On a Resonant Third-order p-Laplacian M-point Boundary Value Problem on the Half-line With Two Dimensional Kernel

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Abstract— **By using a semi-projector and the Re and Gen extension of coincidence degree theory, this work studies the existence of solution for a third-order p-Laplacian boundary value problems at resonance on the half-line with two dimensional kernel. An example is used to show applicability of existence result.**

Index Terms— **Coincidence degree, Integral boundary value problem, M-point, p-Laplacian, Resonance.**

1 Introduction

This work studies the existence of solutions for the following p-Laplacian third-order boundary value problem having integral and m-point boundary conditions at resonance on the half-line with two dimensional kernel:

$$(\sigma(t)\varphi_p(u''(t)))'+f(t,u(t),u'(t),u''(t)) = 0,\ t \in (0,+\infty), \tag{1}$$

$$u(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt,\ u'(0) = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u'(t)dt,$$
$$\lim_{t\to+\infty} (\sigma(t)\varphi_p(u''(t))) = 0 \tag{2}$$

where $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is an $L^1[0,+\infty)$-Carathéodory function, $0 < \xi_1 < \xi_2 < \cdots \leq \xi_m < +\infty$, $0 < \eta_1 < \eta_2 < \cdots \leq \eta_n < +\infty$, $\alpha_i \in \mathbb{R},\ i = 1, 2, \cdots, m$ and $\beta_j \in \mathbb{R},\ j = 1, 2, \cdots, n$. $\sigma \in C[0,+\infty) \cap C^2(0,+\infty)$, $\sigma(t) > 0$ on $[0,+\infty)$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and $\varphi_q\left(\frac{1}{\sigma}\right) \in L^1[0,+\infty)$.

Boundary value problem (1) is to be at resonance if $(\sigma(t)\varphi_p(u''(t)))' = 0$ subject to boundary condition (2) has a non-trivial solution. The Mawhin's coincidence degree theorem [4] has been used by many authors to study resonant problems where the differential operator is linear see [9, 11, 8, 7]. For the case of nonlinear p-Laplacian differential operator, the Ge and Ren [1] extension coincidence degree theory has also been applied see [14, 5, 12, 10, 13].

However, to the best of our knowledge, only few authors in literature have considered p-Laplacian boundary value problems on the half-line.

In section 2 of this work necessary lemmas theorem and definitions will be given, section 3 will be dedicated to stating and proving condition for existence of solutions. Finally an example will be given to demonstrate applicability of results obtained.

2 Preliminaries

In this section, we will give some definitions and lemmas that will be used in this work.

Definition 1. ([14]) A map $h : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is $L^1[0,+\infty)$-Carathéodory, if the following conditions are satisfied:

(i) for each $(q, r, s) \in \mathbb{R}^3$, the mapping $t \to h(t, q, r, s)$ is Lebesgue measurable;

(ii) for a.e. $t \in [0,\infty)$, the mapping $(q, r, s) \to h(t, q, r, s)$ is continuous on $\mathbb{R}^3$;

(iii) for each $k > 0$, there exists $\varphi_k(t) \in L_1[0,+\infty)$ such that, for a.e. $t \in [0,\infty)$ and every $(q, r, s) \in [-k, k]$, we have
$$|h(t, q, r, s)| \leq \varphi_k(t).$$

Definition 2. [1] Let $(U,\ \|\cdot\|_U)$ and $(Z,\ \|\cdot\|_Z)$ be two Banach spaces. The continuous operator $M : U \cap \text{dom } M \to Z$, is quasi-linear if $\text{Im } M = M(U \cap \text{dom } M)$ is a closed subset of Z and $\ker M = \{u \in U \cap \text{dom } M : Mu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < +\infty$.

Definition 3. [2] Let U be a Banach space and $U_1 \subset U$ a subspace. The operator $Q : U \to U_1$ is a semi-projector if $Q^2 = Q$ and $Q(\lambda u) = \lambda Qu$ where $u \in U,\ \lambda \in \mathbb{R}$.

Let $U_1 = \ker M$ and U_2 be the complement space of U_1 in U, then $U = U_1 \oplus U_2$. Similarly, if Z_1 is a subspace

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of Z and Z_2 is the complement space of Z_1 in Z, then $Z = Z_1 \oplus Z_2$. Let $P : U \to U_1$ be a projector, $Q : Z \to Z_1$ be a semi-projector and $\Omega \subset U$ an open bounded set with $\theta \in \Omega$ the origin. Also, Let N_1 be denoted by N, let $N_\lambda : \overline{\Omega} \to Z$, where $\lambda \in [0, 1]$ is a continuous operator and $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$.

Definition 4. [10] Let U be the space of all continuous and bounded vector-valued functions on $[0, +\infty)$ and $X \subset U$. Then X is said to be relatively compact if the following statements hold:

(i) X is bounded in U;

(ii) all functions from X are equicontinuous on any compact subinterval of $[0, +\infty)$;

(ii) all functions from X are equiconvergent at ∞, i.e. $\forall\, \epsilon > 0, \exists$ a $T = T(\epsilon)$ such that $\|A(t) - A(+\infty)\|_{R^n} < \epsilon\ \forall\, t > T$ and $A \in X$.

Definition 5 [1] Let $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ be a continuous operator. The operator N_λ is said to be M-compact in $\overline{\Omega}$ if there exists a vector subspace $Z_1 \in Z$ such that $\dim Z_1 = \dim U_1$ and a compact and continuous operator $R : \overline{\Omega} \times [0, 1] \to U_2$ such that for $\lambda \in [0, 1]$, the following holds

(i) $(I - Q)N_\lambda(\overline{\Omega}) \subset \text{Im } M \subset (I - Q)Z$,

(ii) $QN_\lambda u = 0 \Leftrightarrow QNu = 0,\ \lambda \in (0, 1)$,

(iii) $R(\cdot, u)$ is the zero operator and $R(\cdot, \lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,

(iv) $M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda$.

Lemma 1. [2] The following are true for ϕ_p:

(i) ϕ_p is continuous, invertible and monotonically increasing. In addition, $\phi_p^{-1} = \phi_q$ and for $q > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) For all $y, z, \geq 0$,

$$\begin{aligned} &\phi_p(y + z) \leq \phi_p(y) + \phi_p(z), && \text{if } 1 < p < 2, \\ &\phi_p(y + z) \leq 2^{p-2}(\phi_p(y) + \phi_p(z)), && \text{if } p \geq 2. \end{aligned}$$

Theorem 1 [1] Let $(U,\ \|\cdot\|_U)$ and $(Z,\ \|\cdot\|_Z)$ be two Banach spaces and $\Omega \subset U$ an open and bounded set. If the following holds

(C_1) The operator $M : U \cap \text{dom } M \to Z$ is a quasi-linear,

(C_2) the operator $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is M-compact,

(C_3) $Mu \neq N_\lambda u$, $\lambda \in [0, 1]$, $u \in \partial\Omega$,

(C_4) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$, where $N = N_1$ and the operator $J : Z_1 \to U_1$ is a homeomorphism with $J(\theta) = \theta$.

then the equation $Mu = Nu$ has at least one solution in $\text{dom } M \cap \overline{\Omega}$.

Let $U = \{u \in C^2[0, +\infty) : u, u', \sigma\varphi_p(u'') \in AC[0, +\infty)$, $\lim_{t\to+\infty} e^{-t}|u^{(i)}(t)|$ exist, $i = 0, 1, 2\}$, with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ defined on U where $\|u\|_\infty = \sup_{t\in[0,+\infty)} e^{-t}|u^i|,\ i = 0, 1, 2$. The space $(U, \|\cdot\|)$ by standard argument is a Banach Space.

Let $Z = L^1[0, +\infty)$ with the norm $\|y\|_{L^1} = \int_0^{+\infty} |y(v)|dv$. Define M as a continuous operator such that $M : \text{dom } M \subset U \to Z$ where

$$\text{dom } M = \Bigg\{u \in U : (\varphi_p(u''))' \in L^1[0, +\infty), \\ u(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(t)dt, u'(0) = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u'(t)dt, \\ \lim_{t\to+\infty} (\sigma(t)\varphi_p(u''(t))) = 0, \Bigg\}$$

and $Mu = (\sigma(t)\varphi_p(u''(t)))'$. We will define the operator $N_\lambda u : \overline{\Omega} \to Z$ for $\lambda \in [0, 1]$ by

$$N_\lambda u = -\lambda f(t, u(t), u'(t), u''(t)), \quad t \in [0, +\infty)$$

where $\Omega \subset U$ is an open and bounded set. Then the boundary value problem (1) in abstract form is $Mu = Nu$.
In order to establish conditions for existence of solution of (1.1)-(1.2), we assume the following:
(ϕ_1) $\sum_{j=1}^{n} \beta_j \eta_j = 1,\ \sum_{i=1}^{m} \alpha_i \xi_i = 1,\ \sum_{i=1}^{m} \alpha_i \xi_i^2 = 0;$
(ϕ_2) $W = (Q_1 e^{-t} \cdot Q_2 te^{-t} - Q_2 e^{-t} \cdot Q_1 te^{-t}) := (w_{11} \cdot w_{22} - w_{12} \cdot w_{21}) \neq 0$ where

$$Q_1 y = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^{t} \int_0^{x} \varphi_q\left(\frac{1}{\sigma(t)}\right) \varphi_q\left(\int_s^{+\infty} y(v)dv \right) dsdxdt$$

and

$$Q_2 y = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} \int_0^{t} \varphi_q\left(\frac{1}{\sigma(t)}\right) \varphi_q\left(\int_s^{+\infty} y(v)dv \right) dsdt.$$

Lemma 2 The operator $M : \text{dom } M \subset U \to Z$ is quasi-linear.

Proof Clearly, $\ker M = \{u \in \text{dom } M : u = a + bt,\ a,\ b \in \mathbb{R}\}$. Next, we obtain $\text{Im } M$. Let $u \in \text{dom } M$ and consider the problem

$$(\sigma(t)\varphi_p(u''(t)))' = y,\ t \in [0, +\infty), \tag{3}$$

Integrating (3) from t to $+\infty$, and applying (2) gives

$$\begin{aligned} u''(t) &= -\varphi_p^{-1}\Big(\frac{1}{\sigma(t)}\int_t^{+\infty} y(v)dv\Big) \\ &= -\varphi_q\Big(\frac{1}{\sigma(t)}\Big)\varphi_q\Big(\int_t^{+\infty} y(v)dv\Big). \end{aligned} \tag{4}$$

Integrating (4) from 0 to t yields

$$u'(t) = u'(0) - \int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)ds, \tag{5}$$

applying boundary conditions (2) to (5), then $u'(0) = u'(0)$,

$$\sum_{j=1}^{n}\beta_j \int_0^{\eta_j} u'(t)dt = \sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\Big[u'(0) - \int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)ds\Big]dt$$

and

$$u'(0) = u'(0)\sum_{j=1}^{n}\beta_j\eta_j - \sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdt.$$

Since $\sum_{j=1}^{n}\beta_j\eta_j = 1$,

$$Q_1y = \sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_0^t \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdt = 0.$$

Integrating, (5) from 0 to t gives

$$\begin{aligned} u(t) &= u(0) + u'(0)t \\ &\quad - \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdx. \end{aligned} \tag{6}$$

Applying boundary conditions (2), gives

$$\begin{aligned} u(0) &= \sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\Big(u(0) + u'(0)t - \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdx\Big)dt \\ \Rightarrow u(0) &= u(0)\sum_{i=1}^{m}\alpha_i\xi_i + u'(0)\sum_{i=1}^{m}\alpha_i\xi_i^2 \\ &\quad - \sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdxdt. \end{aligned}$$

Since $\sum_{i=1}^{m}\alpha_i\xi_i = 1$ and $\sum_{i=1}^{m}\alpha_i\xi_i^2 = 0$ then

$$Q_2y = \sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdxdt = 0.$$

Thus Im $M = \{y \in Z : Q_1y = Q_2y = 0\}$ and

$$u(t) = a + bt - \int_0^t\int_0^x \varphi_q\Big(\frac{1}{\sigma(s)}\Big)\varphi_q\Big(\int_s^{+\infty} y(v)dv\Big)dsdx,$$

where a and b are arbitrary constants and $u(t)$ is a solution to (3) satisfying (2). So $\ker M = 2 < \infty$ and $M \subset (U \cap \text{dom } M) \subset Z$ is closed. Therefore, M is quasi-linear.

We will define the projector $P : U \to U_1$ as

$$Pu(t) = u(0) + u'(0)t, \quad u \in U, \tag{7}$$

Similarly, the operator $Q : Z \to Z_1$ will be defined as

$$Qy = (\Delta_1 y) + (\Delta_2 y)\cdot t \tag{8}$$

where

$$\Delta_1 y = \frac{1}{W}(\delta_{11}Q_1y + \delta_{12}Q_2y)e^{-t},$$
$$\Delta_2 y = \frac{1}{W}(\delta_{21}Q_1y + \delta_{22}Q_2y)e^{-t},$$

and δ_{ij} is the co-factor of w_{ij}, $i, j = 1, 2$. The operator $Q : Z \to Z_1$ can be shown to be a semi-projector.

Finally, Let the operator $R : U \times [0,1] \to U_2$ be defined by

$$\begin{aligned} R(u,\lambda)(t) &= \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\int_s^{+\infty}(\lambda(f(v,u(v),u'(v),u''(v)) + QN_\lambda u(v))dv\Big)dsdx \\ &= \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\int_s^{+\infty}(-I + Q)N_\lambda u(v)dv\Big)dsdx, \end{aligned}$$

where U_2 is the complement space of $\ker M$ in U.

Lemma 3 If f is a $L^1[0,+\infty)$-Carathéodory function, then $R : U \times [0,1] \to U_2$ is M-compact.

Proof. Let $\Omega \subset U$ be nonempty, open and bounded, then for $u \in \overline{\Omega}$, there exists a constant $k > 0$ such that $\|u\| < k$. Since f is an $L^1[0,+\infty)$-Carathéodory function, there exists $\psi_k \in L^1[0,+\infty)$ such that for a.e. $t \in [0,+\infty)$ and $\lambda \in [0,1]$, we have

$$\|N_\lambda u\|_{L^1} + \|QN_\lambda u\|_{L^1} \le \|\psi_k\|_{L^1} + \|QNu\|_{L^1},$$

For any $u \in \overline{\Omega}$, $\lambda \in [0,1]$, we have

$$\begin{aligned}\|R(u,\lambda)\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t}|R(u,\lambda)(t)| \\ &\le \frac{1}{e}\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|N_\lambda u\|_{L^1}+\|QN_\lambda u\|_{L^1}) \\ &\le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1}+\|QNu\|_{L^1})<+\infty,\end{aligned} \tag{9}$$

$$\begin{aligned}\|R'(u,\lambda)\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t}|R'(u,\lambda)(t)| \\ &\le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1}+\|QNu\|_{L^1})<+\infty\end{aligned} \tag{10}$$

and

$$\begin{aligned}\|R''(u,\lambda)\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t}|R''(u,\lambda)(t)| \\ &\le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{\infty}\varphi_q(\|\psi_k\|_{L^1}+\|QNu\|_{L^1})<+\infty.\end{aligned} \tag{11}$$

Therefore it follows from (9), (10) and (11) that $R(u,\lambda)\overline{\Omega}$ is uniformly bounded.

Next we will show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous in a compact set. Let $u \in \overline{\Omega}$, $\lambda \in [0,1]$. For any $T \in [0,+\infty)$, with $t_1, t_2 \in [0,T]$ where $t_1 < t_2$, we have

$$\begin{aligned}&|e^{-t_2}R(u,\lambda)(t_2)-e^{-t_1}R(u,\lambda)(t_1)| \\ &= \left|e^{-t_2}\int_0^{t_2}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\right. \\ &\left.\int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)dsdx\right. \\ &-e^{-t_1}\int_0^{t_1}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\\ &\left.\int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)dsdx\right| \\ &\le (e^{-t_2}-e^{t_1})\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1} \\ &+\|QNu\|_{L^1})t_1 \\ &+e^{-t_2}\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1} \\ &+\|QNu\|_{L^1})(t_2-t_1)\ \to 0, \quad \text{as } t_1\to t_2,\end{aligned} \tag{12}$$

$$\begin{aligned}&|e^{t_2}R'(u,\lambda)(t_2)-e^{t_1}R'(u,\lambda)(t_1)| \\ &= \left|e^{-t_2}\int_0^{t_2}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\right. \\ &\int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)ds \\ &-e^{-t_1}\int_0^{t_1}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\\ &\left.\int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)ds\right| \\ &\le (e^{-t_2}-e^{-t_1})\varphi_q(\|\psi_k\|_{L^1} \\ &+\|QNu\|_{L^1})\int_0^{t_1}\left|\varphi_q\left(\frac{1}{\sigma(s)}\right)\right|ds \\ &+e^{-t_2}\varphi_q(\|\psi_k\|_{L^1}+\|QNu\|_{L^1})\int_{t_1}^{t_2}\left|\varphi_q\left(\frac{1}{\sigma(s)}\right)\right|ds \\ &\to 0, \quad \text{as } t_1\to t_2\end{aligned} \tag{13}$$

and

$$\begin{aligned}&|e^{-t_2}R''(u,\lambda)(t_2)-e^{t_1}R''(u,\lambda)(t_1)| \\ &= \left|e^{-t_2}\varphi_q\left(\frac{1}{\sigma(t_2)}\right)\varphi_q\left(\int_{t_2}^{+\infty}(-I+Q)N_\lambda u(v)dv\right)ds\right. \\ &= \left|e^{-t_2}\varphi_q\left(\frac{1}{\sigma(t_2)}\right)\varphi_q\left(\int_{t_2}^{+\infty}(-I+Q)N_\lambda u(v)dv\right)ds\right. \\ &-e^{-t_1}\varphi_q\left(\frac{1}{\sigma(t_1)}\right)\varphi_q\Big(\int_{t_1}^{t_2}(-I+Q)N_\lambda u(v)dv \\ &\left.+\int_{t_2}^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)ds\right| \\ &\to 0, \quad \text{as } t_1\to t_2.\end{aligned} \tag{14}$$

Thus, (12), (13) and (14) show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous on $[0,T]$.

Finaly, We prove equiconvergent at $+\infty$. We will note that since $\lim_{t\to+\infty} e^{-t}=0$ then

$$\begin{aligned}&\lim_{t\to+\infty} e^{-t}R(u,\lambda)(t)=\lim_{t\to+\infty} e^{-t}R'(u,\lambda)(t) \\ &=\lim_{t\to+\infty} e^{-t}R''(u,\lambda)(t)=0.\end{aligned}$$

Then,

$$\begin{aligned}&|e^{-t}R(u,\lambda)(t)-\lim_{t\to+\infty} e^{-t}R(u,\lambda)(t)| \\ &= \left|e^{-t}\int_0^{t}\int_0^{x}\varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\right. \\ &\left.\int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)dsdx-0\right| \\ &\le te^{-t}\le\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|\psi_k\|_{L^1}+\|QNu\|_{L^1}) \\ &\to 0, \text{ uniformly as } t\to+\infty,\end{aligned} \tag{15}$$

$$
\begin{aligned}
&|e^{-t}R'(u,\lambda)(t) - \lim_{t\to+\infty} e^{-t}R'(u,\lambda)(t)| \\
&= \left| e^{-t}\int_0^t \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\right. \\
&\left. \int_s^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)ds - 0 \right| \\
&\le e^{-t}\varphi_q(\|\psi_k\|_{L^1} + \|QN_\lambda u\|_{L^1}) \\
&\to 0, \text{ uniformly as } t\to+\infty
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
&|e^{-t}R''(u,\lambda)(t) - \lim_{t\to+\infty} e^{-t}R''(u,\lambda)(t)| \\
&= \left| e^{-t}\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\Big(\right. \\
&\left. \int_t^{+\infty}(-I+Q)N_\lambda u(v)dv\Big)ds - 0 \right| \\
&\le \left| e^{-t}\varphi_q\frac{1}{\sigma(t)}\right|\varphi_q\Big(\int_t^{+\infty}|Nu(v)+QNu(v)|dv\Big) \\
&\to 0, \text{ unifromly as } t\to+\infty.
\end{aligned} \tag{17}
$$

Therefore $R(u,\lambda)\overline{\Omega}$ is equiconvergent at $+\infty$. It then follows from definition 4 that $R(u,\lambda)$ is compact.

Lemma 4 The operator N_λ is M-compact.

Proof. Since Q is a semi-projector, then $Q(I-Q)N_\lambda(\overline{\Omega}) = 0$. Hence, $(I-Q)N_\lambda(\overline{\Omega}) \subset \ker Q = \operatorname{Im} M$. Conversely, let $y \in \operatorname{Im} M$, then $y = y - Qy = (I-Q)y \in (I-Q)Z$. Hence, condition (i) of definition 5 is satisfied. It can easily be shown that condition (ii) of definition 5 holds.

Let $u \in \Sigma_\lambda$, $Mu = N_\lambda u$, then $N_\lambda u \in \operatorname{Im} M$. Hence, $QN_\lambda u = 0$ and $R(u,\lambda)(t)$ becomes

$$
\begin{aligned}
R(u,\lambda)(t) &= \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\\
&\int_s^{+\infty}\lambda f(v,u(v),u'(v),u''(v))dv\Big)dsdx.
\end{aligned}
$$

Then $R(u,0)(t) = 0$ and

$$
\begin{aligned}
R(u,\lambda)(t) &= \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\\
&\int_s^{+\infty}\lambda(f(v,u(v),u'(v),u''(v))dv\Big)dsdx \\
&= \int_0^t\int_0^x u''(s)dsdx = u(t) - u(0) - u'(0)t \\
&= u(t) - Pu(t) = [(I-P)u](t).
\end{aligned}
$$

Therefore, condition (iii) of definition 5 holds.

Let $u \in \overline{\Omega}$. Since $Mu = (\sigma(t)\varphi_p(u''(t)))'$ we have

$$
\begin{aligned}
&M[Pu + R(u,\lambda)](t) = (\sigma(t)\varphi_p([Pu+R(u,\lambda)]''(t))' \\
&= \Big(\sigma(t)\varphi_p\Big[u(0)+u'(0)t + \int_0^t\int_0^x \varphi_q\left(\frac{1}{\sigma(s)}\right)\varphi_q\Big(\\
&\int_s^{+\infty}(-I+Q)N_\lambda(v)dv\Big)dsdx\Big]''\Big)' \\
&= \Big(\sigma(t)\varphi_p\Big[\varphi_q\Big(\frac{1}{\sigma(t)}\int_t^{+\infty}[(-I+Q)N_\lambda](v))dv\Big)\Big]\Big)' \\
&= \Big(\int_t^{+\infty}[(-I+Q)N_\lambda](v))dv\Big)' \\
&= -[(-I+Q)N_\lambda](t) = [(I-Q)N_\lambda](t),
\end{aligned}
$$

that is condition (iv) of definition 5 holds. Hence, N_λ is M-compact in $\overline{\Omega}$.

3 Existence Results

In this section, the conditions for existence of solutions for boundary value problem (1) and (2) will be stated and proved.

Theorem 2 Suppose the following hypothesis holds:

(H_1) There exists functions $x(t), y(t), z(t), r(t) \in L^1[0,+\infty)$ such that for all $(u,v,w) \in \mathbb{R}^3$ and *a.e.* $t \in [0,+\infty)$,

$$
\begin{aligned}
|f(t,u,v,w)| &\le x(t)|u|^{p-1} + y(t)|v|^{p-1} \\
&+ z(t)|w|^{p-1} + r(t)
\end{aligned} \tag{18}
$$

(H_2) For $u \in \operatorname{dom} M$ there exist a constant $a_0 > 0$, $d > 0$ such that if $|u(t)| > a_0$ for $t \in [0,d]$ or $|u'(t)| > a_0$ for $t \in [0,+\infty)$, then either

$$
Q_1Nu(t) \neq 0 \quad \text{or} \quad Q_2Nu(t) \neq 0, \quad t \in [0,+\infty). \tag{19}
$$

(H_3) There exists a constant $b_0 > 0$ such that for $|a| > b_0$ or $|b| > b_0$ either

$$
Q_1N(a+bt) + Q_2N(a+bt) < 0, \quad t \in (0,+\infty), \tag{20}
$$

or

$$
Q_1N(a+bt) + Q_2N(a+bt) > 0, \quad t \in (0,+\infty), \tag{21}
$$

where $a, b \in \mathbb{R}$, $|a| + |b| > b_0$ and $t \in [0,+\infty)$.
Then the boundary value problem (1) and has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$, provided

$$
\Lambda(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1}) < 1, \text{ for } p > 2
$$

or

$$
2^{2q-4}\Lambda(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1}) < 1, \text{ for } 1 < p \le 2
$$

where $\Lambda = \max\Big\{(2+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, (1+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\Big\}$.

The following lemmas are also needed to prove Theorem 2.

Lemma 6. The set $\Omega_1 = \{u \in \operatorname{dom} M : Mu = N_\lambda u$ for some $\lambda \in (0,1)\}$ is bounded.

Proof Let $u \in \Omega_1$ then $N_\lambda u \in \operatorname{Im} M = \ker Q$. Hence, $QN_\lambda u = 0$ and $QNu = 0$. It follows from H_2 that there exists $t_0 \in [0,d]$, $t_1 \in [0,+\infty)$ such that

$$|u(t_0)| \le a_0, \quad |u'(t_1)| \le a_0.$$

from $u(0) = u(t_0) + \int_0^{t_0} u'(v)dv$, we have

$$|u(0)| = \left|u(t_0) - \int_0^{t_0} u'(v)dv\right| \le a_0 + d\|u'\|_\infty.$$

Also, from $u'(t) = u'(t_1) - \int_t^{t_1} u''(v)dv$,

$$|u'(t)| = \left|u(t_1) - \int_t^{t_1} u'(v)dv\right| \le a_0 + \|u''\|_{L^1},$$

then

$$|u'(0)| \le a_0 + \|u''\|_{L^1} \tag{22}$$

and

$$\|u'\|_\infty = \sup_{t\in[0,+\infty)} e^{-t}|u'(t)| \le a_0 + \|u''\|_{L^1}. \tag{23}$$

Hence, from (22) and (23) we have

$$|u(0)| \le 2a_0 + \|u''\|_{L^1}. \tag{24}$$

Since $Mu = N_\lambda u$, from (4), we then get

$$u''(t) = -\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\Big(\int_t^{+\infty} \lambda f(v,u(v),u'(v),u''(v))dv\Big),$$

hence,

$$\begin{aligned}\|u''\|_{L^1} &= \int_0^{+\infty}\left| -\varphi_q\left(\frac{1}{\sigma(t)}\right)\varphi_q\Big(\int_t^{+\infty} \lambda f(v,u(v),u'(v),u''(v))dv\Big)\right|dt \\ &\le \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}\varphi_q(\|Nu\|_{L^1}).\end{aligned}$$

Since, $QNu = 0$ for $u \in \Omega_1$, it follows from (9), (10) and (11) that

$$\begin{aligned}&\|R(u,\lambda)\| \\ &\le \max\left\{\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}(\varphi_q(\|Nu\|_{L^1}).\end{aligned}$$

Also,

$$\begin{aligned}&\|Pu\| \\ &\le a_0(2+d) + (1+d)\varphi_q(\|Nu\|_{L^1})\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}.\end{aligned} \tag{25}$$

In addition, for $u \in \Omega_1$, we have

$$u(t) = Pu(t) + (I-P)u(t) = Pu(t) + R(u,\lambda)u(t),$$

therefore,

$$\begin{aligned}\|u\| &= \|Pu\| + \|R(u,\lambda)\| \\ &\le a_0(2+d) + (1+d)\varphi_q(\|Nu\|_{L^1})\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} \\ &\quad + \max\left\{\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}\varphi_q(\|Nu\|)_{L^1}) \\ &= a_0(2+d) + \max\left\{(2+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1},\right. \\ &\quad \left.(1+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}\varphi_q(\|Nu\|)_{L^1}).\end{aligned}$$

Let $\Lambda = \max\left\{(2+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1}, (1+d)\left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_{L^1} + \left\|\varphi_q\left(\frac{1}{\sigma}\right)\right\|_\infty\right\}$, then

$$\|u\| \le a_0(2+d) + \Lambda\varphi_q(\|Nu\|)_{L^1}). \tag{26}$$

Considering (H_1), and lemma 5, if $p > 2$, we have

$$\begin{aligned}&\varphi_q(\|Nu\|_{L^1}) \le \varphi_q[\|x\|_{L^1}\varphi_p(\|u\|_\infty) \\ &+ \|y\|_{L^1}\varphi_p(\|u'\|_\infty) + \|z\|_{L^1}\varphi_p(\|u''\|_\infty) + \|r\|_{L^1}] \\ &\le \|u\|(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1}) + \|r\|_{L^1}^{q-1},\end{aligned} \tag{27}$$

if $1 < p \le 2$, then

$$\begin{aligned}&\varphi_q(\|Nu\|_{L^1}) \le \varphi_q[\|x\|_{L^1}\varphi_p(\|u\|_\infty) \\ &+ \|y\|_{L^1}\varphi_p(\|u'\|_\infty) + \|z\|_{L^1}\varphi_p(\|u''\|_\infty) + \|r\|_{L^1}] \\ &\le 2^{2q-4}\|u\|(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1}) \\ &+ 2^{2q-4}\|r\|_{L^1}^{q-1}.\end{aligned} \tag{28}$$

In view of (26), (27) and (28)

$$\|u\| \le \frac{a_0(2+d) + \Lambda\|r\|_{L^1}^{q-1}}{1 - \Lambda(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1})}$$

or

$$\|u\| \le \frac{a_0(2+d) + 2^{2q-4}\Lambda\|r\|_{L^1}^{q-1}}{1 - 2^{2q-4}\Lambda(\|x\|_{L^1}^{q-1} + \|y\|_{L^1}^{q-1} + \|z\|_{L^1}^{q-1})}.$$

Therefore Ω_1 is bounded.

Lemma 7. If $\Omega_2 = \{u \in \ker M : -\lambda u + (1-\lambda)JQNu = 0,\ \lambda \in [0,1]\}$, $J : \operatorname{Im} Q \to \ker M$ is a homomorphisim, then Ω_2 is bounded.

Proof. For $a, b \in R$, let $J : \operatorname{Im} Q \to \ker M$ be defined by

$$J(a+bt) = \frac{1}{W}[\delta_{11}|a|+\delta_{12}|b|+(\delta_{21}|a|+\delta_{22}|b|)t)]e^{-t}, \quad (29)$$

If (20) holds, for any $u(t) = a+bt \in \Omega_2$, from $-\lambda u+(1-\lambda)JQNu = 0$, we obtain

$$\begin{cases} \delta_{11}(-\lambda|a| + (1-\lambda)Q_1N(a+bt)) \\ \quad + \delta_{12}(-\lambda|b| + (1-\lambda)Q_2N(a+bt)) = 0, \\ \delta_{21}(-\lambda|a| + (1-\lambda)Q_1N(a+bt)) \\ \quad + \delta_{22}(-\lambda|b| + (1-\lambda)Q_2N(a+bt)) = 0. \end{cases}$$

Since $B \neq 0$, then

$$\begin{aligned} \lambda|a| &= (1-\lambda)Q_1N(a+bt), \\ \lambda|b| &= (1-\lambda)QN_2(a+bt). \end{aligned} \quad (30)$$

From (30), when $\lambda = 1,\ a = b = 0$. When $\lambda = 0$,

$$Q_1N(a+bt) + Q_2N(a+bt) = 0$$

which contradicts (20) and (21), hence from (H_3), $|a| \leq b_0$ and $|b| \leq b_0$. For $\lambda \in (0,1)$, in view of (20) and (30), we have

$$0 \leq \lambda(|a|+|b|) = (1-\lambda)[Q_1N(a+bt)+Q_2N(a+bt)] < 0,$$

which contradicts $\lambda(|a| + |b|) \geq 0$. Hence, (H_3), $|a| \leq b_0$ and $|b| \leq b_0$, thus $\|u\| \leq 2b_0$. Therefore Ω_2 is bounded.

From lemma 2, we saw that condition (C_1) of theorem 1 holds, lemma 3 proved (C_2). Lemmas 6 and 7 showed that (C_3) holds.

Proof of Theorem 2 Let $\Omega \supset \Omega_1 U \Omega_2$ be a nonempty, open and bounded set, $u \in \operatorname{dom} M \cap \partial\Omega$, $H(u,\lambda) = -\lambda u + (1-\lambda)JQNu$, and J be as defined in Lemma 7 then $H(u,\lambda) \neq 0$. Therefore by the homotopy property of the Brouwer degree

$$\begin{aligned} &\deg\{JQN|_{\overline{\Omega}\cap\ker M}, \Omega\cap\ker M, 0\} \\ &\quad = \deg\{H(\cdot,0), \Omega\cap\ker M, 0\} \\ &\quad = \deg\{H(\cdot,1), \Omega\cap\ker M, 0\} \\ &\quad = \deg\{-I, \Omega\cap\ker M, 0\} \neq 0. \end{aligned}$$

Hence, condition (C_4) of theorem 1 holds.

Since all the conditions of theorem 1 are satisfied, (1) - (2) has at least one solution in $\overline{\Omega} \cap \operatorname{dom} M$.

4 Example

Consider the boundary value problem:

$$(e^{2t+1}\varphi_3(u''(t)))' + f(t, u(t), u'(t), u''(t)) = 0, \quad (31)$$

$$\begin{aligned} u(0) &= 9\int_0^{1/3} u(t)dt - 4\int_0^{1/2} u(t)dt, \\ u'(0) &= \frac{4}{5}\int_0^{5/4} u'(t)dt, \ \lim_{t\to+\infty} e^{2t+1}\varphi_3(u''(t)) = 0. \end{aligned} \quad (32)$$

where $t \in (0, +\infty)$

$$f(t,u(t),u'(t),u''(t)) = \begin{cases} 0, \quad 0 \leq t \leq 1, \\ 2e^{-2t-1}\sin u^{\frac{1}{2}} + e^{-2t-2}\sin v^{\frac{1}{2}} \\ \quad + e^{-3t-3}w^{\frac{1}{2}} + \frac{1}{6}e^{-6t}, \quad t > 1. \end{cases}$$

Here $\sigma(t) = e^{2t+1}$, $p = \frac{3}{2}$, $q = 3$, $\alpha_1 = 9$, $\alpha_2 = -4$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, $\beta_1 = \frac{4}{5}$, $\eta_1 = \frac{5}{4}$,
$\sum_{i=1}^{2}\alpha_i\xi_i = (9)\left(\frac{1}{3}\right) + (-4)\left(\frac{1}{2}\right) = 3 - 2 = 1$,
$\sum_{i=1}^{2}\alpha_i\xi_i^2 = (9)\left(\frac{1}{3}\right)^2 + (-4)\left(\frac{1}{2}\right)^2 = 1 - 1 = 0$,
$\sum_{j=1}^{1}\beta_j\eta_j = \left(\frac{4}{5}\right)\left(\frac{5}{4}\right) = 1$. $W = 0.0009 \neq 0$. Hence, (ϕ_1) and (ϕ_2) holds.
$\|x_1\|_{L^1} = \int_0^{+\infty} 2|e^{-2t-1}|dt = 2\int_0^{+\infty}|e^{-2t-1}|dt = \frac{1}{e}$,
$\|x_2\|_{L^1} = \int_0^{+\infty}|e^{-2t-2}|dt = \frac{1}{2e^2}$,
$\|x_3\|_{L^1} = \int_0^{+\infty}|e^{-3t-3}|dt = \int_0^{+\infty}|e^{-3t-3}|dt = \frac{1}{3e^3}$,
$\|\varphi_q\left(\frac{1}{\sigma}\right)\|_{L^1} = \int_0^{+\infty}|e^{-4t-2}|dt = \frac{1}{4e^2}$,
$\|\varphi_q\left(\frac{1}{\sigma}\right)\|_\infty = \sup_{t\in[0,+\infty)} e^{-t}e^{-2t-2} = 1$.

$$\Lambda = \max\left\{(2+1)\left(\frac{1}{4e^2}\right), (1+1)\left(\frac{1}{4e^2}\right) + 1\right\} = 1.0677.$$

$$\Lambda(\|x_1\|_{L^1}^{q-1} + \|x_2\|_{L^1}^{q-1} + \|x_3\|_{L^1}^{q-1}) = 0.1497 < 1$$

Therefore, (H_1) holds. (H_2) and (H_3) can also be shown to hold. Hence, from Theorem 2, the boundary value problem (31) and (32) has at least one solution in dom $M \cap \overline{\Omega}$.

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