Generalized Simpson-Type Inequalities Considering First Derivatives through the k-Fractional Integrals

Hui Lei, Gou Hu, Jialu Nie, and Tingsong Du*

***Abstract*—Using the k-fractional integrals, we establish several Simpson-type integral inequalities for mappings whose first derivatives belong to the Lebesgue $\mathrm{L_q}$ spaces. We also present certain new inequalities of Simpson-type for mappings whose first derivatives in absolute value at some powers are (s, m)-convex. The inequalities established here generalize some known results in the literature involving Riemann-Liouville fractional integrals.**

***Index Terms*—Simpson-type inequality, (s, m)-convex functions, k-fractional integrals.**

I. Introduction

A Mapping $f : \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is named convex on I, if inequality

$$f\big(t\tau_1 + (1-t)\tau_2\big) \le tf(\tau_1) + (1-t)f(\tau_2)$$

holds for all $\tau_1, \tau_2 \in I$ and $t \in [0, 1]$.

The concept of convex mappings has been extended in different ways by many authors in recent years. In [1], the author introduced the definition of s-convexity as follows.

Definition 1.1: A mapping $f : [0, \infty) \to \mathbb{R}$ is called s-convex in the second sense, for certain fixed $s \in (0, 1]$, if the inequality

$$f\big(t\tau_1 + (1-t)\tau_2\big) \le t^s f(\tau_1) + (1-t)^s f(\tau_2)$$

holds for all $\tau_1, \tau_2 \in [0, \infty)$ and $t \in [0, 1]$.

In [2], Toader also gave the following extension of convex mappings to m-convex mappings.

Definition 1.2: A mapping $f : [0, d] \to \mathbb{R}, d > 0$ is said to be m-convex, for some fixed $m \in (0, 1]$, if the inequality

$$f\big(t\tau_1 + m(1-t)\tau_2\big) \le tf(\tau_1) + m(1-t)f(\tau_2)$$

holds for all $\tau_1, \tau_2 \in [0, d]$ and $t \in [0, 1]$.

By combining the concepts of s-convex and m-convex mappings, Eftekhari [3] introduced a class of (s, m)-convex mappings as follows.

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Definition 1.3: A mapping $f : [0, \infty) \to \mathbb{R}$ is said to be (s, m)-convex in the second sense, for certain fixed $(s, m) \in (0, 1]^2$, if the inequality

$$f\big(t\tau_1 + m(1-t)\tau_2\big) \le t^s f(\tau_1) + m(1-t)^s f(\tau_2)$$

holds for all $\tau_1, \tau_2 \in [0, \infty)$ and $t \in [0, 1]$.

Recently, many authors have studied some integral inequalities associated with these classes of convex mappings. For more information and related results, we refer the interested readers to [4], [5], [6], [7], [8], [9], [10], and the related references therein.

In what follows, we review the space of all complex-valued Lebesgue measurable functions, which will be used subsequently.

Let $\chi_c^p(a, b)$ $(c \in \mathbb{R}, 1 \le p \le \infty)$ be the space of all complex-valued Lebesgue measurable functions f on $[a, b]$ for which $||f||_{\chi_c^p} < \infty$, where the norm $|| \cdot ||_{\chi_c^p}$ is defined with the following expression:

$$||f||_{\chi_c^p} = \left(\int_a^b |t^c f(t)|^p \, \frac{\mathrm{d}t}{t}\right)^{1/p}, \quad (1 \le p < \infty)$$

and

$$||f||_{\chi_c^\infty} = \operatorname{ess} \sup_{a<t<b} \left[t^c |f(t)|\right], \quad (p = \infty),$$

where ess sup stands for essential supremum.

Also, we need the following fractional integral operators, which are essential to our current work.

Definition 1.4: Let $f \in L([a, b])$. The Riemann-Liouville integrals $\mathcal{J}_{a^+}^\mu f$ and $\mathcal{J}_{b^-}^\mu f$ of order $\mu > 0$ with $a \ge 0$ are defined as

$$\mathcal{J}_{a^+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)\mathrm{d}t, \quad (x > a)$$

and

$$\mathcal{J}_{b^-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)\mathrm{d}t, \quad (x < b),$$

where $\Gamma(\mu) = \int_0^\infty e^{-t} t^{\mu-1} \mathrm{d}t$. It is to be noted that $\mathcal{J}_{a^+}^0 f(x) = \mathcal{J}_{b^-}^0 f(x) = f(x)$.

In [11], Mubeen and Habibullah gave the following extension of Riemann-Liouville fractional integrals to k-fractional integrals.

Definition 1.5: Let $f \in L([a, b])$. Then k-fractional integrals ${}_k\mathcal{J}_{a^+}^\mu f(x)$ and ${}_k\mathcal{J}_{b^-}^\mu f(x)$ are defined as

$${}_k\mathcal{J}_{a^+}^\mu f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)\mathrm{d}t, \ (0 \le a < x < b)$$

and

$$ {}_k\mathcal{J}^{\mu}_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)\mathrm{d}t,\ (0 \le a < x < b), $$

where $k > 0$ with $\mu > 0$ and $\Gamma_k(\mu)$ is the k-gamma function given as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{\frac{-t^k}{k}} \mathrm{d}t$, with the property that $\Gamma_k(\mu + k) = \mu\Gamma_k(\mu)$. Note that ${}_k\mathcal{J}^0_{a^+} f(x) = {}_k\mathcal{J}^0_{b^-} f(x) = f(x)$.

For recent results involving the k-fractional integrals, we refer the interested readers to [12], [13], [14], and the references mentioned in these articles.

The following inequality is named the Simpson-type integral inequality:

$$ \left| \frac{1}{6}\left[h(\tau_1) + 4h\left(\frac{\tau_1+\tau_2}{2}\right) + h(\tau_2)\right] - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} h(t)\mathrm{d}t \right| \le \frac{1}{2880}\left\|h^{(4)}\right\|_\infty (\tau_2-\tau_1)^4, \tag{1} $$

where $h : [\tau_1, \tau_2] \to \mathbb{R}$ is a four-order differentiable mapping on (τ_1, τ_2) and $\left\|h^{(4)}\right\|_\infty = \sup_{t\in(\tau_1,\tau_2)} \left|h^{(4)}(t)\right| < \infty$.

Some researchers have generalized and studied the Simpson-type inequality in many different ways. For more information about the Simpson-type inequality, we refer the interested readers to [15], [16], [17], [18], [19], [20]. The authors in [21], [22], [23], [24], [25], [26], [27] also provided some Simpson-type inequalities for several classes of convex functions.

Motivated by the results in the articles above and the recent studies, the main aim of this paper is to establish several k-fractional integral inequalities of Simpson type for mappings whose first derivatives belong to the Lebesgue $\mathrm{L_q}$ spaces or (s, m)-convexity.

II. Main results

The following lemma is of importance to prove our main results.

Lemma 2.1: Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < mb \le b$ for some fixed $m \in (0, 1]$. Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) such that $f' \in L^1([a, mb])$. The following identity for k-fractional integrals with $\mu > 0$ and $k > 0$ exists:

$$ \begin{aligned} &\mathcal{T}_f(\mu, k; m, a, b) \\ &:= \left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+mb}{2}\right) + \frac{1}{6}f(mb)\right] \\ &\quad - \frac{2^{\frac{\mu}{k}-2}\Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}(2^{\frac{\mu}{k}}-1)}\left[{}_k\mathcal{J}^{\mu}_{a^+} f(mb) + {}_k\mathcal{J}^{\mu}_{mb^-} f(a)\right] \\ &\quad - \frac{2^{\frac{\mu}{k}-2}\left(2^{\frac{\mu}{k}}-2\right)\Gamma_k(\mu+k)}{(mb-a)^{\frac{\mu}{k}}\left(2^{\frac{\mu}{k}}-1\right)} \\ &\quad \times \left[{}_k\mathcal{J}^{\mu}_{(\frac{a+mb}{2})^+} f(mb) + {}_k\mathcal{J}^{\mu}_{(\frac{a+mb}{2})^-} f(a)\right] \\ &= \frac{mb-a}{4}[I_1 + I_2 + I_3 + I_4], \end{aligned} \tag{2} $$

where

$$ I_1 = \int_0^1 \left[\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}}\right] f'\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t, $$

$$ I_2 = \int_0^1 \left[\frac{1}{2}(1-t)^{\frac{\mu}{k}} - \frac{1}{6}\right] f'\left(ta + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t, $$

$$ \begin{aligned} I_3 = &\int_0^1 \left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3}\right] \\ &\times f'\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t \end{aligned} $$

and

$$ \begin{aligned} I_4 = &\int_0^1 \left[\frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} + \frac{1}{3}\right] \\ &\times f'\left(ta + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t. \end{aligned} $$

Proof. Integrating by parts, we have that

$$ \begin{aligned} I_1 &= \int_0^1 \left[\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}}\right] f'\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t \\ &= \frac{2}{mb-a}\left[\frac{1}{6}f(mb) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] - \frac{\frac{\mu}{k}}{mb-a} \\ &\quad \times \int_0^1 (1-t)^{\frac{\mu}{k}+1} f\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t \\ &= \frac{2}{mb-a}\left[\frac{1}{6}f(mb) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] - \frac{\frac{\mu}{k}}{mb-a}J_3 \end{aligned} $$

and

$$ \begin{aligned} I_3 = &\int_0^1 \left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3}\right] \\ &\times f'\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t \\ = &\frac{2}{mb-a}\left[\frac{1}{6}f(mb) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] \\ &- \frac{\frac{\mu}{k}}{(mb-a)(2^{\frac{\mu}{k}}-1)} \\ &\times \int_0^1 (1+t)^{\frac{\mu}{k}+1} f\left(tmb + (1-t)\frac{a+mb}{2}\right)\mathrm{d}t \\ = &\frac{2}{mb-a}\left[\frac{1}{6}f(mb) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] \\ &- \frac{\frac{\mu}{k}}{(mb-a)(2^{\frac{\mu}{k}}-1)}J_2. \end{aligned} $$

Analogously,

$$ I_2 = \frac{2}{mb-a}\left[\frac{1}{6}f(a) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] - \frac{\frac{\mu}{k}}{mb-a}J_1 $$

and

$$ \begin{aligned} I_4 = &\frac{2}{mb-a}\left[\frac{1}{6}f(a) + \frac{1}{3}f\left(\frac{a+mb}{2}\right)\right] \\ &- \frac{\frac{\mu}{k}}{(mb-a)(2^{\frac{\mu}{k}}-1)}J_4. \end{aligned} $$

Adding four equalities above, we get that

$$ \begin{aligned} &\frac{4}{mb-a}\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+mb}{2}\right) + \frac{1}{6}f(mb)\right] \\ &- \frac{\frac{\mu}{k}}{(mb-a)(2^{\frac{\mu}{k}}-1)}[J_1 + J_2 + J_3 + J_4] \\ &- \frac{\frac{\mu}{k}(2^{\frac{\mu}{k}}-2)}{(mb-a)(2^{\frac{\mu}{k}}-1)}[J_1 + J_3] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3} $$

Now making suitable substitutions, we have that

$$J_1 = \int_0^1 (1-t)^{\frac{\mu}{k}+1} f\Big(ta + (1-t)\frac{a+mb}{2}\Big)\mathrm{d}t = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_a^{\frac{a+mb}{2}} (u-a)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u$$

and

$$J_2 = \int_0^1 (1+t)^{\frac{\mu}{k}+1} f\Big(tmb + (1-t)\frac{a+mb}{2}\Big)\mathrm{d}t = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_{\frac{a+mb}{2}}^{mb} (u-a)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u.$$

Hence, we obtain that

$$J_1 + J_2 = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_a^{mb} (u-a)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u = \frac{2^{\frac{\mu}{k}} k\Gamma_k(\mu)}{(b-a)^{\frac{\mu}{k}}} {}_k\mathcal{J}^{\mu}_{mb^-} f(a). \tag{4}$$

Likewise, we have that

$$J_3 = \int_0^1 (1-t)^{\frac{\mu}{k}+1} f\Big(tmb + (1-t)\frac{a+mb}{2}\Big)\mathrm{d}t = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_{\frac{a+mb}{2}}^{mb} \big(mb-u\big)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u$$

and

$$J_4 = \int_0^1 (1+t)^{\frac{\mu}{k}+1} f\Big(ta + (1-t)\frac{a+mb}{2}\Big)\mathrm{d}t = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_a^{\frac{a+mb}{2}} \big(mb-u\big)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u.$$

Therefore, we have that

$$J_3 + J_4 = \frac{2^{\frac{\mu}{k}}}{(b-a)^{\frac{\mu}{k}}} \int_a^{mb} \big(mb-u\big)^{\frac{\mu}{k}-1} f(u)\mathrm{d}u = \frac{2^{\frac{\mu}{k}} k\Gamma_k(\mu)}{(b-a)^{\frac{\mu}{k}}} {}_k\mathcal{J}^{\mu}_{a^+} f(mb). \tag{5}$$

Also, we note that

$$J_1 = \frac{2^{\frac{\mu}{k}} k\Gamma_k(\mu)}{(b-a)^{\frac{\mu}{k}}} {}_k\mathcal{J}^{\mu}_{(\frac{a+mb}{2})^-} f(a) \tag{6}$$

and

$$J_3 = \frac{2^{\frac{\mu}{k}} k\Gamma_k(\mu)}{(b-a)^{\frac{\mu}{k}}} {}_k\mathcal{J}^{\mu}_{(\frac{a+mb}{2})^+} f(mb). \tag{7}$$

Using equalities (4), (5), (6) and (7) in identity (3), we get the desired identity in (2). The proof is completed.

Remark 2.1: If we choose $k = 1 = \mu$ in Lemma 2.1, then we have that

$$\frac{1}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{b-a}{2}\int_0^1 \Big[\Big(\frac{t}{2}-\frac{1}{3}\Big)f'\Big(tb + (1-t)\frac{a+b}{2}\Big) + \Big(\frac{1}{3}-\frac{t}{2}\Big)f'\Big(ta + (1-t)\frac{a+b}{2}\Big)\Big]\mathrm{d}t. \tag{8}$$

Noting that $tb + (1-t)\frac{a+b}{2} = \frac{1+t}{2}b + \frac{1-t}{2}a$ and $ta + (1-t)\frac{a+b}{2} = \frac{1+t}{2}a + \frac{1-t}{2}b$, the identity (8) yields Lemma 1 presented by Sarikaya et al. in [18].

The following calculations of definite integrals are used in next result:

$$\mathcal{K}_1 = \int_0^1 \left|\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\mathrm{d}t = \frac{1}{2(\frac{\mu}{k}+1)} - \frac{1}{6} + \Big(1 - \frac{1}{\frac{\mu}{k}+1}\Big)\Big(\frac{1}{3}\Big)^{1+\frac{k}{\mu}} \tag{9}$$

and

$$\mathcal{K}_2 = \int_0^1 \left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3}\right|\mathrm{d}t = \frac{1}{2^{\frac{\mu}{k}}-1}\frac{1}{\frac{\mu}{k}+1}\left[2^{\frac{\mu}{k}} - \Big(\frac{2(2^{\frac{\mu}{k}}-1)}{3}+1\Big)^{1+\frac{k}{\mu}} + \frac{1}{2}\right] + \Big(\frac{1}{2^{\frac{\mu}{k}}-1} + \frac{2}{3}\Big)\left[\Big(\frac{2(2^{\frac{\mu}{k}}-1)}{3}+1\Big)^{\frac{k}{\mu}} - \frac{3}{2}\right]. \tag{10}$$

Theorem 2.1: Under all conditions of Lemma 2.1, suppose that f' is bounded, i.e., $||f'||_\infty = \sup_{t\in(a,mb)} |f'(t)| < \infty$. Then for any $x \in [a, mb]$, the following inequality holds:

$$|\mathcal{T}_f(\mu,k;m,a,b)| \le \frac{mb-a}{2}\left(\mathcal{K}_1 + \mathcal{K}_2\right)||f'||_\infty, \tag{11}$$

where $\mathcal{K}_i (i = 1, 2)$ are defined by equalities (9) and (10), respectively.

Proof. If we use Lemma 2.1 and the property of modulus, then we have that

$$|\mathcal{T}_f(\mu,k;m,a,b)| \le \frac{mb-a}{4}\Big(|I_1| + |I_2| + |I_3| + |I_4|\Big). \tag{12}$$

Here,

$$|I_1| = \left|\int_0^1 \Big[\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}}\Big] \times \Big[f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big)\Big]\mathrm{d}t\right| \le \int_0^1 \left|\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}}\right| \cdot \left|f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big)\right|\mathrm{d}t.$$

Similarly,

$$|I_2| \le \int_0^1 \left|\frac{1}{2}(1-t)^{\frac{\mu}{k}} - \frac{1}{6}\right| \cdot \left|f'\Big(ta + (1-t)\frac{a+mb}{2}\Big)\right|\mathrm{d}t,$$

$$|I_3| \le \int_0^1 \left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3}\right| \times \left|f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big)\right|\mathrm{d}t$$

and

$$|I_4| \le \int_0^1 \left|\frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} + \frac{1}{3}\right| \times \left|f'\Big(ta + (1-t)\frac{a+mb}{2}\Big)\right|\mathrm{d}t.$$

Adding four inequalities above, we get that

$$\begin{aligned}&|I_1|+|I_2|+|I_3|+|I_4|\\&\le 2\bigg(\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\mathrm{d}t+\int_0^1\bigg|\frac{1}{2(2^{\frac{\mu}{k}}-1)}\\&\quad\times(1+t)^{\frac{\mu}{k}}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{3}\bigg|\mathrm{d}t\bigg)||f'||_\infty\\&=2\left(\mathcal{K}_1+\mathcal{K}_2\right)||f'||_\infty.\end{aligned}$$

This ends the proof.

Theorem 2.2: Under all conditions of Lemma 2.1, suppose that $f'\in L^1([a,mb])$. Then for any $x\in[a,mb]$, the following inequality holds:

$$|\mathcal{T}_f(\mu,k;m,a,b)|\le\frac{1}{3}||f'||_1,\tag{13}$$

where $||f'||_1=\int_a^{mb}|f'(u)|\mathrm{d}u<\infty$.

Proof. Similar to the proof of Theorem 2.1. Continuing from inequality (12), we get that

$$\begin{aligned}&|I_1|+|I_2|+|I_3|+|I_4|\\&\le\Bigg\{\sup_{t\in[0,1]}\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\\&\quad+\sup_{t\in[0,1]}\left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}}+\frac{1}{3}\right|\Bigg\}\\&\quad\times\Bigg\{\int_0^1\left|f'\left(tmb+(1-t)\frac{a+mb}{2}\right)\right|\mathrm{d}t\\&\quad+\int_0^1\left|f'\left(ta+(1-t)\frac{a+mb}{2}\right)\right|\mathrm{d}t\Bigg\}.\end{aligned}$$

The desired inequality follows from the inequality above by noting the following facts that

$$\sup_{t\in[0,1]}\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|=\frac{1}{3},$$

$$\sup_{t\in[0,1]}\left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}}+\frac{1}{3}\right|=\frac{1}{3}$$

and

$$\begin{aligned}&\int_0^1\left|f'\left(tmb+(1-t)\frac{a+mb}{2}\right)\right|\mathrm{d}t\\&\quad+\int_0^1\left|f'\left(ta+(1-t)\frac{a+mb}{2}\right)\right|\mathrm{d}t\\&=\frac{2}{mb-a}\int_a^{mb}|f'(u)|\mathrm{d}u\\&=\frac{2}{mb-a}||f'||_1.\end{aligned}$$

This ends the proof.

The following results acquired by the inequality of $(A-B)^\theta\le A^\theta-B^\theta$ for any $A\ge B\ge 0$ with $\theta\ge 1$, will be used subsequently.

$$\begin{aligned}&\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|^p\mathrm{d}t\\&=\left(\frac{1}{6}\right)^p\Bigg\{\int_0^{1-3^{-\frac{k}{\mu}}}\left[3(1-t)^{\frac{\mu}{k}}-1\right]^p\mathrm{d}t\\&\quad+\int_{1-3^{-\frac{k}{\mu}}}^1\left[1-3(1-t)^{\frac{\mu}{k}}\right]^p\mathrm{d}t\Bigg\}\\&\le\left(\frac{1}{6}\right)^p\Bigg\{\int_0^{1-3^{-\frac{k}{\mu}}}\left(\left[3(1-t)^{\frac{\mu}{k}}\right]^p-1\right)\mathrm{d}t\\&\quad+\int_{1-3^{-\frac{k}{\mu}}}^1\left(1-\left[3(1-t)^{\frac{\mu}{k}}\right]^p\right)\mathrm{d}t\Bigg\}\\&=\left(\frac{1}{6}\right)^p\bigg[-3^p\frac{2}{\frac{\mu}{k}p+1}\left(3^{-\frac{k}{\mu}}\right)^{\frac{\mu}{k}p+1}\\&\quad+3^p\frac{1}{\frac{\mu}{k}p+1}-2\left(1-3^{-\frac{k}{\mu}}\right)+1\bigg]:=M_1\end{aligned}\tag{14}$$

and

$$\begin{aligned}&\int_0^1\left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{3}\right|^p\mathrm{d}t\\&=\int_0^{\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{\frac{k}{\mu}}-1}\left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}+\frac{1}{3}-\frac{(1+t)^{\frac{\mu}{k}}}{2(2^{\frac{\mu}{k}}-1)}\right]^p\mathrm{d}t\\&\quad+\int_{\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{\frac{k}{\mu}}-1}^1\left[\frac{(1+t)^{\frac{\mu}{k}}}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{3}\right]^p\mathrm{d}t\\&\le\int_0^{\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{\frac{k}{\mu}}-1}\left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}+\frac{1}{3}\right]^p-\left[\frac{(1+t)^{\frac{\mu}{k}}}{2(2^{\frac{\mu}{k}}-1)}\right]^p\mathrm{d}t\\&+\int_{\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{\frac{k}{\mu}}-1}^1\left(\frac{(1+t)^{\frac{\mu}{k}}}{2(2^{\frac{\mu}{k}}-1)}\right)^p-\left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}+\frac{1}{3}\right]^p\mathrm{d}t\\&=\left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}+\frac{1}{3}\right]\left[2\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{\frac{k}{\mu}}-3\right]\\&\quad+\left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}\right]^p\frac{1}{\frac{\mu}{k}p+1}\\&\quad\times\left[1+2^{\frac{\mu}{k}p+1}-2\left(\frac{2^{\frac{\mu}{k}+1}+1}{3}\right)^{p+\frac{k}{\mu}}\right]\\&:=M_2.\end{aligned}\tag{15}$$

Theorem 2.3: Under all conditions of Lemma 2.1, suppose that $f'\in L^q([a,b])$ with $1<q<\infty$. Then for any $x\in[a,mb]$ with some fixed $m\in(0,1]$, the following inequality holds:

$$\begin{aligned}&|\mathcal{T}_f(\mu,k;m,a,b)|\\&\le\frac{mb-a}{2}\left(\frac{1}{mb-a}\right)^{\frac{1}{q}}\left[(M_1)^{\frac{1}{p}}+(M_2)^{\frac{1}{p}}\right]||f'||_q,\end{aligned}\tag{16}$$

where

$$||f'||_q=\left(\int_a^{mb}|f'(u)|^q\mathrm{d}u\right)^{\frac{1}{q}}<\infty$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $M_i(i=1,2)$ are defined by (14) and (15), respectively.

Proof. Continuing from inequality (12) in the proof of Theorem 2.1, and using the Hölder's inequality, we get that

$$\begin{aligned}|I_1| &\le \left(\int_0^1 \left|\left[\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right]\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_0^1 \left|f'\Big(tmb+(1-t)\frac{a+mb}{2}\Big)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\ &= \Big(\frac{2}{mb-a}\Big)^{\frac{1}{q}} \left(\int_0^1 \left|\left[\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right]\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_{\frac{a+mb}{2}}^{mb} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}|I_2| &\le \left(\int_0^1 \left|\left[\frac{1}{2}(1-t)^{\frac{\mu}{k}}-\frac{1}{6}\right]\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_0^1 \left|f'\Big(ta+(1-t)\frac{a+mb}{2}\Big)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\ &= \Big(\frac{2}{mb-a}\Big)^{\frac{1}{q}} \left(\int_0^1 \left|\left[\frac{1}{2}(1-t)^{\frac{\mu}{k}}-\frac{1}{6}\right]\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times \left(\int_a^{\frac{a+mb}{2}} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.\end{aligned}$$

Adding two inequalities above, we get that

$$\begin{aligned}&|I_1|+|I_2| \\ &\le \Big(\frac{2}{mb-a}\Big)^{\frac{1}{q}}\Big(\int_0^1 \left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|^p \mathrm{d}t\Big)^{\frac{1}{p}} \\ &\times\left\{\Big(\int_{\frac{a+mb}{2}}^{mb} |f'(u)|^q \mathrm{d}u\Big)^{\frac{1}{q}}+\Big(\int_a^{\frac{a+mb}{2}} |f'(u)|^q \mathrm{d}u\Big)^{\frac{1}{q}}\right\}.\end{aligned} \tag{17}$$

Utilizing the inequality $(\rho^r+\sigma^r)\le 2^{1-r}(\rho+\sigma)^r$ for $\rho,\sigma>0$ and $r\le 1$, we have that

$$\begin{aligned}&\left(\int_{\frac{a+mb}{2}}^{mb} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}+\left(\int_a^{\frac{a+mb}{2}} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}} \\ &\le 2^{1-\frac{1}{q}}\left(\int_a^{mb} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.\end{aligned} \tag{18}$$

Using inequality (18) in inequality (17), we obtain that

$$\begin{aligned}&|I_1|+|I_2| \\ &\le 2\Big(\frac{1}{mb-a}\Big)^{\frac{1}{q}}\left(\int_0^1 \left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times\Big(\int_a^{mb} |f'(u)|^q \mathrm{d}u\Big)^{\frac{1}{q}}.\end{aligned} \tag{19}$$

Making use of (14) in inequality (19), we have that

$$|I_1|+|I_2| \le 2\Big(\frac{1}{mb-a}\Big)^{\frac{1}{q}}(M_1)^{\frac{1}{p}}\left(\int_a^{mb} |f'(u)|^q \mathrm{d}u\right)^{\frac{1}{q}}.$$

Analogously,

$$\begin{aligned}&|I_3|+|I_4| \\ &\le 2\|f'\|_q\Big(\frac{1}{mb-a}\Big)^{\frac{1}{q}} \\ &\quad\times\left(\int_0^1\left|\frac{(1+t)^{\frac{\mu}{k}}}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{3}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\le 2\|f'\|_q\Big(\frac{1}{mb-a}\Big)^{\frac{1}{q}}(M_2)^{\frac{1}{p}},\end{aligned}$$

where M_2 is defined by (15). This ends the proof.

Theorem 2.4: If $|f'|^q$ for $q>1$ with $p^{-1}+q^{-1}=1$ is (s,m)-convex on I, then the following inequality with $\mu>0$ and $k>0$ holds:

$$\begin{aligned}&|\mathcal{T}_f(\mu,k;m,a,b)| \\ &\le \frac{mb-a}{4}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\left[(M_1)^{\frac{1}{p}}+(M_2)^{\frac{1}{p}}\right] \\ &\quad\times\left\{\left[m\left|f'\left(\frac{a}{m}\right)\right|^q+\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ &\quad\left.+\left[\left|f'\left(\frac{a+mb}{2}\right)\right|^q+m|f'(b)|^q\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{20}$$

where M_1 and M_2 are defined by (14) with (15), respectively.

Proof. Continuing from inequality (12) in the proof of Theorem 2.1, and using the Hölder's inequality, the (s,m)-convexity of $|f'|^q$ and the inequality (14), we get that

$$\begin{aligned}&|I_1| \\ &\le \left(\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_0^1\left|f'\Big(tmb+(1-t)\frac{a+mb}{2}\Big)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\ &\le \left(\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_0^1\left[mt^s|f'(b)|^q+(1-t)^s\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\ &\le (M_1)^{\frac{1}{p}}\left(\frac{m}{s+1}|f'(b)|^q+\frac{1}{s+1}\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Analogously,

$$\begin{aligned}&|I_2| \\ &\le (M_1)^{\frac{1}{p}}\left(\frac{m}{s+1}\left|f'\left(\frac{a}{m}\right)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}&|I_3| \\ &\le (M_2)^{\frac{1}{p}}\left(\frac{m}{s+1}|f'(b)|^q+\frac{1}{s+1}\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}&|I_4| \\ &\le (M_2)^{\frac{1}{p}}\left(\frac{m}{s+1}\left|f'\left(\frac{a}{m}\right)\right|^q+\frac{1}{s+1}\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

The desired result yields by adding four inequalities above. This ends the proof.

In the next result, we will use the following functions.

(I) The Beta function,

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x,y>0.$$

(II) The incomplete Beta function,

$$\beta(a;x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad 0<a<1,\ x,y>0.$$

Now, we give the calculated definite integrals used as the following Theorem 2.5.

$$\begin{aligned}\mathcal{K}_3 &= \int_0^1 \left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right| t\mathrm{d}t \\ &= \beta\left(1-3^{-\frac{k}{\mu}};2,\frac{\mu}{k}+1\right)-\frac{1}{2}\beta\left(2,\frac{\mu}{k}+1\right) \\ &\quad -\frac{1}{6}\left(1-3^{-\frac{k}{\mu}}\right)^2+\frac{1}{12}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathcal{K}_4 &= \int_0^1 \left|\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}}-\frac{1}{2(2^{\frac{\mu}{k}}-1)}-\frac{1}{3}\right| t\mathrm{d}t \\ &= \frac{1}{2^{\frac{\mu}{k}}-1}\left\{\frac{1}{\frac{\mu}{k}+2}\left[2^{\frac{\mu}{k}+1}-\left(\frac{2(2^{\frac{\mu}{k}}-1)}{3}+1\right)^{1+2\frac{k}{\mu}}+\frac{1}{2}\right]\right. \\ &\quad \left.-\frac{1}{\frac{\mu}{k}+1}\left[2^{\frac{\mu}{k}}-\left(\frac{2(2^{\frac{\mu}{k}}-1)}{3}+1\right)^{1+\frac{k}{\mu}}+\frac{1}{2}\right]\right\} \\ &\quad +\left(\frac{1}{2^{\frac{\mu}{k}}-1}+\frac{2}{3}\right)\left[\frac{1}{2}\left(\left[\frac{2(2^{\frac{\mu}{k}}-1)}{3}+1\right]^{\frac{k}{\mu}}-1\right)^2-\frac{1}{4}\right].\end{aligned} \tag{22}$$

Theorem 2.5: If $|f'|^q$ for $q\geq 1$ is (s,m)-convex on I, then the following inequality with $\mu>0$ and $k>0$ holds:

$$\begin{aligned}&|\mathcal{T}_f(\mu,k;m,a,b)| \\ &\leq \frac{mb-a}{4} \\ &\times\left\{(\mathcal{K}_1)^{1-\frac{1}{q}}\left[\left(\xi_1 m\left|f'\left(\frac{a}{m}\right)\right|^q+\xi_2\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\right. \\ &\left.+\left(\xi_1 m\left|f'(b)\right|^q+\xi_2\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \\ &+(\mathcal{K}_2)^{1-\frac{1}{q}}\left[\left(\xi_3 m\left|f'\left(\frac{a}{m}\right)\right|^q+\xi_4\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.\left.+\left(\xi_3 m\left|f'(b)\right|^q+\xi_4\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]\right\},\end{aligned} \tag{23}$$

where

$$\xi_1=(1-s)\mathcal{K}_1+s\mathcal{K}_3, \quad \xi_2=\mathcal{K}_1-s\mathcal{K}_3,$$

$$\xi_3=(1-s)\mathcal{K}_2+s\mathcal{K}_4, \quad \xi_4=\mathcal{K}_2-s\mathcal{K}_4$$

and $\mathcal{K}_i$ $(i=1,2,3,4)$ are defined by the equalities (9), (10), (21) and (22), respectively.

Proof. Continuing from inequality (12) in the proof of Theorem 2.1, and using the Hölder's inequality in the following way, we get that

$$\begin{aligned}&|I_1| \\ &\leq\left(\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\right. \\ &\quad\left.\times\left|f'\left(tmb+(1-t)\frac{a+mb}{2}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned} \tag{24}$$

Using the (s,m)-convexity of $|f'|^q$, we have that

$$\begin{aligned}&\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\cdot\left|f'\left(tmb+(1-t)\frac{a+mb}{2}\right)\right|^q\mathrm{d}t \\ &\leq\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right| \\ &\quad\times\left[mt^s\left|f'(b)\right|^q+(1-t)^s\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right]\mathrm{d}t \\ &=m\left|f'(b)\right|^q\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|t^s\mathrm{d}t \\ &\quad+\left|f'\left(\frac{a+mb}{2}\right)\right|^q\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|(1-t)^s\mathrm{d}t.\end{aligned} \tag{25}$$

Utilizing the inequality of $u^\varrho\leq(u-1)\varrho+1$ for all $0\leq\varrho\leq 1$ with $u>0$, we have that

$$\begin{aligned}&\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|t^s\mathrm{d}t \\ &\leq\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|\left[(1-s)+st\right]\mathrm{d}t \\ &=(1-s)\mathcal{K}_1+s\mathcal{K}_3\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|(1-t)^s\mathrm{d}t \\ &\leq\int_0^1\left|\frac{1}{6}-\frac{1}{2}(1-t)^{\frac{\mu}{k}}\right|(1-st)\mathrm{d}t \\ &=\mathcal{K}_1-s\mathcal{K}_3.\end{aligned} \tag{27}$$

Using inequalities (25), (26) and (27) in (24), we obtain that

$$\begin{aligned}|I_1|\leq(\mathcal{K}_1)^{1-\frac{1}{q}}&\left\{\left(\mathcal{K}_1-s\mathcal{K}_3\right)\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right. \\ &\left.+\left[(1-s)\mathcal{K}_1+s\mathcal{K}_3\right]m\left|f'(b)\right|^q\right\}^{\frac{1}{q}}.\end{aligned}$$

To obtain the upper bounds of $|I_2|$, $|I_3|$ and $|I_4|$, respectively, by means of the same approach above, we have that

$$\begin{aligned}|I_2|\leq(\mathcal{K}_1)^{1-\frac{1}{q}}&\left\{\left[(1-s)\mathcal{K}_1+s\mathcal{K}_3\right]m\left|f'\left(\frac{a}{m}\right)\right|^q\right. \\ &\left.+\left(\mathcal{K}_1-s\mathcal{K}_3\right)\left|f'\left(\frac{a+mb}{2}\right)\right|^q\right\}^{\frac{1}{q}},\end{aligned}$$

$$|I_3| \leq (\mathcal{K}_2)^{1-\frac{1}{q}} \left\{ \left(\mathcal{K}_2 - s\mathcal{K}_4\right) \left| f'\left(\frac{a+mb}{2}\right) \right|^q + \Big[(1-s)\mathcal{K}_2 + s\mathcal{K}_4\Big] m \left| f'(b) \right|^q \right\}^{\frac{1}{q}}$$

and

$$|I_4| \leq (\mathcal{K}_2)^{1-\frac{1}{q}} \left\{ \Big[(1-s)\mathcal{K}_2 + s\mathcal{K}_4\Big] m \left| f'\left(\frac{a}{m}\right) \right|^q + \left(\mathcal{K}_2 - s\mathcal{K}_4\right) \left| f'\left(\frac{a+mb}{2}\right) \right|^q \right\}^{\frac{1}{q}}.$$

By adding four inequalities above, we obtain the desired result in (23). This ends the proof.

Corollary 2.1: Under all conditions of Theorem 2.5, if we take $q = 1$, then we have the following inequality

$$\begin{aligned} &|\mathcal{T}_f(\mu, k; m, a, b)| \\ &\leq \frac{mb-a}{4} \Bigg\{ \Big[(1-s)(\mathcal{K}_1 + \mathcal{K}_2) + s(\mathcal{K}_3 + \mathcal{K}_4)\Big] \\ &\quad \times m \left(\left| f'\left(\frac{a}{m}\right) \right| + |f'(b)| \right) \\ &\quad + 2\Big[\mathcal{K}_1 + \mathcal{K}_2 - s(\mathcal{K}_3 + \mathcal{K}_4)\Big] \left| f'\left(\frac{a+mb}{2}\right) \right| \Bigg\}, \end{aligned}$$

where $\mathcal{K}_i$ $(i = 1, 2, 3, 4)$ are defined by the equalities (9), (10), (21) and (22), respectively.

By means of Lemma 2.1, we obtain a new simpson-type inequality in the case that first derivative of considered function is Lipschitzian.

Theorem 2.6: Under all conditions of Lemma 2.1, suppose that f' satisfies a Lipschitz condition on $[a, mb]$ for some $L > 0$. Then for $\mu > 0$, $k > 0$ with some fixed $m \in (0, 1]$, we have the following inequality

$$|\mathcal{T}_f(\mu, k; m, a, b)| \leq \frac{L(mb-a)^2}{4} \Big(\mathcal{K}_3 + \mathcal{K}_4\Big), \tag{28}$$

where $\mathcal{K}_i$ $(i = 3, 4)$ are defined by the equalities (21) and (22), respectively.

Proof. If we use Lemma 2.1, then we get that

$$\begin{aligned} &\mathcal{T}_f(\mu, k; m, a, b) \\ &= \frac{mb-a}{4} \Bigg\{ \int_0^1 \left[\frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}} \right] \\ &\quad \times \left[f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big) \right. \\ &\quad \left. - f'\Big(ta + (1-t)\frac{a+mb}{2}\Big) \right] \mathrm{d}t \\ &\quad + \int_0^1 \left[\frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3} \right] \\ &\quad \times \left[f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big) \right. \\ &\quad \left. - f'\Big(ta + (1-t)\frac{a+mb}{2}\Big) \right] \mathrm{d}t \Bigg\}. \end{aligned}$$

Since f' satisfies a Lipschitz condition on $[a, mb]$ for some $L > 0$, we have that

$$\begin{aligned} &\left| f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big) - f'\Big(ta + (1-t)\frac{a+mb}{2}\Big) \right| \\ &\leq L \left| tmb + (1-t)\frac{a+mb}{2} - ta - (1-t)\frac{a+mb}{2} \right| \\ &= L(mb-a)t. \end{aligned}$$

Therefore,

$$\begin{aligned} &|\mathcal{T}_f(\mu, k; m, a, b)| \\ &\leq \frac{(mb-a)}{4} \Bigg\{ \int_0^1 \left| \frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}} \right| \\ &\quad \times \left| f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big) \right. \\ &\quad \left. - f'\Big(ta + (1-t)\frac{a+mb}{2}\Big) \right| \mathrm{d}t \\ &\quad + \int_0^1 \left| \frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3} \right| \\ &\quad \times \left| f'\Big(tmb + (1-t)\frac{a+mb}{2}\Big) \right. \\ &\quad \left. - f'\Big(ta + (1-t)\frac{a+mb}{2}\Big) \right| \mathrm{d}t \Bigg\} \\ &\leq \frac{L(mb-a)^2}{4} \Bigg[\int_0^1 \left| \frac{1}{6} - \frac{1}{2}(1-t)^{\frac{\mu}{k}} \right| t \mathrm{d}t \\ &\quad + \int_0^1 \left| \frac{1}{2(2^{\frac{\mu}{k}}-1)}(1+t)^{\frac{\mu}{k}} - \frac{1}{2(2^{\frac{\mu}{k}}-1)} - \frac{1}{3} \right| t \mathrm{d}t \Bigg]. \end{aligned}$$

The desired inequality follows from the above by noting the equalities (21) and (22). This ends the proof.

Corollary 2.2: In Theorem 2.6, if we take $m = 1$ and $\mu = 1 = k$, then one obtains

$$\begin{aligned} &\left| \left[\frac{1}{6}f(a) + \frac{2}{3}f\Big(\frac{a+b}{2}\Big) + \frac{1}{6}f(b) \right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{2L(b-a)^2}{81}. \end{aligned} \tag{29}$$

III. Conclusion

Based on the presented k-fractional integral identity, several Simpson-type integral inequalities are obtained for mappings whose first derivatives belong to the Lebesgue $\mathrm{L_p}$ spaces or (s, m)-convexity. Some sub-results can be deduced from our main results by considering different special parameter values for k, μ and m. The results established here provide new extensions of those given in earlier works as the estimates of Simpson-type inequalities for k-fractional integrals involving (s, m)-convex mappings doesn't exist previously. With these ideas and the techniques developed in this article, it is possible to investigate further estimates of other type integral inequalities for k-fractional integrals which involve other related classes of mappings.

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