\mathrm{d}z < \infty.$$

So $Z(t)$ is ergodic (Theorem 1.16 in [14]), and with respect to the Lebesgue measure its invariant measure υ has density $(A\sigma_1^2z^2p(z))^{-1}$, where A is a normal constant.

Now, we show that $f(t) = EZ^p(t)$ is uniformly bounded for some $p > 1$ determined later. Applying Itô's formula to Z^p, we have $dZ^p(t) = (\lambda pZ^{p-1} - \delta pZ^p + \frac{p(p-1)\sigma_1^2Z^p}{2})dt + p\sigma_1Z^pdB_1(t)$. Taking expectation of equation above, and using $a^{\frac{1}{p}}b^{\frac{p-1}{p}} \le \frac{a}{p} + \frac{b(p-1)}{p}, a, b > 0$, then $f'(t) \le \lambda^p + p[\frac{p-1}{p} - (\delta - \frac{\sigma_1^2(p-1)}{2})]f(t)$. Choosing $p > 1$ close enough to 1 such that $\frac{p-1}{p} - (\delta - \frac{\sigma_1^2(p-1)}{2}) < 0$, then $\sup_{t\ge0} EZ^p = \sup_{t\ge0} f(t) < \infty$, implying that $\int_0^\infty z^p\upsilon(dz) < \infty$.

By the ergodic theorem, we have

$$p\{\lim_{T\to\infty}\frac{1}{T}\int_0^T Z(t)\,\mathrm{d}t = \int_0^\infty z\upsilon(dz)\} = 1, \tag{10}$$

for all $z \in R_+^1$, with Jensen's inequality yields

$$E[\frac{1}{T}\int_0^T Z(t)\,\mathrm{d}t]^p \le E\frac{1}{T}\int_0^T Z^p(t)\,\mathrm{d}t \le \sup_{t\ge0} EZ^p(t) < \infty.$$

Therefore, $\{\frac{1}{T}\int_0^T Z(t)\,\mathrm{d}t, T \ge 0\}$ is uniformly integrable. Together with (10), so we have

$$E\frac{1}{T}\int_0^T Z(t)\,\mathrm{d}t \to \int_0^\infty z\upsilon(dz). \tag{11}$$

Taking expectation of (7), we have $\frac{EZ(t)}{t} = \lambda - \frac{\delta}{t}E\int_0^t Z(s)\,\mathrm{d}s$. Let $t \to \infty$, taking (11) into account, so yields $\int_0^\infty z\upsilon(dz) = \frac{\lambda}{\delta}$.

At last, we concentrate on $x_1(t)$. We shall show that $x_1(t)$ is stable in distribution. To do this, we introduce a new stochastic process $x_\epsilon(t)$ which is defined by its initial condition $x_1(0) = x_\epsilon(0)$ and the stochastic differential equation

$$dx_\epsilon(t) = (\lambda - (\delta+\varepsilon)x_\epsilon(t))dt + \sigma_1x_\epsilon(t)dB_1(t).$$

First we prove that $\liminf_{t\to\infty}(x_1(t) - x_\epsilon(t)) \ge 0, \quad a.s.$

Therefore, we consider

$$\begin{aligned} & d(x_1(t) - x_\epsilon(t)) \\ = \ & [-(\delta+\varepsilon)(x_1(t) - x_\epsilon(t)) + (\varepsilon - \frac{\beta x_4(t)}{1+mx_4(t)})x_1(t)]dt \\ & +\sigma_1(x_1(t) - x_\epsilon(t))dB_1(t). \end{aligned}$$

The solution is given by

$$x_1(t) - x_\epsilon(t) = \psi(t)\int_0^t \psi^{-1}(s)(\varepsilon - \frac{\beta x_4(s)}{1+mx_4(s)})x_1(s)\,\mathrm{d}s,$$

where $\psi(t) = \exp\{-(\delta+\varepsilon+\frac{1}{2}\sigma_1^2)t + \sigma_1B_1(t)\}$. Due to Theorem 2, it has been shown $x_4(t) \to 0$ $a.s.$ as $t \to \infty$. For almost all $\omega \in \Omega, \exists T = T(\omega)$ such that $x_4(t) < \frac{\epsilon}{\beta - m\epsilon}, \forall t \ge T$. Hence for all $\omega \in \Omega$, if $t > T$, then

$$\begin{aligned} & x_1(t) - x_\epsilon(t) \\ = \ & \psi(t)(\int_0^T \psi^{-1}(s)(\varepsilon - \frac{\beta x_4(t)}{1+mx_4(t)})x_1(t)\,\mathrm{d}s \\ & + \int_T^t \psi^{-1}(s)(\varepsilon - \frac{\beta x_4(t)}{1+mx_4(t)})x_1(t)\,\mathrm{d}s). \end{aligned}$$

Hence $x_1(t) - x_\epsilon(t) \ge \psi(t)\kappa(T)$, where $\kappa(T) = \int_0^T \psi^{-1}(s)(\varepsilon - \frac{\beta x_4(s)}{1+mx_4(s)})x_1(s)\,\mathrm{d}s$. So, $\mid \kappa(T) \mid < \infty$ and $\psi(t) \to 0$ $a.s.$

Therefore,

$$\liminf_{t\to\infty}(x_1(t) - x_\epsilon(t)) \ge 0,\ a.s.$$

Next, we consider

$$\begin{aligned} & d(Z(t) - x_\epsilon(t)) \\ = \ & [-\delta Z(t) + (\delta+\varepsilon)x_\epsilon(t)]dt + \sigma_1(Z(t) - x_\epsilon(t))dB_1(t). \end{aligned}$$

Taking expectation of above equation, we see

$$\begin{aligned} & E \mid Z(T) - x_\epsilon(T) \mid \\ = \ & E\int_0^T [\varepsilon x_\epsilon(t) - \delta(Z(t) - x_\epsilon(t))]dt \\ \le \ & E\int_0^T [\varepsilon Z(t) - \delta \mid Z(t) - x_\epsilon(t) \mid]dt, \end{aligned}$$

where the last inequality is using the fact that $Z(t) \ge x_\epsilon(t)$. Hence, we have

$$E \mid Z(T) - x_\epsilon(T) \mid \le \frac{\epsilon \sup_{u\ge0} EZ(u)}{\delta}(1 - exp(-\delta T)).$$

This implies,

$$\liminf_{\epsilon\to0}\lim_{T\to\infty} E \mid Z(T) - x_\epsilon(T) \mid = 0. \tag{12}$$

Combining (12) and the fact that $x_1(t) \le Z(t)$, we get

$$\lim_{t\to\infty}(x_1(t) - Z(t)) = 0 \quad \text{in probability.}$$

Since $Z(t)$ converges weakly to distribution ν, so does $x_1(t)$ as $t \to \infty$.

V. PERSISTENCE

In this section, we can further have the almost sure persistence of the total CD4$^+$ T cells of model (2).

Set $X(t) = x_1(t) + x_2(t) + x_3(t)$.

Theorem 4 *If $\lambda > 0$, then for any given initial $(x_1(0), x_2(0), x_3(0), x_4(0))$ the solution of model (2) obeys $0 < \liminf_{t\to\infty} X(t) \le \limsup_{t\to\infty} X(t) < +\infty$.*

Proof: It follows from that

$$\begin{aligned} dX(t) = {} & (\lambda - \delta x_1(t) - \mu x_2(t) - a x_3(t))dt \\ & + \sigma_1 x_1(t) dB_1(t) + \sigma_2 x_2(t) dB_2(t) \\ & + \sigma_3 x_3(t) dB_3(t) \\ = {} & [\lambda - \delta X(t) + (\delta - \mu) x_2(t) + (\delta - a) x_3(t)]dt \\ & + \sigma_1 x_1(t) dB_1(t) + \sigma_2 x_2(t) dB_2(t) \\ & + \sigma_3 x_3(t) dB_3(t). \end{aligned}$$

Define a function $V(X(t)) = \frac{1}{X(t)}$ and by the Itô's formula, we have

$$\begin{aligned} dV(X(t)) = {} & -\frac{1}{X^2(t)} dX(t) + \frac{1}{X^3(t)} (dX(t))^2 \\ = {} & -\frac{1}{X^2(t)} (\lambda - \delta X(t) + (\delta - \mu) x_2(t) \\ & + (\delta - a) x_3(t))dt + \frac{1}{X^3(t)} (\sigma_1^2 x_1^2(t) \\ & + \sigma_2^2 x_2^2(t) + \sigma_3^2 x_3^2(t))dt \\ & - \frac{1}{X^2(t)} (\sigma_1 x_1(t) dB_1(t) + \sigma_2 x_2(t) dB_2(t) \\ & + \sigma_3 x_3(t) dB_3(t)). \end{aligned}$$

Since $\frac{x_1(t)}{X(t)} \le 1$, $\frac{x_2(t)}{X(t)} \le 1$ and $\frac{x_3(t)}{X(t)} \le 1$, then

$$\begin{aligned} d\tfrac{1}{X(t)} e^t \le {} & [-\frac{\lambda}{X^2(t)} + \frac{a + \mu - \delta}{X(t)} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{X(t)} \\ & + \frac{1}{X(t)}] e^t dt - \frac{e^t}{X^2(t)} (\sigma_1 x_1(t) dB_1(t) \\ & + \sigma_2 x_2(t) dB_2(t) + \sigma_3 x_3(t) dB_3(t)). \end{aligned}$$

It is easy to obtain

$$\begin{aligned} \tfrac{1}{X(t)} \le {} & \frac{1}{X(0)} e^{-t} + e^{-t} \int_0^t e^u H(X(t))\, \mathrm{d}u \\ & - e^{-t} [\int_0^t \frac{e^u}{X^2(u)} \sigma_1 x_1(u) dB_1(u) \\ & + \int_0^t \frac{e^u}{X^2(u)} \sigma_2 x_2(u) dB_2(u) \\ & + \int_0^t \frac{e^u}{X^2(u)} \sigma_3 x_3(u) dB_3(u)]. \end{aligned}$$

Obviously, if $\lambda > 0$, then we have $H(X(t)) \le M_1$ for all $X(t) > 0$.

In fact, we only need to prove the next two equations

$$P(\liminf_{t\to\infty} X(\omega, t) = 0) = 0 \tag{13}$$

and

$$P(\limsup_{t\to\infty} X(\omega, t) = +\infty) = 0. \tag{14}$$

We now begin to prove assertion (13). If it is not true, then there is a sufficiently small $\epsilon \in (0, 1)$ such that

$$P(\Omega_1) > \epsilon, \tag{15}$$

where $\Omega_1 = \{\omega \mid \liminf_{t\to\infty} X(\omega, t) = 0\}$. Define the stopping time

$$\tau_k = \inf\{t \ge 0 \mid X(\omega, t) \le \frac{1}{k},\ \omega \in \Omega_1\},\ k \in Z^+.$$

According to the definition, τ_k is increasing and $\tau_k \to \infty$ as $k \to \infty$. It then follows that

$$\begin{aligned} & E(\frac{1}{X(\tau_k)} 1_{\Omega_1}) \\ \le {} & E(\frac{1}{X(0)} e^{-\tau_k} + e^{-\tau_k} \int_0^{\tau_k} e^u H(X(t))\, \mathrm{d}u) \qquad (16) \\ \le {} & \frac{1}{X(0)} + M_1. \end{aligned}$$

However,

$$E(\frac{1}{X(\tau_k)} 1_{\Omega_1}) \ge k E(1_{\Omega_1}) \ge k\epsilon.$$

Letting $k \to \infty$, leads to yield $E(\frac{1}{X(\tau_k)} 1_{\Omega_1}) \ge k E(1_{\Omega_1}) \ge k\epsilon \to \infty$, as $k \to \infty$.

But this contradicts (16). We therefore have the desired assertion (13), namely,

$$\liminf_{t\to\infty} X(\omega, t) > 0\ a.s.$$

On the other hand, it follows

$$\begin{aligned} X(t) = {} & X(0) e^{-\mu t} + e^{-\mu t} \int_0^t e^{-\mu u} [\lambda - \delta X(t) \\ & + (\delta - \mu) x_2(t) + (\delta - a) x_3(t)]\, \mathrm{d}u \\ & + e^{-\mu t} \int_0^t e^{-\mu u} \sigma_1 x_1(u) dB_1(u) \\ & + e^{-\mu t} \int_0^t e^{-\mu u} \sigma_2 x_2(u) dB_2(u) \\ & + e^{-\mu t} \int_0^t e^{-\mu u} \sigma_3 x_3(u) dB_3(u). \end{aligned}$$

If it is not true, then there is a sufficiently small $\delta \in (0, 1)$ such that

$$P(\Omega_2) > \delta, \tag{17}$$

where $\Omega_2 = \{\omega \mid \limsup_{t\to\infty} X(\omega, t) = +\infty\}$. Define the stopping time

$$\tau_k = \inf\{t \ge 0 \mid X(\omega, t) \ge k, \omega \in \Omega_1\}, k \in Z^+.$$

Integrating both sides and taking the expectations, we obtain

$$\begin{aligned} E(X(\tau_k)) \le {} & E(X(0)) e^{-\tau_k \delta} + e^{-\tau_k \delta} \int_0^{\tau_k} e^{u\delta} [\lambda \\ & + (\delta - \mu) x_2(u) + (\delta - a) x_3(u)]\, \mathrm{d}u \qquad (18) \\ \le {} & X(0) + M_2. \end{aligned}$$

Obviously, if $\lambda > 0$, then we have $\lambda + (\delta - \mu) x_2(t) + (\delta - a) x_3(t) \le M_2$ for all $x_2(t) > 0$ and $x_3(t) > 0$.
However, $E(X(\tau_k) 1_{\Omega_2}) \ge k E(1_{\Omega_2}) \ge k\delta \to \infty$, as $k \to \infty$.

But this contradicts (18). Therefore, we have the desired assertion (14), namely,

$$\limsup_{t\to\infty} X(\omega, t) < +\infty\ a.s.$$

This completes the proof.

VI. Asymptotic behavior around E_0 of deterministic model (1)

As mentioned in model (1), if $R_0 < 1$, then model (1) has a globally asymptotically stable disease-free equilibrium E_0, which means the disease will die out with the advancement of time. Noting that E_0 is not an equilibrium of stochastic model (2), it is natural to ask whether the disease will go to extinction in the population. In this section we mainly use the way of estimating the oscillation around E_0 to reflect how the solution of model (2) spirals closely around E_0. We have the following Theorem.

Theorem 5 *Let $(x_1(t), x_2(t), x_3(t), x_4(t))$ be the solution of model (2) with initial value $x(0) \in R_+^4$. If $R_0 < 1, \delta > 2\sigma_1^2 + \frac{a+\mu}{2}, \mu > \frac{\sigma_2^2(1+\eta^2)+\eta(c-a\eta)+\delta}{2\eta^2+1}, a > \frac{\sigma_3^2(1+\eta^2)+c+3c\eta+\delta}{1+\eta+3\eta^2}, \gamma > \frac{1}{2}\sigma_4^2 + \frac{c-a\eta}{2}$, then*

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [(\delta - 2\sigma_1^2 - \frac{a+\mu}{2})(x_1 - \frac{\lambda}{\delta})^2\\ &+(\mu\eta^2 - \frac{1}{2}\sigma_2^2\eta^2 - \frac{\eta(c-a\eta)}{2} - \frac{\delta-\mu}{2} - \frac{1}{2}\sigma_2^2)x_2^2\\ &+(-\frac{3\eta(c-a\eta)}{2} - \frac{c-a\eta}{2} - \frac{1}{2}\sigma_3^2\eta^2 - \frac{\delta-a}{2} - \frac{1}{2}\sigma_3^2)x_3^2\\ &+(\gamma - \frac{1}{2}\sigma_4^2 - \frac{c-a\eta}{2})x_4^2]dt \le 2\sigma_1^2(\frac{\lambda}{\delta})^2.\end{aligned}$$

Proof: Define the function $V(t)$ as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t),$$

where $V_1(t) = \frac{(x_1 - \frac{\lambda}{\delta})^2}{2}, V_2(t) = \frac{(x_1 - \frac{\lambda}{\delta} + x_2 + x_3)^2}{2}, V_3(t) = \frac{(\eta(x_2+x_3)+x_4)^2}{2}$.

Applying Itô's formula, we can obtain

$$\begin{aligned}LV_1(x) =& (x_1 - \frac{\lambda}{\delta})(\lambda - \frac{\beta x_1 x_4}{1+mx_4} - \delta x_1) + \frac{1}{2}\sigma_1^2 x_1^2\\ =& -\delta(x_1 - \frac{\lambda}{\delta})^2 - \frac{\beta(x_1 - \frac{\lambda}{\delta})^2 x_4}{1+mx_4}\\ &-\frac{\beta\lambda(x_1 - \frac{\lambda}{\delta})x_4}{\delta(1+mx_4)} + \frac{\sigma_1^2 x_1^2}{2}\\ \le& -\delta(x_1 - \frac{\lambda}{\delta})^2 + \frac{\sigma_1^2 x_1^2}{2},\end{aligned}$$

$$\begin{aligned}LV_2(x) =& (x_1 - \frac{\lambda}{\delta} + x_2 + x_3)(\lambda - \delta x_1 - \mu x_2 - a x_3)\\ &+\frac{1}{2}\sigma_1^2 x_1^2 + \frac{1}{2}\sigma_2^2 x_2^2 + \frac{1}{2}\sigma_3^2 x_3^2\\ =& -\delta(x_1 - \frac{\lambda}{\delta})^2 - (\delta+\mu)x_2(x_1 - \frac{\lambda}{\delta})\\ &-(a+\mu)x_2 x_3 - (a+\delta)x_3(x_1 - \frac{\lambda}{\delta})\\ &-\mu x_2^2 - a x_3^2 + \frac{1}{2}\sigma_1^2 x_1^2 + \frac{1}{2}\sigma_2^2 x_2^2 + \frac{1}{2}\sigma_3^2 x_3^2\\ \le& (\frac{a+\mu}{2} + \sigma_1^2)(x_1 - \frac{\lambda}{\delta})^2 + (\frac{\delta-\mu}{2} + \frac{1}{2}\sigma_2^2)x_2^2\\ &+(\frac{\delta-a}{2} + \frac{1}{2}\sigma_3^2)x_3^2 + \sigma_1^2(\frac{\lambda}{\delta})^2,\end{aligned}$$

$$\begin{aligned}LV_3(x) =& (\eta(x_2+x_3)+x_4)(-\mu\eta x_2 + (c-a\eta)x_3\\ &-\gamma x_4) + \frac{1}{2}\sigma_2^2\eta^2 x_2^2 + \frac{1}{2}\sigma_3^2\eta^2 x_3^2 + \frac{1}{2}\sigma_4^2 x_4^2\\ \le& -\mu\eta^2 x_2^2 + \eta(c-a\eta)x_2 x_3 + \eta(c-a\eta)x_3^2\\ &+(c-a\eta)x_4 x_3 - \gamma x_4^2\\ &+\frac{1}{2}\sigma_2^2\eta^2 x_2^2 + \frac{1}{2}\sigma_3^2\eta^2 x_3^2 + \frac{1}{2}\sigma_4^2 x_4^2\\ \le& -\mu\eta^2 x_2^2 + \frac{\eta(c-a\eta)}{2}x_2^2 + \frac{\eta(c-a\eta)}{2}x_3^2\\ &+\eta(c-a\eta)x_3^2 + \frac{c-a\eta}{2}x_4^2 + \frac{c-a\eta}{2}x_3^2 - \gamma x_4^2\\ &+\frac{1}{2}\sigma_2^2\eta^2 x_2^2 + \frac{1}{2}\sigma_3^2\eta^2 x_3^2 + \frac{1}{2}\sigma_4^2 x_4^2\\ =& (-\mu\eta^2 + \frac{\eta(c-a\eta)}{2} + \frac{1}{2}\sigma_2^2\eta^2)x_2^2\\ &+(\frac{c-a\eta}{2} + \frac{3\eta(c-a\eta)}{2} + \frac{1}{2}\sigma_3^2\eta^2)x_3^2\\ &+(\frac{c-a\eta}{2} - \gamma + \frac{1}{2}\sigma_4^2)x_4^2.\end{aligned}$$

By computing, we have

$$\begin{aligned}LV(x) \le& (-\delta + 2\sigma_1^2 + \frac{a+\mu}{2})(x_1 - \frac{\lambda}{\delta})^2 + (-\mu\eta^2\\ &+\frac{1}{2}\sigma_2^2\eta^2 + \frac{\eta(c-a\eta)}{2} + \frac{\delta-\mu}{2}\\ &+\frac{1}{2}\sigma_2^2)x_2^2 + (\frac{3\eta(c-a\eta)}{2} + \frac{c-a\eta}{2} + \frac{1}{2}\sigma_3^2\eta^2\\ &+\frac{\delta-a}{2} + \frac{1}{2}\sigma_3^2)x_3^2\\ &+(-\gamma + \frac{1}{2}\sigma_4^2 + \frac{c-a\eta}{2})x_4^2]dt + 2\sigma_1^2(\frac{\lambda}{\delta})^2.\end{aligned}$$

Taking expectation above, yields

$$\begin{aligned}&EV(t) - V(0)\\ =& E\int_0^t LV(r)\,\mathrm{d}r\\ \le& (-\delta + 2\sigma_1^2 + \frac{a+\mu}{2})E\int_0^t (x_1(r) - \frac{\lambda}{\delta})^2\,\mathrm{d}r\\ &+(-\mu\eta^2 + \frac{1}{2}\sigma_2^2\eta^2 + \frac{\eta(c-a\eta)}{2} + \frac{\delta-\mu}{2}\\ &+\frac{1}{2}\sigma_2^2)E\int_0^t x_2^2(r)\,\mathrm{d}r + (\frac{3\eta(c-a\eta)}{2} + \frac{c-a\eta}{2}\\ &+\frac{1}{2}\sigma_3^2\eta^2 + \frac{\delta-a}{2} + \frac{1}{2}\sigma_3^2)E\int_0^t x_3^2(r)\,\mathrm{d}r\\ &+(-\gamma + \frac{1}{2}\sigma_4^2 + \frac{c-a\eta}{2})E\int_0^t x_4^2(r)\,\mathrm{d}r + 2\sigma_1^2(\frac{\lambda}{\delta})^2 t.\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [(\delta - 2\sigma_1^2 - \frac{a+\mu}{2})(x_1 - \frac{\lambda}{\delta})^2\\ &+(\mu\eta^2 - \frac{1}{2}\sigma_2^2\eta^2 - \frac{\eta(c-a\eta)}{2} - \frac{\delta-\mu}{2} - \frac{1}{2}\sigma_2^2)x_2^2\\ &+(-\frac{3\eta(c-a\eta)}{2} - \frac{c-a\eta}{2} - \frac{1}{2}\sigma_3^2\eta^2 - \frac{\delta-a}{2} - \frac{1}{2}\sigma_3^2)x_3^2\\ &+(\gamma - \frac{1}{2}\sigma_4^2 - \frac{c-a\eta}{2})x_4^2]dt \le 2\sigma_1^2(\frac{\lambda}{\delta})^2.\end{aligned}$$

This completes the proof.

VII. Asymptotic behavior around E^* of deterministic model (1)

In studying virus infection model, we are interested in two problems. One is the occurring of extinction, which has been shown in the above part, another is the persistent presence in a population. In the deterministic models, the second problem is solved by showing that the endemic equilibrium of corresponding model is a global attractor or is globally asymptotic stable. But, there is none of endemic equilibrium in model (2). We obtain a unique stationary distribution of model (2) instead of the endemic equilibrium (see [34]). Furthermore, since model (2) is the perturbed model, model (1) has an endemic equilibrium E^*, it seems reasonable to

consider the disease will prevail if the solution of model (2) has the ergodic property. Before giving the main Theorem, we first give a lemma (see [35]).

Let $X(t)$ be a regular temporally homogeneous Markov process in $E_l \subset R^l$ described by the stochastic differential equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} f_r dB_r(t),$$

and the diffusion matrix is defined as follows:

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k} f_r^i(x) f_r^j(x).$$

Lemma 2 *We assume that there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

$(B.1)$ *It suffices to prove V is uniformly elliptical in U, where $Vu = b(x)\nu_x + [tr(A(x))u_{xx}]/2$, i.e. there is a positive number M such that*

$$\sum_{i,j=1}^{k} a_{i,j}\xi_i\xi_j \geq M \mid \xi \mid^2$$

(see [33] and Rayleigh's principle in [36]).

$(B.2)$ *It is sufficient to show that there exists some neighborhood U and a non-negative C^2-function such that LV is negative at every point $x \in E_l \backslash U$ like in [37].*

Then, the Markov process $X(t)$ has a stationary distribution $\nu(.)$ with density in E_l such that for any Borel set $B \subset E_l$, $\lim_{t\to\infty} P(t, x, B) = \nu(B)$, and

$$P_x\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))\,\mathrm{d}t\} = \int_{E_l} f(x)\nu(dx)\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the measure ν.

Theorem 6 *Let $(x_1(t), x_2(t), x_3(t), x_4(t))$ be the solution of model (2) with initial value $x(0) \in \Gamma$. If $R_0 > 1$ and $0 < \rho < \min(k_1x_1^{*2}, k_2x_2^{*2}, k_3x_3^{*2}, k_4x_4^{*2})$, then there is a unique stationary distribution ν for model (2) and the ergodicity holds. Here $(x_1^*, x_2^*, x_3^*, x_4^*)$ is the unique endemic equilibrium of model (1), where*

$$\begin{aligned}
&k_1 = \delta - \sigma_1^2 > 0,\\
&k_2 = \mu(1+\eta) - \tfrac{\eta(c-a\eta)}{2} - \sigma_2^2(\eta^2+1) > 0,\\
&k_3 = a - \tfrac{3\eta(c-a\eta)}{2} - \tfrac{c-a\eta}{2} - \sigma_3^2(\eta^2+1) > 0,\\
&k_4 = \gamma - \sigma_4^2 - \tfrac{c-a\eta}{2} > 0,\\
&\rho = \sigma_1^2x_1^{*2} + \sigma_2^2(1+\eta^2)x_2^{*2} + \sigma_3^2(1+\eta^2)x_3^{*2} + \sigma_4^2x_4^{*2}.
\end{aligned}$$

Especially, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \tfrac{1}{t}E\textstyle\int_0^t [k_1(x_1-x_1^*)^2 + k_2(x_2-x_2^*)^2\\
&+k_3(x_3-x_3^*)^2 + k_4(x_4-x_4^*)^2]dr \leq \rho.
\end{aligned}$$

Proof: As $R_0 > 1$, there is the unique endemic equilibrium $E^*(x_1^*, x_2^*, x_3^*, x_4^*)$ such that

$$\begin{aligned}
&\lambda = \delta x_1^* + \tfrac{\beta x_1^* x_4^*}{1+mx_4^*},\ \tfrac{k\beta x_1^* x_4^*}{1+mx_4^*} = (\alpha+u)x_2^*,\\
&\tfrac{(1-k)\beta x_1^* x_4^*}{1+mx_4^*} = ax_3^* - \alpha x_2^*,\ \tfrac{\eta\beta x_1^* x_4^*}{1+mx_4^*} = cx_3^* - \gamma x_4^*.
\end{aligned}$$

Define the function $V(t)$ as follows

$$V(t) = V_1(t) + V_2(t),$$

where

$$\begin{aligned}
V_1(t) &= \tfrac{(x_1+x_2+x_3-x_1^*-x_2^*-x_3^*)^2}{2},\\
V_2(t) &= \tfrac{(\eta x_2+\eta x_3+x_4-\eta x_2^*-\eta x_3^*-x_4^*)^2}{2}.
\end{aligned}$$

Applying Itô's formula, we can obtain

$$\begin{aligned}
LV_1(x) = \ & (x_1+x_2+x_3-x_1^*-x_2^*-x_3^*)\\
&\times(\lambda - \delta x_1 - \mu x_2 - ax_3)\\
&+\frac{1}{2}\sigma_1^2x_1^2 + \frac{1}{2}\sigma_2^2x_2^2 + \frac{1}{2}\sigma_3^2x_3^2\\
< \ & -\delta(x_1-x_1^*)^2 - \mu(x_2-x_2^*)^2 - a(x_3-x_3^*)^2\\
&+\frac{1}{2}\sigma_1^2x_1^2 + \frac{1}{2}\sigma_2^2x_2^2 + \frac{1}{2}\sigma_3^2x_3^2,
\end{aligned}$$

$$\begin{aligned}
LV_2(x) = \ & (\eta(x_2-x_2^*) + \eta(x_3-x_3^*) + (x_4-x_4^*))\\
&\times(-\mu\eta(x_2-x_2^*) + (c-a\eta)(x_3-x_3^*)\\
&-\gamma(x_4-x_4^*)) + \frac{1}{2}\sigma_2^2\eta^2x_2^2 + \frac{1}{2}\sigma_3^2\eta^2x_3^2\\
&+\frac{1}{2}\sigma_4^2x_4^2\\
\leq \ & (-\mu\eta + \frac{\eta(c-a\eta)}{2} + \sigma_2^2\eta^2)(x_2-x_2^*)^2\\
&+(\frac{3\eta(c-a\eta)}{2} + \frac{c-a\eta}{2} + \sigma_3^2\eta^2)(x_3-x_3^*)^2\\
&+(\frac{c-a\eta}{2} - \gamma + \sigma_4^2)(x_4-x_4^*)^2\\
&+\sigma_2^2\eta^2x_2^{*2} + \sigma_3^2\eta^2x_3^{*2} + \sigma_4^2x_4^{*2}.
\end{aligned}$$

By computing,

$$\begin{aligned}
LV(x) \leq \ & -(\delta-\sigma_1^2)(x_1-x_1^*)^2 - (\mu(1+\eta)\\
&-\frac{\eta(c-a\eta)}{2} - \sigma_2^2(\eta^2+1))(x_2-x_2^*)^2\\
&-(a - \frac{3\eta(c-a\eta)}{2} - \frac{c-a\eta}{2} - \sigma_3^2(\eta^2+1))\\
&\times(x_3-x_3^*)^2 - (\gamma - \sigma_4^2 - \frac{c-a\eta}{2})(x_4-x_4^*)^2\\
&+\sigma_1^2x_1^{*2} + \sigma_2^2(1+\eta^2)x_2^{*2} + \sigma_3^2(1+\eta^2)x_3^{*2}\\
&+\sigma_4^2x_4^{*2}\\
= \ & -k_1(x_1-x_1^*)^2 - k_2(x_2-x_2^*)^2\\
&-k_3(x_3-x_3^*)^2 - k_4(x_4-x_4^*)^2 + \rho.
\end{aligned}$$

Taking the expectation of above equation, we see

$$\begin{aligned}
&EV(t) - V(0)\\
= \ & E\int_0^t LV(r)\,\mathrm{d}r\\
\leq \ & -k_1E\int_0^t (x_1-x_1^*)^2\,\mathrm{d}r - k_2E\int_0^t (x_2-x_2^*)^2\,\mathrm{d}r\\
&-k_3E\int_0^t (x_3-x_3^*)^2\,\mathrm{d}r\\
&-k_4E\int_0^t (x_4-x_4^*)^2\,\mathrm{d}r + \rho t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty} \tfrac{1}{t}E\textstyle\int_0^t [k_1(x_1-x_1^*)^2 + k_2(x_2-x_2^*)^2\\
&+k_3(x_3-x_3^*)^2 + k_4(x_4-x_4^*)^2]dr \leq \rho.
\end{aligned}$$

Noting that $0 < \rho < \min\{k_1x_1^{*2}, k_2x_2^{*2}, k_3x_3^{*2}, k_4x_4^{*2}\}$, then the ellipsoid

$$\begin{aligned}
&-k_1(x_1-x_1^*)^2 - k_2(x_2-x_2^*)^2 - k_3(x_3-x_3^*)^2\\
&-k_4(x_4-x_4^*)^2 + \rho = 0
\end{aligned}$$

lies entirely in R^4_+. We can take U to be any neighborhood of the ellipsoid such that $\bar{U} \subseteq E_l = R^4_+$, so for $x \in U \backslash E_l, LV(t) \leq -C$ (C is a positive constant), which implies condition $(B.2)$ is satisfied.

On the other hand, we can write model (2) as the form of model,

$$
\begin{aligned}
& d\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \\
= & \begin{pmatrix} \lambda - \frac{\beta x_1 x_4}{1+mx_4} - \delta x_1 \\ \frac{k\beta x_1 x_4}{1+mx_4} - \mu x_2 - \alpha x_2 \\ \frac{(1-k)\beta x_1 x_4}{1+mx_4} - a x_3 + \alpha x_2 \\ N a x_3 - \gamma x_4 - \frac{\beta x_1 x_4}{1+mx_4} \end{pmatrix} dt \\
& + \begin{pmatrix} \sigma_1 x_1(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_3 x_3(t) \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ \sigma_2 x_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_4 x_4(t) \end{pmatrix} dB_2(t) \\
& + \begin{pmatrix} \sigma_1 x_1(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_3 x_3(t) \\ 0 \end{pmatrix} dB_3(t) + \begin{pmatrix} 0 \\ \sigma_2 x_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_4 x_4(t) \end{pmatrix} dB_4(t).
\end{aligned}
$$

Here the diffusion matrix is

$$
A = \begin{pmatrix} \sigma_1^2 x_1^{*2} & 0 & 0 & 0 \\ 0 & \sigma_2^2 x_2^{*2} & 0 & 0 \\ 0 & 0 & \sigma_3^2 x_3^{*2} & 0 \\ 0 & 0 & 0 & \sigma_4^2 x_4^{*2} \end{pmatrix}.
$$

There is a $M = \min\{\sigma_1^2 {x_1}^2, \sigma_2^2 {x_1}^2, \sigma_3^2 {x_1}^2, \sigma_4^2 {x_1}^2\}$, such that for all $(x_1, x_2, x_3, x_4) \in \bar{U}$ and $\xi \in R^4_+$,

$$
\begin{aligned}
\sum_{i,j=1}^{k} a_{i,j}\xi_i\xi_j = & \ \sigma_1^2 {x_1}^2 \xi_1^2 + \sigma_2^2 {x_2}^2 \xi_2^2 + \sigma_3^2 {x_3}^2 \xi_3^2 \\
& + \sigma_4^2 {x_4}^2 \xi_4^2 \\
\geq & \ \min\{\sigma_1^2 {x_1}^2, \sigma_2^2 {x_2}^2, \sigma_3^2 {x_3}^2, \sigma_4^2 {x_4}^2\} \mid \xi \mid^2 \\
= & \ M \mid \xi \mid^2,
\end{aligned}
$$

which shows that condition (B.1) is also satisfied. Therefore, we can conclude that stochastic model (2) has a stationary distribution $\nu(.)$. This completes the proof.

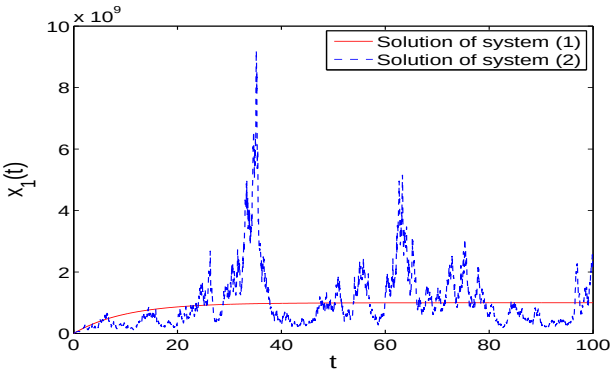
VIII. Numerical simulations

In this section, we use the Milstein method (see, e.g., [39]) to substantiate our main results. Consider the following discretization equations

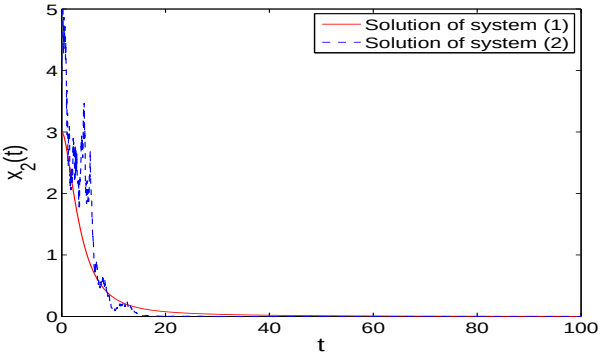
$$
\begin{aligned}
x_{1_{k+1}} = & \ x_{1_k} + (\lambda - \frac{\beta x_{1_k} x_{4_k}}{1 + m x_{4_k}} - \delta x_{1_k})\Delta t \\
& - x_{1_k}(\sigma_1 \xi_{1,k} \sqrt{\Delta t} + \tfrac{1}{2}\sigma_1^2(\xi_{1,k}^2 - 1)\Delta t), \\
x_{2_{k+1}} = & \ x_{2_k} + (\frac{k_1 \beta x_{1_k} x_{4_k}}{1 + m x_{4_k}} - \mu x_{2_k} - \alpha x_{2_k})\Delta t \\
& - x_{2_k}(\sigma_2 \xi_{2,k} \sqrt{\Delta t} + \tfrac{1}{2}\sigma_2^2(\xi_{2,k}^2 - 1)\Delta t), \\
x_{3_{k+1}} = & \ x_{3_k} + (\frac{(1-k_1)\beta x_{1_k} x_{4_k}}{1 + m x_{4_k}} + \alpha x_{2_k} - a x_{3_k})\Delta t \\
& - x_{3_k}(\sigma_3 \xi_{3,k} \sqrt{\Delta t} + \tfrac{1}{2}\sigma_3^2(\xi_{3,k}^2 - 1)\Delta t), \\
x_{4_{k+1}} = & \ x_{4_k} + (c x_{4_k} - \gamma x_{4_k}) - \frac{\eta \beta x_{1_k} x_{4_k}}{1 + m x_{4_k}} \Delta t \\
& - x_{4_k}(\sigma_4 \xi_{4,k} \sqrt{\Delta t} + \tfrac{1}{2}\sigma_4^2(\xi_{4,k}^2 - 1)\Delta t),
\end{aligned}
$$

where Δt is time increment $\xi_{1,k}$, $\xi_{2,k}$, $\xi_{3,k}$ and $\xi_{4,k} (k = 0, 1, 2, 3 \cdots)$ are N(0,1)-distributed independent random variables. In the figures, the red lines and the blue lines represent solutions of deterministic model (1) and stochastic model (2) respectively.

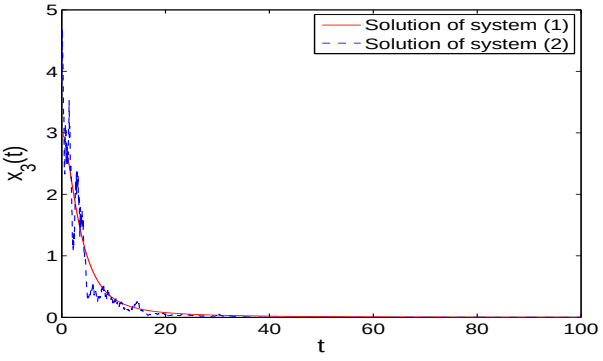
In Fig. 1, we choose the parameters $\lambda = 10^8$, $\beta = 10^{-9}$, $\delta = 0.1$, $k = 0.2$, $m = 0.0001$, $\mu = 0.2$, $\alpha = 0.5$, $a = 0.5$, $c = 5$, $\gamma = 10$, $\eta = 0.8$, $\sigma_1 = 0.53$, $\sigma_2 = 0.67$, $\sigma_3 = 0.42$, $\sigma_4 = 0.34$. In Fig. 2, the only difference $\beta = 10^{-8}$, $c = 50$, parameter values chosen above are consistent with the conditions required for the population densities fluctuate around the deterministic steady-state values (see Theorem 5,6). In Fig. 3 and Fig. 4, we represent the histograms of the values of the uninfected cells, the latently infected cells, the infected cells and the free virus of model (1) and model (2).



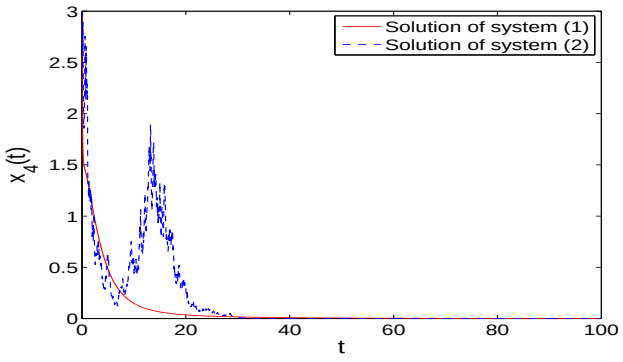
(a)



(b)

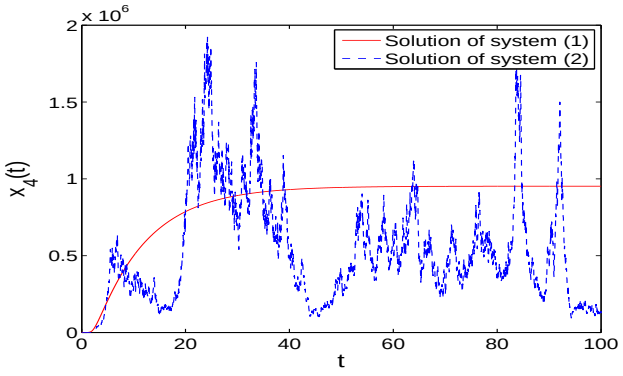


(c)



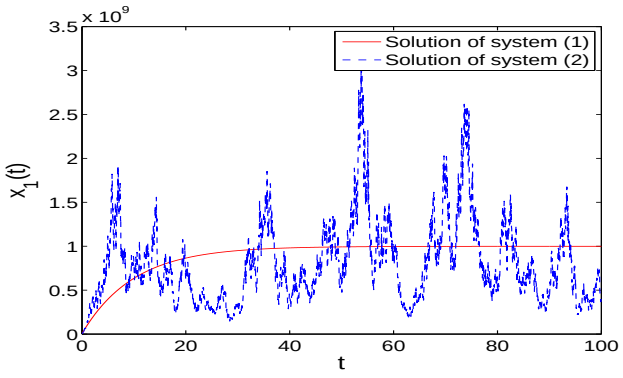
(d)

Fig.1. The solutions of model (1) and model (2) with $R_0 < 1$. The uninfected cells, the latently infected cells, the infected cells and the free virus of model (1) and model (2) are represented by figure (a), figure (b), figure (c), figure (d), respectively.

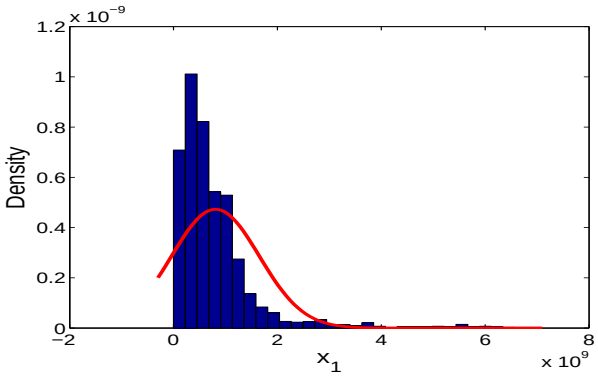


(d)

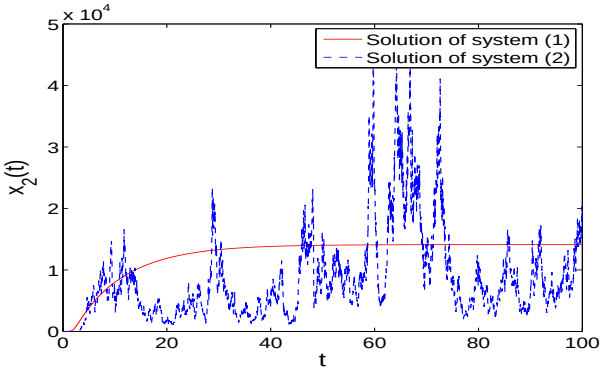
Fig.2. The solutions of model (1) and model (2) with $R_0 > 1$. The uninfected cells, the latently infected cells, the infected cells and the free virus of model (1) and model (2) are represented by figure (a), figure (b), figure (c), figure (d), respectively.



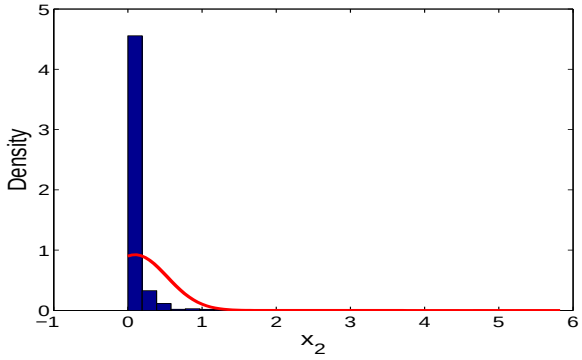
(a)



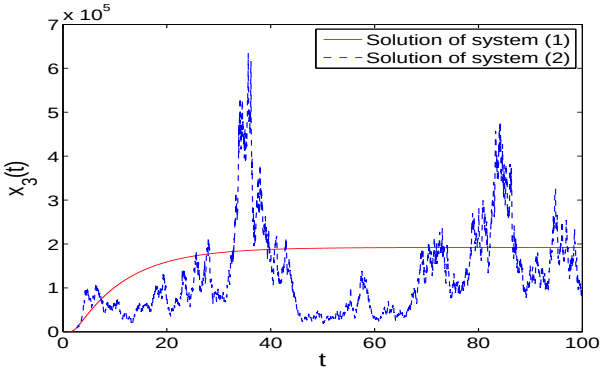
(a)



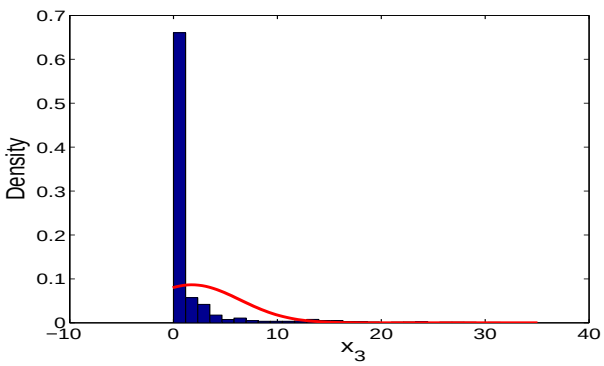
(b)



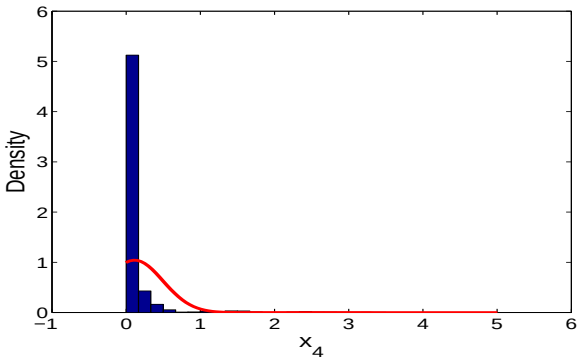
(b)



(c)

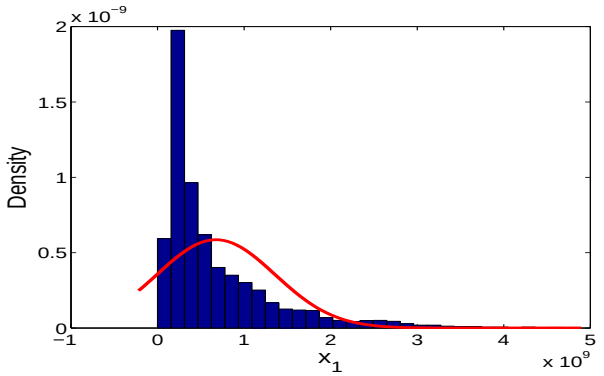


(c)

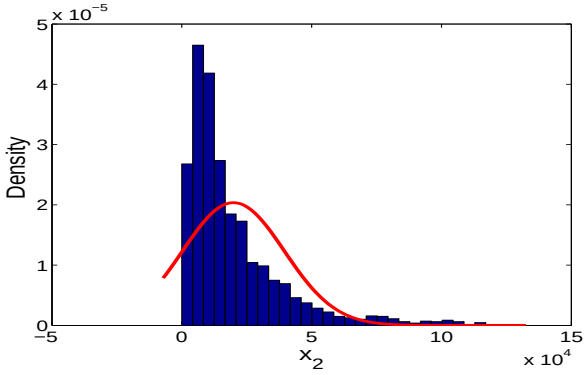


(d)

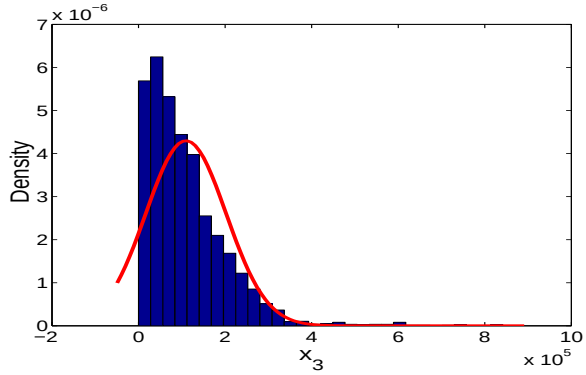
Fig.3. The density distribution of model (1) and model (2) with $R_0 < 1$. The uninfected cells, the latently infected cells, the infected cells and the free virus of model (1) and model (2) are represented by figure (a), figure (b), figure (c), figure (d), respectively.



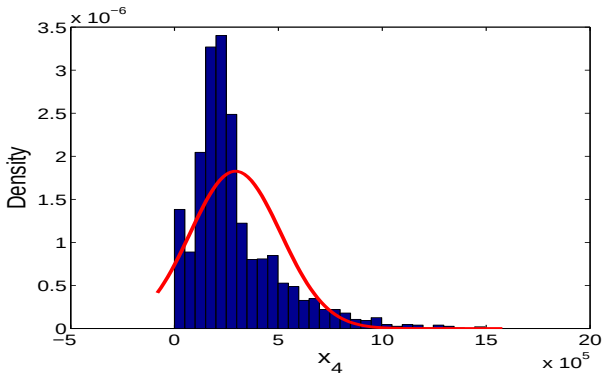
(a)



(b)



(c)



(d)

Fig.4. The density distribution of model (1) and model (2) with $R_0 > 1$. The uninfected cells, the latently infected cells, the infected cells and the free virus of model (1) and model (2) are represented by figure (a), figure (b), figure (c), figure (d), respectively.

IX. Discussion

In the real world, the population may suffer the occurrence of a catastrophe, e.g., earthquakes, hurricanes, volcanic eruption or tsunami, etc. However, the deterministic infection dynamical models may be difficult to explain the above phenomena. In this paper, we have considered the dynamics of stochastic HIV model and analyzed to study the effect of environmental white noise on the dynamics of model (2). By means of the theory of stochastic differential equations, Itô's formula, the method of Lyapunov functions and certain long-run-average limits to examined the solutions of the model from virous perspective and derived some sufficient conditions on the boundedness, extinction, persistence of the solutions and asymptotic behavior around equilibria for infection-free and endemic of deterministic model (1).

The deterministic HIV infection model, for example, the model (1) assumes that the parameters of the death rate and transmission rate in the model are all deterministic and the model irrespective of the environmental fluctuations and changes. Hence, model (1) has some limitations in mathematical modeling of epidemic models, besides model (1) is quite difficult to fit data perfectly and to predict the future dynamics of the model accurately. May [38] pointed out the fact that due to environmental noise, the birth rate, carrying capacity, death rate and other parameters involved in the model exhibit random fluctuation to a greater or lesser extent. In this paper, we incorporate white noise in model (1) and consider that the population lives in an environment subjected to random fluctuations which affect mainly death rate.

We have an interesting topics deserving further investigation, such as the persistence, extinction, and global attractively of a stochastic HIV epidemic model with delays. We leave this topics for our future work.

References

[1] M.A. Nowak, "Virus Dynamics: Mathematical Principles of Immunology and Virology," *Oxford University: Oxford*, 2000.

[2] R.V. Culshaw, S. Ruan, "A delay-differential equation model of HIV infection of CD4$^+$ T-cells," *Mathematical Biosciences*, vol. 165, no. 1, pp. 27-39, 2000.

[3] V.C. Rebecca, S. Ruan, W. Glenn, "A mathematical model of cell-to-cell spread of HIV-1 that includes a time delay," *Journal of mathematical biology*, vol. 46, no. 5, pp. 425-444, 2003.

[4] H. Miao, C. Kang, "Stability and Hopf bifurcation analysis for an HIV infection model with Beddington-DeAngelis incidence and two delays," *Journal of Applied Mathematics and Computing*, vol. 60, no. 1-2, pp. 265-290, 2019.

[5] U. Muhammad, S. Zulqurnain, A. Fazli, L. Juan, A. Muhammad, "Stochastic numerical technique for solving HIV infection model of CD4+ T cells," *The European Physical Journal Plus*, vol. 135, no. 6, pp. 5876-5890, 2020.

[6] S. Tang, Z. Teng, H. Miao, "Global dynamics of a reaction-diffusion virus infection model with humoral immunity and nonlinear incidence," *Computers and Mathematics with Applications*, vol. 78, no. 3, pp. 786-806, 2019.

[7] Z. Yuan, X. Zou, "Global threshold dynamics in an HIV virus model with nonlinear infection rate and distributed invasion and production delays," *Mathematical Biosciences and Engineering*, vol. 10, no. 10, pp. 483-498, 2013.

[8] P.K. Roy, A.N. Chatterjee, D. Greenhalgh, Q. Khan, "Long term dynamics in a mathematical model of HIV-1 infection with delay in different variants of the basic drug therapy model," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 3, pp. 1621-1633, 2013.

[9] J. Cai, P. Chen, X. Mei, "Range-Based Threshold Spot Volatility Estimation for Jump Diffusion Models," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 47, pp. 43-48, 2017.

[10] A.M. Elaiw, S.A. Azoz, "Global properties of a class of HIV infection models with Beddington-DeAngelis functional response," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 4, pp. 383-394, 2013.

[11] L. Imhof, S. Walcher, "Exclusion and persistence in deterministic and stochastic chemostat models," *Journal of Differential Equations*, vol. 217, no. 1, pp. 26-53, 2005.

[12] L. Arnold, "Stochastic Differential Equations: Theory and Applications," *Wiley*, New York, 1972.

[13] K.B. Gopal, N.B. Rabi, "Stability in distribution for a class of singular diffusions," *The Annals of Probability*, vol. 20, no. 1, pp. 312-321, 1992.

[14] Y.A. Kutoyants, "Statistical Inference for Ergodic Diffusion Processes," *Springer*, London, 2003.

[15] D. Jiang, J. Yu, C. Ji, N. Shi, "Asymptotic behavior of global positive solution to a stochastic SIR model," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 221-232, 2011.

[16] N. Dalal, D. Greenhalgh , X. Mao, "A stochastic model for internal HIV dynamics," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1084-1101, 2008.

[17] H. Huo, L. Feng, "Global stability for an HIV/AIDS epidemic model with different latent stages and treatment," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1480-1489, 2013.

[18] M.U. Nsuami, P.J. Witbooi, "A stochastic model for HIV epidemic with treatment and inflow of infectives," *International Journal of Applied Mathematics*, vol. 31, no. 5, pp. 545-568, 2018.

[19] W. Tan, X. Zhu, "A Stochastic Model of the HIV Epidemic for Heterosexual Transmission Involving Married Couples and Prostitutes: I. The Probabilities of HIV Transmission and Pair Formation," *Mathematical and computer modelling*, vol. 24, no. 11, pp. 47-107, 1996.

[20] W. Tan, Z. Xiang, "A Stochastic Model for the HIV Epidemic in Homosexual Populations Involving Age and Race," *Mathematical and computer modelling*, vol. 24, no. 12, pp. 67-105, 1996.

[21] X. Meng, "Stability of a novel stochastic epidemic model with double epidemic hypothesis," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 506-515, 2010.

[22] A. Lahrouz, A. Settati, "Necessary and sufficient condition for extinction and persistence of SIRS model with random perturbation," *Applied Mathematics and Computation*, vol. 233, no. 1, pp. 10-19, 2014.

[23] W. Zhang, X. Meng, Y. Dong, "Periodic Solution and Ergodic Stationary Distribution of Stochastic SIRI Epidemic Systems with Nonlinear Perturbations," *Journal of Systems Science and Complexity*, vol. 32, no. 4, pp. 1104-1124, 2019.

[24] K. Mahieddine, B. Bilal, A. Bedreddine, "Mathematical analysis of an HIV infection model including quiescent cells and periodic antiviral therapy," *International Journal of Biomathematics*, vol. 10, no. 5, pp. 1750-1765, 2017.

[25] Y. Zhao, D. Jiang, D. O'Regan, "The extinction and persistence of the stochastic SIS epidemic model with vaccination," *Physica. Section A*, vol. 392, no. 20, pp. 4916-4927, 2013.

[26] G. Chen, T. Li, "Stability of stochastic delayed SIR model," *Stochastics and Dynamics*, vol. 9, no. 2, pp. 231-252, 2009.

[27] C. Kang, H. Miao, X. Chen, "Global Stability Analysis for a Delayed HIV Infection Model with General Incidence Rate and Cell Immunity," *Engineering Letters*, vol. 24, no. 4, pp. 392-398, 2016.

[28] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, "A stochastic differential equation SIS epidemic," *SIAM Journal on Applied Mathematics* vol. 71, no. 3, pp. 876-902, 2011.

[29] C. Ji, D. Jiang, "Threshold behaviour of a stochastic SIR model," *Applied Mathematical Modelling*, vol. 38, no. 21-22, pp. 5067-5079, 2014.

[30] X. Mao, "Stochastic Differential Equations and Applications," *Horwood*, Chichester, 1997.

[31] T. Nguyen, "Asymptotic Properties of a Stochastic SIR Epidemic Model with BeddingtonCDeAngelis Incidence Rate," *Journal of Dynamics and Differential Equations*, vol. 30, no. 1, pp. 93-106, 2018.

[32] M. Jin, Y. Lin, "Periodic solution of a stochastic SIRS epidemic model with seasonal variation," *Journal Of Biological Dynamics*, vol. 12, no. 1, pp. 1-10, 2018.

[33] R.Z. Hasminskii, "Stochastic Stability of Differential equations," *Alphen aan den Rijn*, The Netherlands, 1980.

[34] Q. Liu, D. Jiang, T. Hayat, A. Alsaedi, "Threshold behavior in a stochastic delayed SIS epidemic model with vaccination and double diseases," *Journal of the Franklin Institute*, vol. 356, no. 13, pp. 7466-7485, 2019.

[35] T. Caraballo, P.E. Kloeden, "The persistence of synchronization under environmental noise," *Proceedings: Mathematical, Physical and Engineering Sciences*, vol. 461, no. 2059, pp. 2257-2267, 2005.

[36] T.C. Gard, "Introduction to Stochastic Differential Equations," *Marcel Dekker*, New York, 1987.

[37] G. Strang, "Linear Algebra and Its Applications," *Thomson Learning Inc.*, 1988.

[38] R.M. May, "Stability and complexity in model ecosystems," *Princeton University Press*, 1973.

[39] D.J. Higham, "An algorithmic introduction to numerical simulation of stochastic differential equations," *SIAM Review*, vol. 43, no. 3, pp. 525-546, 2001.