A Class of Polynomial Spline Curve with Free Parameters that Naturally Interpolates the Data Points

Juncheng Li, and Chengzhi Liu

Abstract—**In this paper, we present a class of polynomial spline curve that can naturally interpolate the given data points. The proposed curve is C^3 and can be adjusted by two free parameters when the data points are kept unchanged. In addition, we give three selection schemes of the free parameters for constructing the curve with approximate shortest arc length, the curve with approximate smallest strain energy and the curve with approximate shortest arc length and smallest strain energy.**

***Index Terms*—Arc length, free parameter, interpolating spline, polynomial spline, strain energy**

I. Introduction

In geometric modeling, the shape of curves/surfaces often needs to be adjusted freely. In order to achieve this goal, the parametric curves/surfaces with free parameters have been widely concerned. Some examples are the curves/surfaces with free parameters that similar to the Bézier curves/surfaces (see [1-4]), the curves/surfaces with free parameters that similar to the B-spline curves/surfaces (see [5-8]), the trigonometric curves/surfaces with free parameters that similar to the Bézier curves/surfaces or the B-spline curves/surfaces (see [9-13]). Due to the free parameters, those curves/surfaces have better performance than the classical Bézier curves/surfaces or B-spline curves/surfaces.

As we know, the construction of interpolating curves is a basic problem in CAD and related fields. When we use Bézier curve or B-spline curve to construct interpolating curves, the general way is to reverse a linear equations system to get the control points. In order to construct curves that can naturally interpolate the data points, the interpolation splines with free parameters are presented in [14, 15]. However, the splines presented in [14, 15] are trigonometric polynomials. It would not convenient to use the non-polynomial splines in CAD systems. In addition, the splines presented in [14, 15] can only achieve C^2 continuity. Higher-order continuity may be required in some applications. Therefore, it is necessary to construct the polynomial splines with free parameters that can achieve higher-order continuity and naturally interpolate the data points. The main purpose of this paper is to present a class of polynomial spline curve with two free parameters that can achieve C^3 continuity and naturally interpolate the data points, and give some schemes for selecting the free parameters of the curve.

The rest of this paper is organized as follows. In Section II, the new curve is presented and some properties of the curve are given. In Section III, three selection schemes of the free parameters are discussed. A short conclusion is given in Section IV.

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II. The Interpolating Spline Curve with Two Free Parameters

A. The Basis Functions

Definition 1. *For $0 \le t \le 1$, $\alpha, \beta \in \mathbf{R}$, the basis functions are expressed by*

$$\begin{cases} f_0(t) = a_0(t)\alpha + b_0(t)\beta, \\ f_1(t) = a_1(t)\alpha + b_1(t)\beta + c_1(t), \\ f_2(t) = -a_0(t)\alpha + b_2(t)\beta + (1 - c_1(t)), \\ f_3(t) = -a_1(t)\alpha - (b_0(t) + b_1(t) + b_2(t))\beta, \end{cases} \tag{1}$$

where

$$\begin{cases} a_0(t) = -t + 20t^4 - 45t^5 + 36t^6 - 10t^7, \\ a_1(t) = 15t^4 - 39t^5 + 34t^6 - 10t^7, \\ b_0(t) = t^2 - 10t^4 + 20t^5 - 15t^6 + 4t^7, \\ b_1(t) = -2t^2 + 25t^4 - 54t^5 + 43t^6 - 12t^7, \\ b_2(t) = t^2 - 20t^4 + 48t^5 - 41t^6 + 12t^7, \\ c_1(t) = 1 - 35t^4 + 84t^5 - 70t^6 + 20t^7. \end{cases}$$

By simple calculation, we can easily get that the proposed basis functions have the following properties,

(A) The sum of the basis functions is 1, that is

$$f_0(t) + f_1(t) + f_2(t) + f_3(t) \equiv 1 .$$

(B) The basis functions have symmetry, that is

$$f_i(1-t) = f_{3-i}(t) \ (i = 0,1,2,3) .$$

(C) The basis functions satisfy that

$$\begin{cases} f_0(0) = 0, \quad f_1(0) = 1, \quad f_2(0) = 0, \quad f_3(0) = 0, \\ f_0(1) = 0, \quad f_1(1) = 0, \quad f_2(1) = 1, \quad f_3(1) = 0. \end{cases} \tag{2}$$

$$\begin{cases} f_0'(0) = -\alpha, \quad f_1'(0) = 0, \quad f_2'(0) = \alpha, \quad f_3'(0) = 0, \\ f_0'(1) = 0, \quad f_1'(1) = -\alpha, \quad f_2'(1) = 0, \quad f_3'(1) = \alpha. \end{cases} \tag{3}$$

$$\begin{cases} f_0''(0)=2\beta, & f_1''(0)=-4\beta, & f_2''(0)=2\beta, & f_3''(0)=0, \\ f_0''(1)=0, & f_1''(1)=2\beta, & f_2''(1)=-4\beta, & f_3''(1)=2\beta. \end{cases} \quad (4)$$

$$\begin{cases} f_0'''(0)=f_1'''(0)=f_2'''(0)=f_3'''(0)=0, \\ f_0'''(1)=f_1'''(1)=f_2'''(1)=f_3'''(1)=0. \end{cases} \quad (5)$$

B. The Interpolating Spline Curve

Definition 2. *Given data points* $\boldsymbol{q}_i$ $(i=0,1,\cdots,n; n\ge 3)$ *, the interpolating spline curve is expressed by*

$$\boldsymbol{R}_i(t)=\sum_{j=0}^{3} f_j(t)\boldsymbol{q}_{i+j} \quad (i=0,1,\cdots,n-3), \quad (6)$$

where $f_j(t)$ $(j=0,1,2,3)$ *are the basis functions expressed in Eq. (1).*

From the properties of the basis functions, we can get that the proposed curve has the following properties,

(A) Since the basis functions have symmetry, we have

$$\begin{aligned} &\boldsymbol{R}_i(1-t;\boldsymbol{q}_{i+3},\boldsymbol{q}_{i+2},\boldsymbol{q}_{i+1},\boldsymbol{q}_i) \\ &= f_3(t)\boldsymbol{q}_{i+3}+f_2(t)\boldsymbol{q}_{i+2}+f_1(t)\boldsymbol{q}_{i+1}+f_0(t)\boldsymbol{q}_i \\ &= \boldsymbol{R}_i(t;\boldsymbol{q}_i,\boldsymbol{q}_{i+1},\boldsymbol{q}_{i+2},\boldsymbol{q}_{i+3}). \end{aligned} \quad (7)$$

Eq. (7) shows that both $\boldsymbol{q}_i$ and $\boldsymbol{q}_{n-i}$ $(i=0,1,\cdots,n)$ define the same curves in different parameterizations. That means the curve has symmetry.

(B) From Eq. (2) and Eq. (6), we have

$$\begin{cases} \boldsymbol{R}_i(0)=\boldsymbol{q}_{i+1}, \\ \boldsymbol{R}_i(1)=\boldsymbol{q}_{i+2}, \end{cases} \quad (8)$$

where $i=0,1,\cdots,n-3$. Eq. (8) shows that the curve naturally interpolates the given data points except $\boldsymbol{q}_0$ and $\boldsymbol{q}_n$.

(C) By Eq. (3) and Eq. (6), we have

$$\begin{cases} \boldsymbol{R}_i'(0)=\alpha(\boldsymbol{q}_{i+2}-\boldsymbol{q}_i), \\ \boldsymbol{R}_i'(1)=\alpha(\boldsymbol{q}_{i+3}-\boldsymbol{q}_{i+1}). \end{cases} \quad (9)$$

By Eq. (4) and Eq. (6), we have

$$\begin{cases} \boldsymbol{R}_i''(0)=2\beta(\boldsymbol{q}_i-2\boldsymbol{q}_{i+1}+\boldsymbol{q}_{i+2}), \\ \boldsymbol{R}_i''(1)=2\beta(\boldsymbol{q}_{i+1}-2\boldsymbol{q}_{i+2}+\boldsymbol{q}_{i+3}). \end{cases} \quad (10)$$

By Eq. (5) and Eq. (6), we have

$$\begin{cases} \boldsymbol{R}_i'''(0)=0, \\ \boldsymbol{R}_i'''(1)=0. \end{cases} \quad (11)$$

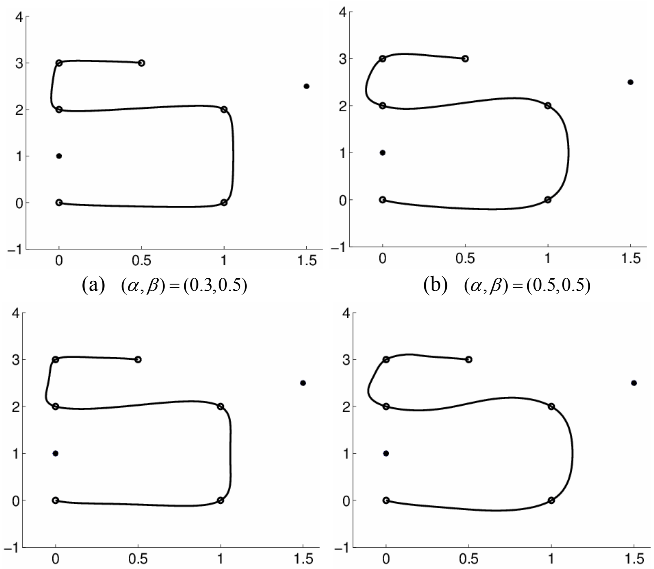
From Eqs. (9)-(11), we have

$$\boldsymbol{R}_i^{(k)}(1)=\boldsymbol{R}_{i+1}^{(k)}(0), \; k=0,1,2,3. \quad (12)$$

Eq. (12) shows that the curve achieves C^3 continuity.

(D) Since Eq. (6) contains two free parameters α and β, shape of the curve can be modified by the two free parameters.

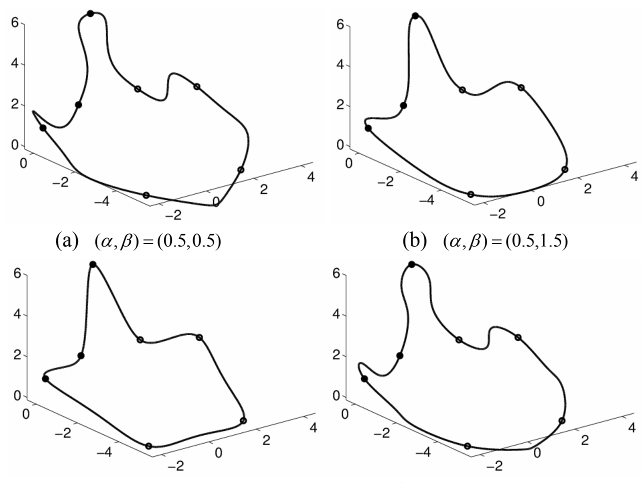
Because the curve naturally interpolates the given data points except $\boldsymbol{q}_0$ and $\boldsymbol{q}_n$, we would get the open curve that can naturally interpolate all the data points if two auxiliary points $\boldsymbol{q}_{-1}$ and $\boldsymbol{q}_{n+1}$ are added to the data points $\boldsymbol{q}_i$ $(i=0,1,\cdots,n)$. Not hard to find that the two auxiliary points would only influence the first and the last segments of the curve. Thus the open curve would be uniquely determined by the two free parameters when all the data points including the two auxiliary points are fixed, see Fig. 1 as an illustration.



(a) $(\alpha,\beta)=(0.3,0.5)$ (b) $(\alpha,\beta)=(0.5,0.5)$ (c) $(\alpha,\beta)=(0.5,1.2)$ (d) $(\alpha,\beta)=(0.7,1.2)$

Fig. 1. The open curves with different free parameters

If three auxiliary points $\boldsymbol{q}_{-1}=\boldsymbol{q}_n$, $\boldsymbol{q}_{n+1}=\boldsymbol{q}_0$ and $\boldsymbol{q}_{n+2}=\boldsymbol{q}_1$ are added to the data points $\boldsymbol{q}_i$ $(i=0,1,\cdots,n)$, we would get the closed curve that can naturally interpolate all the data points. The closed curve could be uniquely determined by the two free parameters when all the data points including the three auxiliary points are fixed, see Fig. 2 as an illustration.



(a) $(\alpha,\beta)=(0.5,0.5)$ (b) $(\alpha,\beta)=(0.5,1.5)$ (c) $(\alpha,\beta)=(1.0,1.5)$ (d) $(\alpha,\beta)=(1.0,0.5)$

Fig. 2. The closed curves with different free parameters

As we know, the B-spline and the NURBS are basic spline models in most CAD systems. Compared with the B-spline and the NURBS, the proposed spline has the following advantages,

(A) When using the B-spline or the NURBS to construct C^3 interpolating curve, we often need to reverse an equation system. However, we can naturally get the C^3 interpolating curve by using the proposed spline without solving the equation system.

(B) The shape of the B-spline would be unique if the control points are fixed. Although the shape of the NURBS can be modified by changing the weights, the rational structure of the NURBS leads to a large amount of calculation. However, we can easily modify the shape of the proposed spline by changing the two free parameters even if all the data points are fixed. Moreover the proposed spline is polynomial, so its structure is simpler than the NURBS.

Compared with the interpolating spline presented in [14, 15], the proposed spline has the following advantages,

(A) The proposed spline is polynomial, while the splines presented in [14, 15] are trigonometric. Hence the proposed spline is more suitable for CAD systems.

(B) The proposed spline achieves C^3 continuity, while the splines presented in [14, 15] can only achieve C^2 continuity. Hence the proposed spline is more suitable for applications requiring higher-order continuity.

III. Parameters Selection Schemes of the Curve

As shown in Fig. 1 and Fig. 2, we can change the shape of the proposed curve by the two free parameters. Alternatively, we can concretely select the two parameters according to some goals. Here, we present the selection schemes of the free parameters for constructing the curve with approximate shortest arc length, the curve with approximate smallest strain energy and the curve with approximate shortest arc length and smallest strain energy.

(A) Parameters selection scheme for constructing the curve with approximate shortest arc length.

Generally, the arc length of the curve $\boldsymbol{r}(t)$ $(a \le t \le b)$ can be described by

$$L = \int_a^b |\boldsymbol{r}'(t)| \mathrm{d}t . \tag{13}$$

In order to facilitate the calculation, the arc length (13) can be expressed as the following approximate form (see [16]),

$$\hat{s}_1(\mathbf{b}) = \int_0^1 \|\mathbf{b}'(t)\|^2 \,\mathrm{d}t . \tag{14}$$

Then the arc length of the proposed open curve can be approximately expressed by

$$L(\alpha,\beta) = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{R}_i'(t)\right)^2 \mathrm{d}t . \tag{15}$$

By Eq. (1), we can rewrite (6) as follows,

$$\boldsymbol{R}_i(t) = \boldsymbol{A}_i(t)\alpha + \boldsymbol{B}_i(t)\beta + \boldsymbol{C}_i(t) , \tag{16}$$

where

$\boldsymbol{A}_i(t) = a_0(t)\boldsymbol{q}_i + a_1(t)\boldsymbol{q}_{i+1} - a_0(t)\boldsymbol{q}_{i+2} - a_1(t)\boldsymbol{q}_{i+3}$,

$\boldsymbol{B}_i(t) = b_0(t)\boldsymbol{q}_i + b_1(t)\boldsymbol{q}_{i+1} + b_2(t)\boldsymbol{q}_{i+2} - (b_0(t) + b_1(t) + b_2(t))\boldsymbol{q}_{i+3}$,

$\boldsymbol{C}_i(t) = c_1(t)\boldsymbol{q}_{i+1} + (1 - c_1(t))\boldsymbol{q}_{i+2}$.

By taking Eq. (16) into Eq. (15), we have

$$L(\alpha,\beta) = \lambda_1\alpha^2 + \lambda_2\beta^2 + 2\lambda_3\alpha\beta + 2\lambda_4\alpha + 2\lambda_5\beta + \lambda_6 , \tag{17}$$

where

$\lambda_1 = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{A}_i'(t)\right)^2 \mathrm{d}t$, $\lambda_2 = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{B}_i'(t)\right)^2 \mathrm{d}t$,

$\lambda_3 = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{A}_i'(t) \cdot \boldsymbol{B}_i'(t)\right) \mathrm{d}t$, $\lambda_4 = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{A}_i'(t) \cdot \boldsymbol{C}_i'(t)\right) \mathrm{d}t$,

$\lambda_5 = \sum_{i=-1}^{n-2} \int_0^1 \left(\boldsymbol{B}_i'(t) \cdot \boldsymbol{C}_i'(t)\right) \mathrm{d}t$, $\lambda_6 = \sum_{i=0}^{n-2} \int_0^1 \left(\boldsymbol{C}_i'(t)\right)^2 \mathrm{d}t$.

In order to get the approximate shortest arc length, it must have

$$\begin{cases} \dfrac{\partial L(\alpha,\beta)}{\partial \alpha} = 0, \\ \dfrac{\partial L(\alpha,\beta)}{\partial \beta} = 0. \end{cases} \tag{18}$$

From Eq. (17), we can rewrite Eq. (18) as follows,

$$\begin{cases} \lambda_1\alpha + \lambda_3\beta + \lambda_4 = 0, \\ \lambda_3\alpha + \lambda_2\beta + \lambda_5 = 0. \end{cases} \tag{19}$$

By solving Eq. (19), we have

$$\begin{cases} \alpha = \dfrac{\lambda_3\lambda_5 - \lambda_2\lambda_4}{\lambda_1\lambda_2 - \lambda_3^2}, \\ \beta = \dfrac{\lambda_3\lambda_4 - \lambda_1\lambda_5}{\lambda_1\lambda_2 - \lambda_3^2}, \end{cases} \tag{20}$$

where $\lambda_1\lambda_2 - \lambda_3^2 \ne 0$.

It should be noted that Eq. (19) would have no solution if $\lambda_1\lambda_2 - \lambda_3^2 = 0$. In this case, we could modify the auxiliary points to make $\lambda_1\lambda_2 - \lambda_3^2 \ne 0$ hold.

Given all the points shown in Fig. 1, the open curve with approximate shortest arc length is illustrated in Fig. 3.

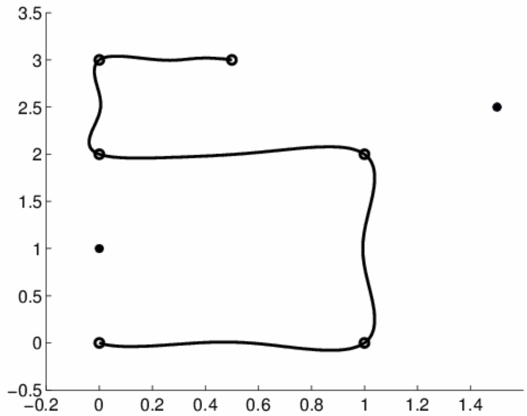


Fig. 3. The open curve with approximate shortest arc length

Similarly, given all the points shown in Fig. 2, the closed curve with approximate shortest arc length is illustrated in Fig. 4.

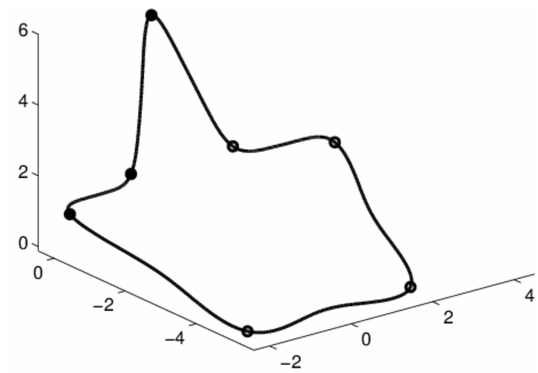


Fig. 4. The closed curve with approximate shortest arc length

(B) Parameters selection scheme for constructing the curve with approximate smallest strain energy.

In general, the fairness of a curve can be approximately measured by its strain energy (see [17]). The smaller strain energy the curve has, the fairer the curve is. The strain energy of the curve $\boldsymbol{r}(t)$ $(a \le t \le b)$ can be approximately described by $E=\int_a^b (\boldsymbol{r}''(t))^2 \mathrm{d}t$ (see [18]). Then the strain energy of the proposed open curve can be approximately expressed by

$$E(\alpha,\beta)=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{R}_i''(t)\right)^2 \mathrm{d}t . \tag{21}$$

By taking Eq. (16) into Eq. (21), we have

$$E(\alpha,\beta)=\mu_1\alpha^2+\mu_2\beta^2+2\mu_3\alpha\beta+2\mu_4\alpha+2\mu_5\beta+\mu_6 , \tag{22}$$

where

$$\mu_1=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{A}_i''(t)\right)^2 \mathrm{d}t ,\ \mu_2=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{B}_i''(t)\right)^2 \mathrm{d}t ,$$

$$\mu_3=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{A}_i''(t)\cdot\boldsymbol{B}_i''(t)\right)\mathrm{d}t ,\ \mu_4=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{A}_i''(t)\cdot\boldsymbol{C}_i''(t)\right)\mathrm{d}t ,$$

$$\mu_5=\sum_{i=-1}^{n-2}\int_0^1 \left(\boldsymbol{B}_i''(t)\cdot\boldsymbol{C}_i''(t)\right)\mathrm{d}t ,\ \mu_6=\sum_{i=0}^{n-2}\int_0^1 \left(\boldsymbol{C}_i''(t)\right)^2 \mathrm{d}t .$$

In order to get the approximate smallest strain energy, it must have

$$\begin{cases}\dfrac{\partial E(\alpha,\beta)}{\partial\alpha}=0,\\[2mm] \dfrac{\partial E(\alpha,\beta)}{\partial\beta}=0.\end{cases} \tag{23}$$

From Eq. (22), we can rewrite Eq. (23) as follows,

$$\begin{cases}\mu_1\alpha+\mu_3\beta+\mu_4=0,\\ \mu_3\alpha+\mu_2\beta+\mu_5=0.\end{cases} \tag{24}$$

By solving Eq. (24), we have

$$\begin{cases}\alpha=\dfrac{\mu_3\mu_5-\mu_2\mu_4}{\mu_1\mu_2-\mu_3^2},\\[2mm] \beta=\dfrac{\mu_3\mu_4-\mu_1\mu_5}{\mu_1\mu_2-\mu_3^2},\end{cases} \tag{25}$$

where $\mu_1\mu_2-\mu_3^2\ne 0$.

It should be noted that Eq. (24) would have no solution if $\mu_1\mu_2-\mu_3^2=0$. In this case, we could modify the auxiliary points to make $\mu_1\mu_2-\mu_3^2\ne 0$ hold.

Given all the points shown in Fig. 1, the open curve with approximate smallest strain energy is illustrated in Fig. 5.

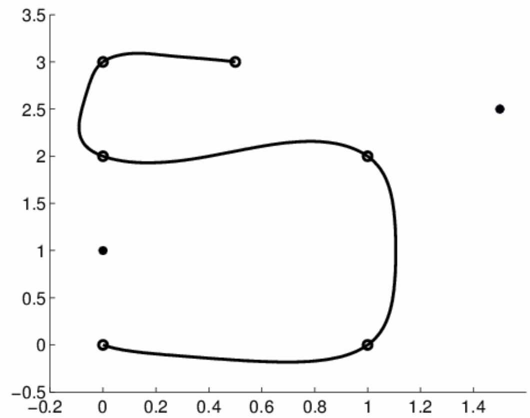


Fig. 5. The open curve with approximate smallest strain energy

Similarly, given all the points shown in Fig. 2, the closed curve with approximate smallest strain energy is illustrated in Fig. 6.

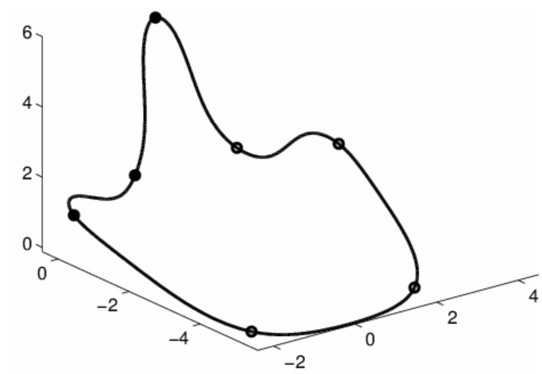


Fig. 6. The closed curve with approximate smallest strain energy

(C) Parameters selection scheme for constructing the curve with approximate shortest arc length and smallest strain energy.

If we need to construct the curve with shortest arc length and smallest strain energy, the objective functional could be described by

$$M(\alpha,\beta)=L(\alpha,\beta)+E(\alpha,\beta) . \tag{26}$$

In order to get the approximate shortest arc length and smallest strain energy, it must have

$$\begin{cases} \dfrac{\partial M(\alpha,\beta)}{\partial \alpha} = 0, \\ \dfrac{\partial M(\alpha,\beta)}{\partial \beta} = 0. \end{cases} \tag{27}$$

From Eq. (17) and Eq. (22), we can rewrite Eq. (27) as follows,

$$\begin{cases} (\lambda_1+\mu_1)\alpha + (\lambda_3+\mu_3)\beta + (\lambda_4+\mu_4) = 0, \\ (\lambda_3+\mu_3)\alpha + (\lambda_2+\mu_2)\beta + (\lambda_5+\mu_5) = 0. \end{cases} \tag{28}$$

By solving Eq. (28), we have

$$\begin{cases} \alpha = \dfrac{(\lambda_3+\mu_3)(\lambda_5+\mu_5) - (\lambda_2+\mu_2)(\lambda_4+\mu_4)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2) - (\lambda_3+\mu_3)^2} \\ \beta = \dfrac{(\lambda_3+\mu_3)(\lambda_4+\mu_4) - (\lambda_1+\mu_1)(\lambda_5+\mu_5)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2) - (\lambda_3+\mu_3)^2} \end{cases} \tag{29}$$

where $(\lambda_1+\mu_1)(\lambda_2+\mu_2) - (\lambda_3+\mu_3)^2 \neq 0$.

It should be noted that Eq. (28) would have no solution if $(\lambda_1+\mu_1)(\lambda_2+\mu_2) - (\lambda_3+\mu_3)^2 = 0$. In this case, we could modify the auxiliary points to make $(\lambda_1+\mu_1)(\lambda_2+\mu_2) - (\lambda_3+\mu_3)^2 \neq 0$ hold.

Given all the points shown in Fig. 1, the open curve with approximate shortest arc length and smallest strain energy is illustrated in Fig. 7.

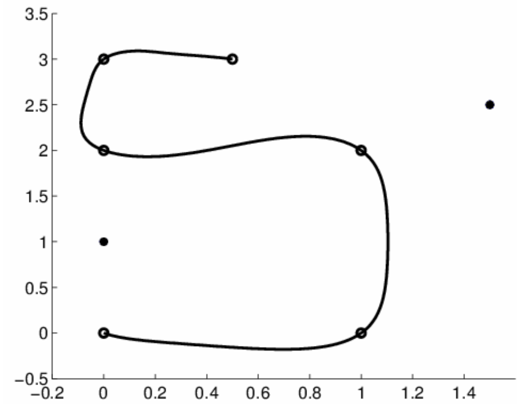


Fig. 7. The open curve with approximate shortest arc length and smallest strain energy

Similarly, given all the points shown in Fig. 2, the closed curve with approximate shortest arc length and smallest strain energy is illustrated in Fig. 8.

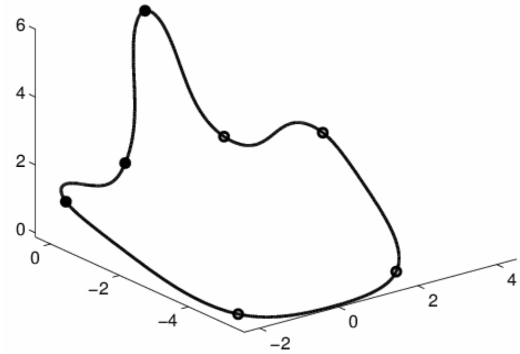


Fig. 8. The closed curve with approximate shortest arc length and smallest strain energy

IV. Conclusion

In this paper, we present a class of polynomial spline curve with two free parameters that can naturally interpolate the data points. The proposed curve can achieve C^3 continuity and be modified by the two free parameters. Furthermore, we give the selection schemes of the free parameters for constructing the curve with approximate shortest arc length, the curve with approximate smallest strain energy and the curve with approximate shortest arc length and smallest strain energy. Since the construction of interpolation curve is a basic problem in CAD and related fields, the proposed spline and the selection schemes of the free parameters may be helpful.

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