A Class of Bivariate Rational Interpolation Surfaces with C^2 Continuity

Xiangbin Qin and Yuanpeng Zhu*

***Abstract*—Based on two new kinds of Hermite-type interpolation basis functions, a class of piecewise bivariate rational interpolation surface scheme with bi-cubic denominator and four parameters is constructed in a rectangular domain. The given interpolation surface is proved to be bounded and its error formula is provided. The conditions for the resulting interpolation surface to be C^2 continuous in the whole rectangular domain are developed. Several numerical examples are given and the numerical results show that the given scheme is effective and practical.**

***Index Terms*—Hermite-type interpolation basis, Interpolation surface, Bounded property, Error estimate, C^2 continuity**

I. Introduction

Modeling smooth interpolation surfaces to given data in rectangular grid is an essential issue in industrial design and scientific data visualization. Generally speaking, for most applications, C^1 smoothness is sufficient, and there are many schemes to tackle this problem, see for example the classical Coos surface schemes [1], the bi-cubic blending rational interpolation schemes [2], [3], [4], the bivariate rational interpolation schemes [5], [6], [7], [8], [9], [10], and the bivariate rational Hermite interpolation schemes [11]. In some practical applications, curvature continuity is needed sometimes and this leads to the need for C^2 smoothness.

By using the classical Coons surface scheme, it is a more difficult task to construct C^2 interpolation surfaces for 3D data defined over rectangular grid. For example, to generate a C^2 bi-quintic Coons surface, it needs to provide the second and higher mixed partial derivatives at the data points in advance. In practical applications, however, the second and higher mixed partial derivatives are hard to estimate and control, and there may also exist compatibility problem in generating the classical C^2 bi-quintic Coons surface, see [12]. In [13], by taking the Boolean sum of two rational cubic/quadratic Hermite-type blending functions and solving two linear systems of equations with respect to the first partial derivative values on a rectangular grid, a kind of C^2 rational bi-cubic spline interpolants was proposed. In [14], a class of rational bi-quintic interpolation splines with two parameters was constructed. For generating interpolation surfaces, the given interpolant only use the values of the interpolated function and can be C^2 continuous for equally spaced knots. And the shape of the generated C^2 rational interpolation surfaces can be modified conveniently by using the parameters for the unchanged interpolating data. Later, in [15], Fan and his colleagues showed that in the applications of image interpolation, the C^2 rational interpolants have lower time complexity and can preserve image details well. In [16], a new kind of C^2 piecewise bivariate rational interpolation scheme with two parameters was constructed. In [17], a class of C^2 bi-quintic partially blended rational quartic/cubic interpolation surfaces were constructed. Recently, in [18], Yuan and Ma construct a kind of C^2 truncated interpolation basis functions over tensor product meshes and they showed that the new bases have some merits in Isogeometric analysis.

The purpose of this paper is to present a class of piecewise bivariate rational interpolation surface scheme with bi-cubic denominator and four parameters over rectangular domain. The given interpolation surface can be C^2 continuous in the whole rectangular domain without using the second or higher mixed partial derivatives at the knots. The values of the generated interpolation surface are bounded and stable no matter what the four parameters might be. It improves on the existing schemes in some ways: (1) The classical C^2 bi-quintic Coons surface has to estimate the second or higher mixed partial derivatives at the knots in advance, while the new given C^2 rational interpolation spline surface is based on the interpolated function only; (2) Compared with the rational interpolation spline surface with two parameters developed in [14] and [16], the new given C^2 rational interpolation spline surface provides four parameters, which is more flexible in adjusting the shape of surface; (3) For special values of the four parameters, the new given C^2 rational interpolation spline surface includes the C^2 rational interpolation spline surface given in [14] as a special case. (4) Compared with the C^2 bi-quintic partially blended rational quartic/cubic interpolation spline surface with 20 terms in each sub-rectangular domain $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ developed in [17], the new givenC^2 rational interpolation spline surface only has 12 terms in each sub-rectangular domain $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and thus it has a more concise expression and lower computation cost. The rest of this paper is organized as follows. In section II, the construction of the new C^2 piecewise bivariate rational interpolation surface is described. Section III discusses the properties of the interpolation surface in detail, including C^2 continuity property, bounded property, and error formula. In section IV, several numerical examples are given to prove the effectiveness and practicability of the new developed schemes. Conclusion is given in the section V.

II. New piecewise bivariate rational interpolation surfaces

In this section, we firstly construct a class of C^2 x-direction interpolation curve with two parameters based on a

Manuscript received August 13, 2019; revised October 21, 2019.

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new kind of Hermite-type interpolation basis functions. And then by using another new kind of Hermite-type interpolation basis functions, we construct a class of C^2 piecewise bivariate rational interpolation surface scheme with bi-cubic denominator and four parameters in a rectangular domain.

Let $\{(x_i, y_i, F_{ij}), i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ be a given set of data points defined over the rectangular domain $R = [x_1, x_n] \times [y_1, y_m]$, where $\pi_x : x_1 < x_2 < \ldots < x_n$ is the partition of $[x_1, x_n]$ and $\pi_y : y_1 < y_2 < \ldots < y_m$ is the partition of $[y_1, y_m]$. $D^x_{i,j}$ and $D^y_{i,j}$ are known as the first partial derivatives at the grid point (x_i, y_j). Denote $h^x_i = x_{i+1} - x_i$, $h^y_j = y_{j+1} - y_j$, $R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, and for any $(x, y) \in R_{i,j}$, let $t = (x - x_i)/h^x_i$, $s = (y - y_j)/h^y_j$, and

$$\Delta^x_{i,j} = \frac{F_{i+1,j} - F_{i,j}}{h^x_i}, \quad \Delta^y_{i,j} = \frac{F_{i,j+1} - F_{i,j}}{h^y_j}.$$

For each $y = y_j,\ j = 1, 2, \ldots, m$, and $x \in [x_i, x_{i+1}]$, a kind of x-direction interpolation curve with two local free parameters $\alpha^x_{i,j}$ and $\beta^x_{i,j}$ is constructed as follows

$$\begin{aligned} P^*_{i,j}(x) = {} & H_0(t; \alpha^x_{i,j}, \beta^x_{i,j}) F_{i,j} + H_1(t; \alpha^x_{i,j}, \beta^x_{i,j}) F_{i+1,j} \\ & + H_2(t; \alpha^x_{i,j}, \beta^x_{i,j}) h^x_i D^x_{i,j} \\ & + H_3(t; \alpha^x_{i,j}, \beta^x_{i,j}) h^x_i D^x_{i+1,j}, \end{aligned} \tag{1}$$

where the four new Hermite-type interpolation basis functions $H_k(t; \alpha, \beta)$, $k = 0, 1, 2, 3$ are defined as

$$\begin{aligned} & H_0(t; \alpha^x_{i,j}, \beta^x_{i,j}) \\ & = \frac{(1-t)^5 + (\alpha^x_{i,j}+2)(1-t)^4 t + (2\alpha^x_{i,j}+\beta^x_{i,j})(1-t)^3 t^2 + (1-t)^2 t^3}{(1-t)^3 + \alpha^x_{i,j}(1-t)^2 t + \beta^x_{i,j}(1-t)t^2 + t^3}, \\ & H_1(t; \alpha^x_{i,j}, \beta^x_{i,j}) \\ & = \frac{t^5 + (\beta^x_{i,j}+2)(1-t)t^4 + (\alpha^x_{i,j}+2\beta^x_{i,j})(1-t)^2 t^3 + (1-t)^3 t^2}{(1-t)^3 + \alpha^x_{i,j}(1-t)^2 t + \beta^x_{i,j}(1-t)t^2 + t^3}, \\ & H_2(t; \alpha^x_{i,j}, \beta^x_{i,j}) \\ & = \frac{(1-t)^4 t + \alpha^x_{i,j}(1-t)^3 t^2}{(1-t)^3 + \alpha^x_{i,j}(1-t)^2 t + \beta^x_{i,j}(1-t)t^2 + t^3}, \\ & H_3(t; \alpha^x_{i,j}, \beta^x_{i,j}) \\ & = \frac{-[(1-t)t^4 + \beta^x_{i,j}(1-t)^2 t^3]}{(1-t)^3 + \alpha^x_{i,j}(1-t)^2 t + \beta^x_{i,j}(1-t)t^2 + t^3}, \end{aligned}$$

with $\alpha^x_{i,j}, \beta^x_{i,j} \geq 0$.

For the four basis functions $H_k(t; \alpha^x_{i,j}, \beta^x_{i,j})$, $k = 0, 1, 2, 3$, by directly computing, we get

$$\begin{aligned} & H_0(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 1,\ H_0'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_0''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = -2, \\ & H_0(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_0'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_0''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 2, \\ & H_1(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_1'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_1''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 2, \\ & H_1(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 1, H_1'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_1''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = -2, \\ & H_2(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_2'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 1, \\ & H_2''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = -2, \\ & H_2(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_2'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_2''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_3(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_3'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_3''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0, \\ & H_3(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ H_3'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 1, \\ & H_3''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 2, \end{aligned}$$

it follows that

$$\begin{aligned} & P^*_{i,j}(x^+_i) = F_{i,j},\ {P^*_{i,j}}'(x^+_i) = D^x_{i,j}, \\ & {P^*_{i,j}}''(x^+_i) = \frac{2(\Delta^x_{i,j} - D^x_{i,j})}{h^x_i}, \\ & P^*_{i,j}(x^-_{i+1}) = F_{i+1,j},\ {P^*_{i,j}}'(x^-_{i+1}) = D^x_{i+1,j}, \\ & {P^*_{i,j}}''(x^-_{i+1}) = \frac{2(D^x_{i+1,j} - \Delta^x_{i,j})}{h^x_i}. \end{aligned}$$

Thus, we can see that if the first partial derivative values $D^x_{i,j}$, $i = 2, 3, \ldots, n-1$ are chosen as follows

$$D^x_{i,j} = \frac{h^x_{i-1}\Delta^x_{i,j} + h^x_i \Delta^x_{i-1,j}}{h^x_{i-1} + h^x_i}, \tag{2}$$

then for $i = 2, 3, \ldots, n-1$, we have

$$\begin{aligned} & P^*_{i,j}(x^+_i) = P^*_{i,j}(x^-_i) = F_{i,j}, \\ & {P^*_{i,j}}'(x^+_i) = {P^*_{i,j}}'(x^-_i) = D^x_{i,j}, \\ & {P^*_{i,j}}''(x^+_i) = {P^*_{i,j}}''(x^-_i) = \frac{2\left(\Delta^x_{i,j} - \Delta^x_{i-1,j}\right)}{h^x_{i-1} + h^x_i}, \end{aligned}$$

which implies that the resulting interpolation function $P^*_{i,j}(x)$ defined by (1) is C^2 continuous in $[x_1, x_n]$. At the end knots x_1 and x_n, the derivative values are computed by the following formulas

$$\begin{cases} D^x_{1,j} = \Delta^x_{1,j} - \frac{h^x_1}{h^x_1 + h^x_2}\left(\Delta^x_{2,j} - \Delta^x_{1,j}\right), \\ D^x_{n,j} = \Delta^x_{n-1,j} + \frac{h^x_{n-1}}{h^x_{n-2} + h^x_{n-1}}\left(\Delta^x_{n-1,j} - \Delta^x_{n-2,j}\right). \end{cases} \tag{3}$$

Remark 1: It is interesting to note that for $\alpha^x_{i,j} = \beta^x_{i,j}$, the four Hermite-type interpolation basis functions $H_k(t; \alpha, \beta)$, $k = 0, 1, 2, 3$ will return to the four interpolation basis functions $\omega_{i,j}(t), i, j = 0, 1$ given in [14]. And when $\alpha^x_{i,j} = \beta^x_{i,j} \to +\infty$, the four Hermite-type interpolation basis functions $H_k(t; \alpha, \beta)$, $k = 0, 1, 2, 3$ will give approximation to the four standard cubic Hermite interpolation basis functions.

For any $(x, y) \in R_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$, we now use the x-direction interpolant $P^*_{i,j}(x)$ given in (1) to construct a new kind of piecewise bivariate rational interpolation surfaces $P_{i,j}(x)$ as follows

$$\begin{aligned} P_{i,j}(x, y) = {} & V_0(s; \alpha^y_{i,j}, \beta^y_{i,j}) P^*_{i,j}(x) \\ & + V_1(s; \alpha^y_{i,j}, \beta^y_{i,j}) P^*_{i,j+1}(x) \\ & + V_2(s; \alpha^y_{i,j}, \beta^y_{i,j}) h^y_j \phi_{i,j}(x) \\ & + V_3(s; \alpha^y_{i,j}, \beta^y_{i,j}) h^y_j \phi_{i,j+1}(x), \end{aligned} \tag{4}$$

where the four new Hermite-type interpolation basis functions $V_k(s; \alpha^y_{i,j}, \beta^y_{i,j})$, $k = 0, 1, 2, 3$ are defined as

$$\begin{aligned} & V_0(s; \alpha^y_{i,j}, \beta^y_{i,j}) \\ & = \frac{(1-s)^5 + (\alpha^y_{i,j}+2)(1-s)^4 s + (2\alpha^y_{i,j}+\beta^y_{i,j}+1)(1-s)^3 s^2}{(1-s)^3 + \alpha^y_{i,j}(1-s)^2 s + \beta^y_{i,j}(1-s)s^2 + s^3}, \\ & V_1(s; \alpha^y_{i,j}, \beta^y_{i,j}) \\ & = \frac{s^5 + (\beta^y_{i,j}+2)(1-s)s^4 + (\alpha^y_{i,j}+2\beta^y_{i,j}+1)(1-s)^2 s^3}{(1-s)^3 + \alpha^y_{i,j}(1-s)^2 s + \beta^y_{i,j}(1-s)s^2 + s^3}, \\ & V_2(s; \alpha^y_{i,j}, \beta^y_{i,j}) \\ & = \frac{(1-s)^4 s + (\alpha^y_{i,j}+1)(1-s)^3 s^2}{(1-s)^3 + \alpha^y_{i,j}(1-s)^2 s + \beta^y_{i,j}(1-s)s^2 + s^3}, \\ & V_3(s; \alpha^y_{i,j}, \beta^y_{i,j}) \\ & = \frac{-[(1-s)s^4 + (\beta^y_{i,j}+1)(1-s)^2 s^3]}{(1-s)^3 + \alpha^y_{i,j}(1-s)^2 s + \beta^y_{i,j}(1-s)s^2 + s^3}, \end{aligned}$$

with $\alpha^y_{i,j}, \beta^y_{i,j} \geq 0$, and the functions $\phi_{i,l}(x), l = j, j+1$ are given by

$$\phi_{i,l}(x) = (1-t)^3 \left(1 + 4t + 9t^2\right) D^y_{i,l} + t^3 \left(6 - 8t + 3t^2\right) D^y_{i+1,l}.$$

From (1) and (4), after some manipulations, we can also rewrite the interpolation surfaces $P_{i,j}(x)$ as the following form

$$P_{i,j}(x,y) = \sum_{k=i}^{i+1} \sum_{l=j}^{j+1} \Big[A_{k,l}(t,s) F_{k,l} + B_{k,l}(t,s) h^x_i D^x_{k,l} + C_{k,l}(t,s) h^y_j D^y_{k,l} \Big], \quad (5)$$

where

$$\begin{aligned}
A_{i,j}(t,s) &= H_0\left(t; \alpha^x_{i,j}, \beta^x_{i,j}\right) V_0\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
A_{i,j+1}(t,s) &= H_0\left(t; \alpha^x_{i,j+1}, \beta^x_{i,j+1}\right) V_1\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
A_{i+1,j}(t,s) &= H_1\left(t; \alpha^x_{i,j}, \beta^x_{i,j}\right) V_0\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
A_{i+1,j+1}(t,s) &= H_1\left(t; \alpha^x_{i,j+1}, \beta^x_{i,j+1}\right) V_1\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
B_{i,j}(t,s) &= H_2\left(t; \alpha^x_{i,j}, \beta^x_{i,j}\right) V_0\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
B_{i,j+1}(t,s) &= H_2\left(t; \alpha^x_{i,j+1}, \beta^x_{i,j+1}\right) V_1\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
B_{i+1,j}(t,s) &= H_3\left(t; \alpha^x_{i,j}, \beta^x_{i,j}\right) V_0\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
B_{i+1,j+1}(t,s) &= H_3\left(t; \alpha^x_{i,j+1}, \beta^x_{i,j+1}\right) V_1\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
C_{i,j}(t,s) &= (1-t)^3 \left(1 + 4t + 9t^2\right) V_2\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
C_{i,j+1}(t,s) &= (1-t)^3 \left(1 + 4t + 9t^2\right) V_3\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
C_{i+1,j}(t,s) &= t^3 \left(6 - 8t + 3t^2\right) V_2\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right),\\
C_{i+1,j+1}(t,s) &= t^3 \left(6 - 8t + 3t^2\right) V_3\left(s; \alpha^y_{i,j}, \beta^y_{i,j}\right).
\end{aligned}$$

We call the terms $A_{k,l}, B_{k,l}$ and $C_{k,l}$, $k = i, i+1$, $l = j, j+1$, as the basis functions of the interpolation surface defined by (5). Before further discussion, we want to give the following end-point properties of the four basis functions $V_k(s; \alpha^y_{i,j}, \beta^y_{i,j})$, $k = 0, 1, 2, 3$, which is quite useful for discussing the C^2 continuous property of the interpolation surface $P_{i,j}(x, y)$

$$\begin{aligned}
&V_0(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 1,\ V_0'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_0''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_0(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_0'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_0''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_1(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_1'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_1''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_1(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 1, V_1'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_1''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_2(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_2'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 1,\\
&V_2''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_2(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_2'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_2''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_3(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_3'(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_3''(0; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\\
&V_3(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,\ V_3'(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 1,\\
&V_3''(1; \alpha^x_{i,j}, \beta^x_{i,j}) = 0,
\end{aligned}$$

***Remark** 2:* It is interesting to note that for $\alpha^x_{i,j} = \beta^x_{i,j}$ and $\alpha^y_{i,j} = \beta^y_{i,j}$, the given interpolation surface $P_{i,j}(x, y)$ will return to the interpolation surface given in [14].

III. Properties of the interpolation surfaces

In this section, we shall discuss the properties of the interpolation surfaces in detail, including the bounded property, the error formula and the C^2 continuous property.

A. Bounded property

We denote

$$\begin{aligned}
M &= \max\left\{|F_{k,l}|, k = i, i+1, l = j, j+1\right\},\\
Q_1 &= \max\left\{h^x_i \left|D^x_{k,l}\right|, k = i, i+1, l = j, j+1\right\},\\
Q_2 &= \max\left\{h^y_j \left|D^y_{k,l}\right|, k = i, i+1, l = j, j+1\right\}.
\end{aligned}$$

By directly computing, we can obtain the following properties of the basis functions

$$\begin{aligned}
&A_{i,j}(t,s) + A_{i,j+1}(t,s) + A_{i+1,j}(t,s) + A_{i+1,j+1}(t,s)\\
&= 1,\\
&B_{i,j}(t,s) + B_{i,j+1}(t,s) - B_{i+1,j}(t,s) - B_{i+1,j+1}(t,s)\\
&= (1-t)t,\\
&C_{i,j}(t,s) - C_{i,j+1}(t,s) + C_{i+1,j}(t,s) - C_{i+1,j+1}(t,s)\\
&= \frac{\left(1+t-10t^3+15t^4-6t^5\right)s\left[1-3s+4s^2-2s^3+\alpha^y_{i,j}(1-s)^3 s+\beta^y_{i,j}(1-s)^2 s^2\right]}{(1-s)^3+\alpha^y_{i,j}(1-s)^2 s+\beta^y_{i,j}(1-s)s^2+s^3}.
\end{aligned}$$

Thus, for the given data, from the expression of the interpolation surface $P_{i,j}(x, y)$ given in (5), we have

$$\begin{aligned}
&|P_{i,j}(x,y)|\\
&\leq M \sum_{k=i}^{i+1} \sum_{l=j}^{j+1} |A_{k,l}(t,s)| + Q_1 \sum_{k=i}^{i+1} \sum_{l=j}^{j+1} |B_{k,l}(t,s)|\\
&\quad + Q_2 \sum_{k=i}^{i+1} \sum_{l=j}^{j+1} |C_{k,l}(t,s)|\\
&= M + Q_1 (1-t)t + Q_2 \sum_{k=i}^{i+1} \sum_{l=j}^{j+1} |C_{k,l}(t,s)|\\
&\leq M + 0.25 Q_1\\
&\quad + Q_2 \left(1 + t - 10t^3 + 15t^4 - 6t^5\right) \frac{s\left(1-3s+4s^2-2s^3\right)}{1-3s+3s^2}.
\end{aligned}$$

Since

$$\max_{t\in[0,1]} \left(1 + t - 10t^3 + 15t^4 - 6t^5\right) = 1.14675,$$

$$\max_{s\in[0,1]} \frac{s\left(1 - 3s + 4s^2 - 2s^3\right)}{1 - 3s + 3s^2} = 0.5,$$

we can conclude the following theorem.

***Theorem** 1:* For any nonnegative free parameters $\alpha^x_{i,j}, \beta^x_{i,j}, \alpha^y_{i,j}, \beta^y_{i,j}$, the values of the resulting interpolation surface $P_{i,j}(x, y)$ on $R_{i,j}$ are bounded by

$$|P_{i,j}(x,y)| \leq M + 0.25 Q_1 + 0.573375 Q_2.$$

B. Error formula

For any $(x, y) \in R_{i,j}$, let $F_{i,j} = F(x_i, y_j), D^x_{i,j} = \frac{\partial F(x_i,y_j)}{\partial x}, D^y_{i,j} = \frac{\partial F(x_i,y_j)}{\partial y}$, and denote

$$\left\|\frac{\partial F(x,y)}{\partial x}\right\| = \max_{(x,y)\in R_{i,j}} \left|\frac{\partial F(x,y)}{\partial x}\right|,$$

$$\left\|\frac{\partial F(x,y)}{\partial y}\right\| = \max_{(x,y)\in R_{i,j}} \left|\frac{\partial F(x,y)}{\partial y}\right|.$$

For any $(x, y) \in R_{i,j}$, by using the Taylor formula of $F(x, y)$ at the points (x_k, y_l), $k = i, i+1$, $l = j, j+1$, we have

$$\begin{aligned}&|F(x,y) - F(x_k, y_l)| \\ &= (x - x_k)\tfrac{\partial F(\theta_k,\eta_l)}{\partial x} + (y - y_l)\tfrac{\partial F(\theta_k,\eta_l)}{\partial y},\end{aligned}$$

where θ_k and η_l are between x and x_k, y and y_l, respectively. It follows that

$$\begin{aligned}&\max_{(x,y)\in R_{i,j}} |F(x,y) - F(x_k, y_l)| \\ &\le h_i^x \left\|\tfrac{\partial F(x,y)}{\partial x}\right\| + h_j^y \left\|\tfrac{\partial F(x,y)}{\partial y}\right\|.\end{aligned}$$

Thus for any $(x, y) \in R_{i,j}$, we have

$$\begin{aligned}&|F(x,y) - P_{i,j}(x,y)| \\ &= \Bigg|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1} \big[A_{k,l}(t,s)\left(F(x,y) - F(x_k,y_l)\right) \\ &\quad + B_{k,l}(t,s)\, h_i^x \tfrac{\partial F(x_k,y_l)}{\partial x} + C_{k,l}(t,s)\, h_j^y \tfrac{\partial F(x_k,y_l)}{\partial y}\big]\Bigg| \\ &\le \max_{(x,y)\in R_{i,j}} |F(x,y) - F(x_k,y_l)| \left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1} A_{k,l}(t,s)\right| \\ &\quad + h_i^x \left\|\tfrac{\partial F(x,y)}{\partial x}\right\| \left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1} B_{k,l}(t,s)\right| \\ &\quad + h_j^y \left\|\tfrac{\partial F(x,y)}{\partial y}\right\| \left|\sum_{k=i}^{i+1}\sum_{l=j}^{j+1} C_{k,l}(t,s)\right| \\ &\le 1.25 h_i^x \left\|\tfrac{\partial F(x,y)}{\partial x}\right\| + 1.573375 h_j^y \left\|\tfrac{\partial F(x,y)}{\partial y}\right\|.\end{aligned}$$

Summarize the above analysis, we have the following theorem.

***Theorem** 2:* Let $F(x, y) \in C^1(R)$ be the interpolated function with $P_{i,j}(x, y)$ is compared. Then for any $(x, y) \in R_{i,j}$, the following error formula holds

$$\begin{aligned}&|F(x,y) - P_{i,j}(x,y)| \\ &\le 1.25 h_i^x \left\|\tfrac{\partial F(x,y)}{\partial x}\right\| + 1.573375 h_j^y \left\|\tfrac{\partial F(x,y)}{\partial y}\right\|.\end{aligned}$$

From Theorems 1 and 2, we can see that the generated interpolation surface is stable for the parameters.

C. C^2 continuity property

For any $(x, y) \in R_{i,j}$, from the interpolation surface $P_{i,j}(x, y)$ given in (4), direct computation gives that

$$\begin{aligned}P_{i,j}(x, y_j^+) &= P_{i,j}^*(x), \quad P_{i,j}(x, y_{j+1}^-) = P_{i,j+1}^*(x), \\ P_{i,j}(x_i^+, y) &= V_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)F_{i,j} + V_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)F_{i,j+1} \\ &\quad + V_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y D_{i,j}^y \\ &\quad + V_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y D_{i,j+1}^y, \\ P_{i,j}(x_{i+1}^-, y) &= V_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)F_{i+1,j} \\ &\quad + V_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)F_{i+1,j+1} \\ &\quad + V_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y D_{i+1,j}^y \\ &\quad + V_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y D_{i+1,j+1}^y,\end{aligned}$$

Thus we have $P_{i,j}(x, y_j^+) = P_{i,j}(x, y_j^-)$, $P_{i,j}(x_i^+, y) = P_{i,j}(x_i^-, y)$.

Furthermore,

$$\begin{aligned}\frac{\partial P_{i,j}(x, y_j^+)}{\partial x} &= \frac{dP_{i,j}^*(x)}{dx}, \\ \frac{\partial P_{i,j}(x, y_{j+1}^-)}{\partial x} &= \frac{dP_{i,j+1}^*(x)}{dx}, \\ \frac{\partial P_{i,j}(x_i^+, y)}{\partial x} &= V_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)D_{i,j}^x \\ &\quad + V_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)D_{i,j+1}^x \\ &\quad + V_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y \frac{D_{i,j}^y}{h_i^x} \\ &\quad + V_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y \frac{D_{i,j+1}^y}{h_i^x}, \\ \frac{\partial P_{i,j}(x_{i+1}^-, y)}{\partial x} &= V_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)D_{i+1,j}^x \\ &\quad + V_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)D_{i+1,j+1}^x \\ &\quad + V_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y \frac{D_{i+1,j}^y}{h_i^x} \\ &\quad + V_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)h_j^y \frac{D_{i+1,j+1}^y}{h_i^x},\end{aligned}$$

thus we have $\frac{\partial P_{i,j}(x,y_j^+)}{\partial x} = \frac{\partial P_{i,j}(x,y_j^-)}{\partial x}$ and $\frac{\partial P_{i,j}(x_i^+,y)}{\partial x} = \frac{\partial P_{i,j}(x_i^-,y)}{\partial x}$ if $h_{i-1}^x = h_i^x$, $\alpha_{i-1,j}^y = \alpha_{i,j}^y$ and $\beta_{i-1,j}^y = \beta_{i,j}^y$.

Similarly, we have

$$\begin{aligned}\frac{\partial P_{i,j}(x, y_j^+)}{\partial y} &= \phi_{i,j}(x), \quad \frac{\partial P_{i,j}(x, y_{j+1}^-)}{\partial y} = \phi_{i,j+1}(x), \\ \frac{\partial P_{i,j}(x_i^+, y)}{\partial y} &= \frac{dV_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{F_{i,j}}{h_j^y} \\ &\quad + \frac{dV_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{F_{i,j+1}}{h_j^y} \\ &\quad + \frac{dV_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}D_{i,j}^y \\ &\quad + \frac{dV_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}D_{i,j+1}^y, \\ \frac{\partial P_{i,j}(x_{i+1}^-, y)}{\partial y} &= \frac{dV_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{F_{i+1,j}}{h_j^y} \\ &\quad + \frac{dV_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{F_{i+1,j+1}}{h_j^y} \\ &\quad + \frac{dV_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}D_{i+1,j}^y \\ &\quad + \frac{dV_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}D_{i+1,j+1}^y,\end{aligned}$$

it follows that $\frac{\partial P_{i,j}(x,y_j^+)}{\partial y} = \frac{\partial P_{i,j}(x,y_j^-)}{\partial y}$ and $\frac{\partial P_{i,j}(x_i^+,y)}{\partial y} = \frac{\partial P_{i,j}(x_i^-,y)}{\partial y}$ if $\alpha_{i-1,j}^y = \alpha_{i,j}^y$ and $\beta_{i-1,j}^y = \beta_{i,j}^y$.

From the above analysis, we can see that the interpolation surface $P_{i,j}(x, y)$ is C^1 continuous in whole rectangular domain R if h_i^x =constant, $\alpha_{i,j}^y$ =constant and $\beta_{i,j}^y$ =constant for each $j \in \{1, 2, \dots, m-1\}$ and all $i = 1, 2, \dots, n-1$, no matter what the parameters $\alpha_{i,j}^x$ and $\beta_{i,j}^x$ might be. In the following, we shall further discuss the C^2 continuous property of the interpolation surface.

For any $(x, y) \in R_{i,j}$, straightforward computation gives the mixed partial derivatives $\frac{\partial^2 P_{i,j}(x,y)}{\partial x \partial y}$ and $\frac{\partial^2 P_{i,j}(x,y)}{\partial y \partial x}$ as follows

$$\begin{aligned}\frac{\partial^2 P_{i,j}(x,y)}{\partial x \partial y} &= \frac{1}{h_j^y}\frac{dV_0\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds}\frac{dP_{i,j}^*(x)}{dx} \\ &+ \frac{1}{h_j^y}\frac{dV_1\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds}\frac{dP_{i,j+1}^*(x)}{dx} \\ &+ \frac{dV_2\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds}\frac{d\phi_{i,j}(x)}{dx} \\ &+ \frac{dV_3\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds}\frac{d\phi_{i,j+1}(x)}{dx} \\ &= \frac{\partial^2 P_{i,j}(x,y)}{\partial y \partial x}.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x \partial y} &= \frac{d\phi_{i,j}(x)}{dx} = \frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y \partial x}, \\ \frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial x \partial y} &= \frac{d\phi_{i,j+1}(x)}{dx} = \frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial y \partial x}, \\ \frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x \partial y} &= \frac{dV_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i,j}^x}{h_j^y} \\ &+ \frac{dV_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i,j+1}^x}{h_j^y} \\ &+ \frac{dV_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i,j}^y}{h_i^x} \\ &+ \frac{dV_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i,j+1}^y}{h_i^x} \\ &= \frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y \partial x}, \\ \frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial x \partial y} &= \frac{dV_0(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i+1,j}^x}{h_j^y} \\ &+ \frac{dV_1(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i+1,j+1}^x}{h_j^y} \\ &+ \frac{dV_2(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i+1,j}^y}{h_i^x} \\ &+ \frac{dV_3(s;\alpha_{i,j}^y,\beta_{i,j}^y)}{ds}\frac{D_{i+1,j+1}^y}{h_i^x} \\ &= \frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial y \partial x}.\end{aligned}$$

These imply that $\frac{\partial^2 P_{i,j}\left(x,y_i^+\right)}{\partial x \partial y} = \frac{\partial^2 P_{i,j}\left(x,y_i^-\right)}{\partial x \partial y} = \frac{\partial^2 P_{i,j}\left(x,y_i^+\right)}{\partial y \partial x} = \frac{\partial^2 P_{i,j}\left(x,y_i^-\right)}{\partial y \partial x}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x \partial y} = \frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial x \partial y} = \frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y \partial x} = \frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial y \partial x}$ if $h_{i-1}^x = h_i^x$, $\alpha_{i-1,j}^y = \alpha_{i,j}^y$ and $\beta_{i-1,j}^y = \beta_{i,j}^y$.

For $\frac{\partial^2 P_{i,j}(x,y)}{\partial x^2}$, since the x-direction interpolation curve $P_{i,j}^*(x)$ is C^2 continuous if the first partial derivative values $D_{i,j}^x$, $i = 2,3,\dots,n-1$ are given by (2) and

$$\frac{d^2\phi_{i,l}\left(x_i^+\right)}{dx^2} = \frac{d^2\phi_{i,l}\left(x_{i+1}^-\right)}{dx^2} = 0, \quad l = j, j+1,$$

we have

$$\begin{aligned}&\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x^2} = \frac{d^2 P_{i,j}^*(x)}{dx^2}, \\ &\frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial x^2} = \frac{d^2 P_{i,j+1}^*(x)}{dx^2}, \\ &\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x^2} \\ &= V_0\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)\frac{2\left(\Delta_{i,j}^x - \Delta_{i-1,j}^x\right)}{h_{i-1}^x + h_i^x} \\ &\quad + V_1\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)\frac{2\left(\Delta_{i,j+1}^x - \Delta_{i-1,j+1}^x\right)}{h_{i-1}^x + h_i^x}, \\ &\frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial x^2} \\ &= V_0\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)\frac{2\left(\Delta_{i+1,j}^x - \Delta_{i,j}^x\right)}{h_i^x + h_{i+1}^x} \\ &\quad + V_1\left(s;\alpha_{i,j}^y,\beta_{i,j}^y\right)\frac{2\left(\Delta_{i+1,j+1}^x - \Delta_{i,j+1}^x\right)}{h_i^x + h_{i+1}^x},\end{aligned}$$

it follows that $\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial x^2} = \frac{\partial^2 P_{i,j}\left(x,y_j^-\right)}{\partial x^2}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial x^2} = \frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial x^2}$ if $\alpha_{i-1,j}^y = \alpha_{i,j}^y$ and $\beta_{i-1,j}^y = \beta_{i,j}^y$.

Finally, for $\frac{\partial^2 P_{i,j}(x,y)}{\partial y^2}$, we have

$$\begin{aligned}\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y^2} &= 0, \frac{\partial^2 P_{i,j}\left(x,y_{j+1}^-\right)}{\partial y^2} = 0, \\ \frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y^2} &= \frac{d^2V_0\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{F_{i,j}}{\left(h_j^y\right)^2} \\ &+ \frac{d^2V_1\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{F_{i,j+1}}{\left(h_j^y\right)^2} \\ &+ \frac{d^2V_2\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{D_{i,j}^y}{h_j^y} \\ &+ \frac{d^2V_3\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{D_{i,j+1}^y}{h_j^y}, \\ \frac{\partial^2 P_{i,j}\left(x_{i+1}^-,y\right)}{\partial y^2} &= \frac{d^2V_0\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{F_{i+1,j}}{\left(h_j^y\right)^2} \\ &+ \frac{d^2V_1\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{F_{i+1,j+1}}{\left(h_j^y\right)^2} \\ &+ \frac{d^2V_2\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{D_{i+1,j}^y}{h_j^y} \\ &+ \frac{d^2V_3\left(s,\alpha_{i,j}^y,\beta_{i,j}^y\right)}{ds^2}\frac{D_{i+1,j+1}^y}{h_j^y},\end{aligned}$$

it follows that $\frac{\partial^2 P_{i,j}\left(x,y_j^+\right)}{\partial y^2} = \frac{\partial^2 P_{i,j}\left(x,y_j^-\right)}{\partial y^2}$ and $\frac{\partial^2 P_{i,j}\left(x_i^+,y\right)}{\partial y^2} = \frac{\partial^2 P_{i,j}\left(x_i^-,y\right)}{\partial y^2}$ if $\alpha_{i-1,j}^y = \alpha_{i,j}^y$ and $\beta_{i-1,j}^y = \beta_{i,j}^y$. Summarizing the above discussion, we can conclude the following theorem.

Theorem 3: If the knots are equally spaced for variable x, that is h_i^x =constant, and the first partial derivative values $D_{i,j}^x$, $i = 2,3,\dots,n-1$ are given by (2), then a sufficient condition for the interpolation surface $P_{i,j}(x,y)$ to be C^2 continuous in the whole rectangular domain R is that $\alpha_{i,j}^y$ =constant and $\beta_{i,j}^y$ =constant for each $j \in \{1,2,\dots,m-1\}$ and all $i = 1,2,\dots,n-1$, no matter what the parameters $\alpha_{i,j}^x$ and $\beta_{i,j}^x$ might be.

For generating the interpolation surface $P_{i,j}(x,y)$, we also need to provide the first partial derivative values $D_{i,j}^y$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$ in advance. In this paper, they are computed by the following formula

$$\begin{cases} D^y_{i,1} = \Delta^y_{i,1} - \frac{h^y_1}{h^y_1+h^y_2}\left(\Delta^y_{i,2} - \Delta^y_{i,1}\right), \\ D^y_{i,j} = \frac{h^y_{j-1}\Delta^y_{i,j}+h^y_j\Delta^y_{i,j-1}}{h^y_{j-1}+h^y_j}, \quad j = 2, 3, \ldots, m-1, \\ D^y_{i,m} = \Delta^y_{i,m-1} \\ \qquad + \frac{h^y_{m-1}}{h^y_{m-2}+h^y_{m-1}}\left(\Delta^y_{i,m-1} - \Delta^y_{i,m-2}\right), \end{cases} \tag{6}$$

where $i = 1, 2, \ldots, n$.

IV. Numerical examples

In this section, we shall give two numerical examples to show that the proposed C^2 interpolation surface scheme can give a good approximation to the interpolated function. And for the unchanged interpolating data, the shape of the interpolation surface can be modified by changing the four parameters according to the control need. In the following figures, the interpolating data points have been marked with solid black dots.

***Example** 1:* Let the interpolated function be $F(x, y) = x^2 + y^2, (x, y) \in [0, 1] \times [0, 1]$, and $x_i = 0.2(i-1), y_j = 0.2(j-1), i, j = 1, 2, \ldots, 6$. The parameters are chosen as $\alpha^x_{i,j} = 10 + 10i + 20j, \beta^x_{i,j} = 20 + 20i + 10j, \alpha^y_{i,j} = 10 + 20j, \beta^y_{i,j} = 20 + 10j, i, j = 1, 2, \ldots, 6$. Fig. 1 shows the resulting interpolation surface $P(x, y)$ defined by (4) and the error surface $F(x, y) - P(x, y)$. From the results, we can see that the interpolation surface gives a good approximation to the interpolated function.

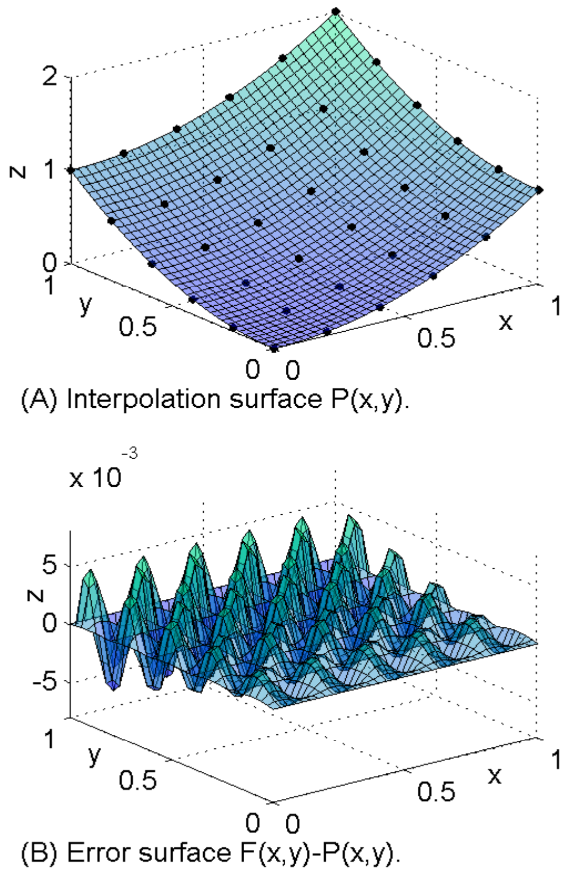

Fig. 1. Interpolation surface and the error surface.

***Example** 2:* Fig. 3 shows the C^2 interpolation surfaces with different parameters for the 3D data set given in Tab. I. It can be seen that the interpolation surface can be modified conveniently by selecting suitable parameters according to needs of practical design.

***Example** 3:* Fig. 3 shows the comparison between the C^2 interpolation surfaces generated by the new developed method and the one given in [16] for the 3D data set given in Tab. II. Our method provides four parameters, which is more flexible in adjusting the shape of surface than the the one given in [16] with two parameters.

TABLE I
The 3D data set given in [14].

y/x	0	0.5	1	1.5
0	3	2	4	3
0.5	2	1	3	2
1	3	3	1	3
1.5	2	4	2	3

TABLE II
The 3D data set given in [16].

y/x	0	1	2	3	4	5	6
0	0	1	3	4	3	1	0
1	1	2	4	5	4	2	1
3	3	4	5	8	5	4	3
5	1	2	4	5	4	2	1
6	0	1	3	4	3	1	0

V. Conclusion

As stated above, the developed interpolation surface is C^2 continuous and include the interpolation scheme given in [14] as a special case. The shape of the interpolation surface can be modified conveniently by using the parameters under the unchanged interpolating data. And the interpolation surface is bounded and stable for the four parameters. There are still some problems worthy of further study, such as the convexity control of the new constructed C^2 interpolation spline surface. These will be our future work.

Acknowledgements

The research is supported by the Doctoral Research Project of Chongqing Technology and Business University (Grant No. 1956003), the National Natural Science Foundation of China (Grant No. 61802129), the Postdoctoral Science Foundation of China (Grant No. 2015M571931), the Fundamental Research Funds for the Central Universities (Grant No. 2017MS121) and the Natural Science Foundation Guangdong Province, China (Grant No. 2018A030310381).

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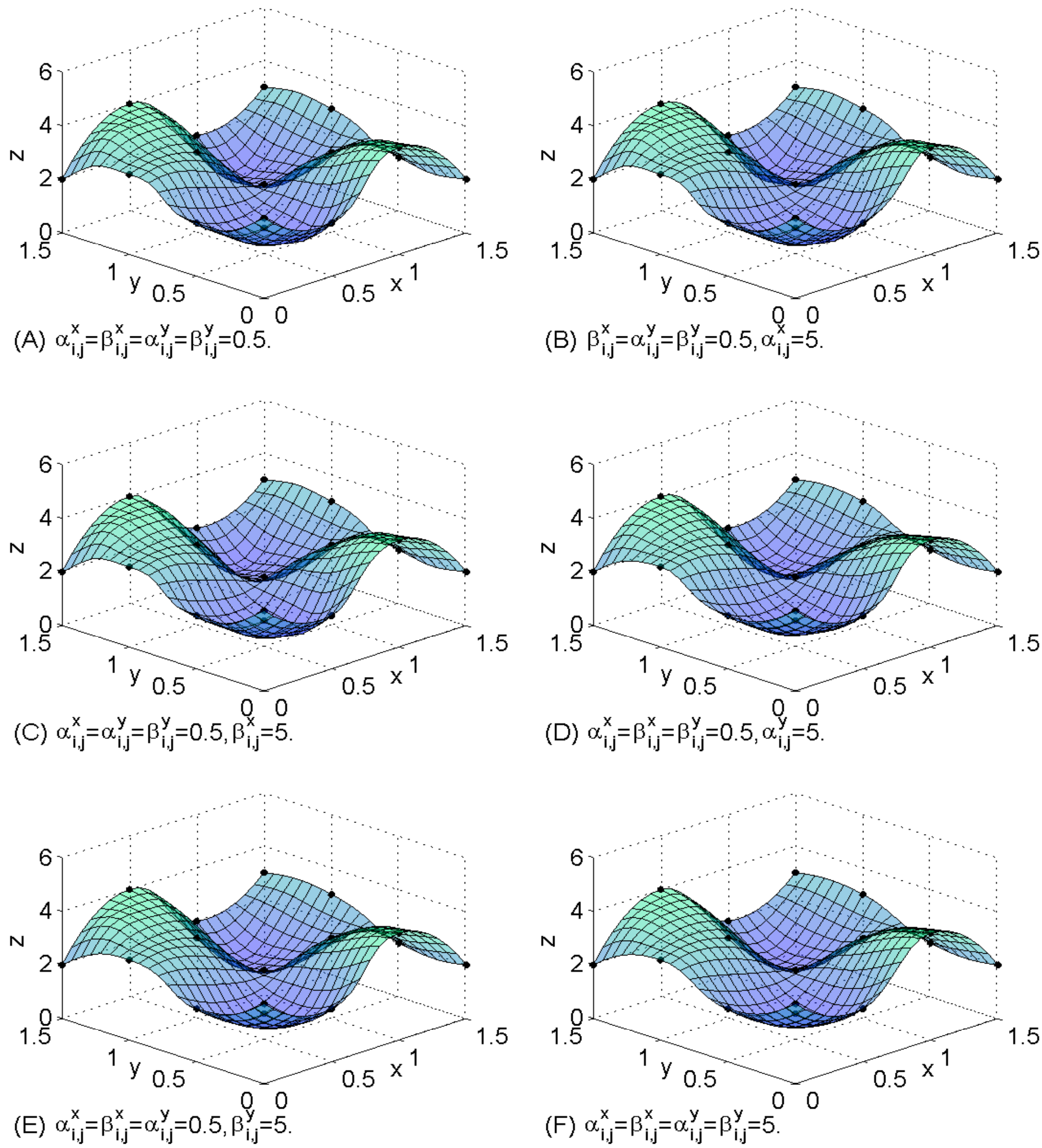

Fig. 2. C^2 interpolation surfaces with different parameters for the data set given in Tab. I.

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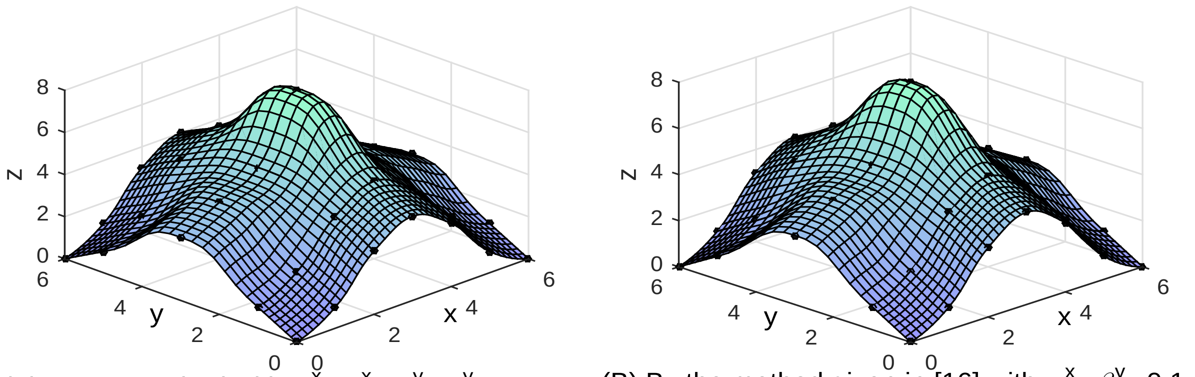

(A) By our method with $\alpha^x_{i,j}=\beta^x_{i,j}=\alpha^y_{i,j}=\beta^y_{i,j}=1$.

(B) By the method given in [16] with $\alpha^x_{i,j}=\beta^y_{i,j}=0.1$.

Fig. 3. Comparison between the C^2 interpolation surfaces generated by our method and the method given in [16] for the 3D data set given in Tab. II.

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