The Lattices of Monadic Filters in Monadic BL-algebras

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***Abstract*—In this paper, we focus on lattice structures of the set of monadic filters (monadic filters, stable monadic filters, involutory monadic filters) of monadic BL-algebras and prove that**

(a) the classes of all monadic filters in monadic BL-algebras forms a complete Heyting algebras with respect to inclusion;

(b) the class of all stable monadic filters relative a monadic filter F in monadic BL-algebras is a complete Boolean algebra with respect to inclusion;

(c) the class of all involutory monadic filters relative a monadic filter F in monadic BL-algebras is a complete Boolean algebra with respect to inclusion.

These results also provide the solid foundation to study the variety of monadic BL-algebras.

***Index Terms*—monadic BL-algebras, filters, Heyting algebras, Boolean algebras.**

I. Introduction

MONADIC algebra, in the sense of Halmos [1], is a Boolean algebra equipped with a closure operator $\exists$, which abstracts algebraic properties of the standard existential quantifier "for some". The name "monadic" comes from the connection with predicate logics for languages having one placed predicates and a single quantifier. After then, monadic MV-algebras, the algebraic counterpart of monadic Łukasiewicz predicate logic, were introduced and studied in [2]. Subsequently, monadic BL-algebras, monadic NM-algebras, monadic bounded hoops and monadic BCI-algebras were introduced and investigated in [3], [4], [5], [6].

Nowadays filters are tools of extreme importance in many areas of classical mathematics. For example, in topology they enhance the concept of convergence and, in measure theory, prime theory can be interpreted as basic components of probability measures and in fuzzy mathematics, filters have been conceived in various manners. Hájek [8] introduced the notion of filters in BL-algebras and proved the completeness of Basic Logic, $\mathbf{BL}$. Broumand Saeid [7], [12], [14], [15] proposed some types of filters in BL-algebras and gave some characterization of them. Buşneag [11], [16] studied the lattice of filters of BL-algebras and obtained some interesting results. Monadic filters also play an important role in studying monadic logical algebras. Note that the notion of monadic filters in monadic BL-algebras is not the same as that of filters in BL-algebras. So, it is interesting to study the lattice of monadic filters of monadic BL-algebras and obtain some generalized result. These results also can provide the solid foundation to study the variety of monadic BL-algebras.

The paper is organized as follows: In Section 2, we review some result on monadic BL-algebras and obtain some new results. In Section 3, we study the lattice structure of some types of monadic filters in monadic BL-algebras.

II. Monadic BL-algebras

Adapting for the propositional case the axiomatization of $\mathbf{S5}(\mathbf{BL})$, D. Castaño et al [6] also proposed a simplified set of axioms for the above calculus, called $\mathbf{S5'}(\mathbf{BL})$, whose axiom schemata are all the ones for logic $\mathbf{BL}$ together with the following axiom schemata:

(S1) $\Box\alpha \Rightarrow \alpha$,
(S2) $\Box(\alpha \Rightarrow \Box\beta) \equiv (\Diamond\alpha \Rightarrow \Box\beta)$,
(S3) $\Box(\Box\alpha \Rightarrow \beta) \equiv (\Box\alpha \Rightarrow \Box\beta)$,
(S4) $\Box(\Diamond\alpha \sqcup \beta) \equiv (\Diamond\alpha \sqcup \Box\beta)$
(S5) $\Diamond(\alpha\&\alpha) \equiv (\Diamond\alpha\&\Diamond\alpha)$

and closed under Modus Ponens $(\alpha, \alpha \Rightarrow \beta/\beta)$ and Necessitation Rules $(\alpha/\Box\alpha)$.

In order to show that $\mathbf{S5'}(\mathbf{BL})$ is complete, we apply a general result from Abstract Algebraic Logic (shortly $\mathbf{AAL}$). We start from by showing that $\mathbf{S5'}(\mathbf{BL})$ is an implicative logic in the sense of Rasiowa [10], which is a logic if there is a binary (either primitive or definable by a formula) connective $\Rightarrow$ of its language such that the following hold:

(R) $\vdash \alpha \Rightarrow \alpha$,
(MP) $\alpha, \alpha \Rightarrow \beta \vdash \beta$,
(T) $\alpha \Rightarrow \beta,\ \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \gamma$,
(Cong) $\alpha \Rightarrow \beta, \beta \Rightarrow \alpha \vdash c(\gamma_1, \cdots, \gamma_i, \alpha, \cdots, \gamma_n) \Rightarrow c(\gamma_1, \cdots, \gamma_i, \beta, \cdots, \gamma_n)$,
(W) $\alpha \vdash \beta \Rightarrow \alpha$.

Most of these properties hold for $\mathbf{BL}$. In order to show that $\mathbf{S5'}(\mathbf{BL})$ is an implicative logic, we only prove that two new connectives $\Box$ and $\Diamond$ and $\Rightarrow$ are compatible.

Proposition 2.1: The following formulas are provable in $\mathbf{S5'}(\mathbf{BL})$:

1) $\vdash \Box\bar{1}$,
2) $\vdash \Box\alpha \Rightarrow (\Box\alpha\&\Box\alpha)$,
3) $\alpha \Rightarrow \beta \vdash \Box\alpha \Rightarrow \Box\beta$,
4) $\Box\alpha, \alpha \Rightarrow \beta \vdash \Box\beta$,
5) $\Box\alpha, \Box(\alpha \Rightarrow \beta) \vdash \Box\beta$,
6) $\Box\alpha \Rightarrow \beta \vdash \Box\alpha \Rightarrow \Box\beta$,
7) $\vdash \Box(\alpha \Rightarrow \beta) \Rightarrow (\Box\alpha \Rightarrow \Box\beta)$,
8) $\vdash \Diamond\alpha \equiv \Box\Diamond\alpha$,
9) $\alpha \Rightarrow \Diamond\alpha$,
10) $\vdash \Diamond\bar{1}$,
11) $\alpha \Rightarrow \beta \vdash \Diamond\alpha \Rightarrow \Diamond\beta$,
12) $\neg\alpha \vdash \neg\Diamond\alpha$,

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13) $\Diamond\alpha, \alpha \Rightarrow \beta \vdash \Diamond\beta$.

Proposition 2.1(3),(11) show that (Cong) is satisfies for $\Box$ and $\Diamond$. Thus $\mathbf{S5}'(\mathbf{BL})$ is an implicative logic and hence is algebraizable in the sense of Blok and Pigozzi [4]. This gives us immediately the completeness with respect to its associated class of algebras, which will be called monadic BL-algebras [6].

***Definition** 2.2:* A *monadic BL-algebra* is a pair $(L, \forall, \exists)$, where is L a BL-algebra, $\forall : L \to L$ and $\exists : L \to L$ are two unary operations on L such that the following conditions hold, for any $x, y \in L$,

1) $\forall(x) \leq x$,
2) $\forall(x \to \forall y) = \exists x \to \forall y$,
3) $\forall(\forall x \to y) = \forall x \to \forall y$,
4) $\forall(\exists x \vee y) = \exists x \vee \forall y$,
5) $\exists(x \odot x) = \exists x \odot \exists x$.

Algebraizability implies completeness of $\mathbf{S5}'(\mathbf{BL})$ with respect to the variety of monadic BL-algebras.

***Theorem** 2.3:* Let T be a theory and α be a formula over $\mathbf{S5}'(\mathbf{BL})$. Then the following statements are equivalent: (1)

1) $T \vdash \alpha$,
2) *for each monadic BL-algebra* $(L, \forall, \exists)$ *and for every model* e *of* T, $e(\alpha) = 1$,
3) $[\alpha]_T = [\overline{1}]_T$ *in* $\mathbf{S5}'(\mathbf{BL})$.

In $\mathbf{S5}'(\mathbf{BL})$, the usual form of the deduction theorem does not hold. Indeed,

$$\alpha \vdash \Box\alpha, \text{ but } \nvdash \alpha \Rightarrow \Box\alpha,$$

see the following example.

***Example** 2.4:* Let L_3 be a three-element G-algebra. If we define $\forall$ and $\exists$ on L_3 in the following way:

$$\forall x = \begin{cases} 0, & x = 0 \\ 0, & x = \frac{1}{2} \\ 1, & x = 1 \end{cases} \qquad \exists x = \begin{cases} 0, & x = 0 \\ 1, & x = \frac{1}{2} \\ 1, & x = 1 \end{cases}.$$

Then $(L_3, \forall, \exists)$ is a monadic G-algebra. Also, for any model e in this algebra, if $e(\alpha) = 1$, then $e(\forall\alpha) = 1$. But for $e(\alpha) = \frac{1}{2}$ we have $e(\forall\alpha) = 0$, and hence $e(\alpha \Rightarrow \forall\alpha) = 0$.

However, $\mathbf{S5}'(\mathbf{BL})$ enjoys the same form of deduction theorem holding for logics with the $\triangle$ operator in [8].

***Theorem** 2.5:* $T, \alpha \vdash \beta$ if and only if $T \vdash \Box\alpha \Rightarrow \beta$.

Proof: We prove by induction on every formula α_i ($1 \leq i \leq n$) of the given derivation of β from $T \cup \alpha$ that $T \vdash \Box\alpha \Rightarrow \alpha_i$.

If $\alpha_i = \alpha$, then the result follows due to (S1). If $\alpha_i \in T$ or is an instance of an axiom, then the result follows using Modus Ponens and the derivability of the schema $\alpha_i \Rightarrow (\Box\alpha \Rightarrow \alpha_i)$.

If α_i comes by application of modus ponens on previous formulas in the derivation, then the result follows, because from $\Box\alpha \Rightarrow \alpha_k$ and $\Box\alpha \Rightarrow (\alpha_k \Rightarrow \alpha_i)$ we may derive $(\Box\alpha \& \Box\alpha) \Rightarrow (\alpha_k \& (\alpha_k \Rightarrow \alpha_i)$ and hence also $\Box\alpha \Rightarrow \alpha_i$, using transitivity of $\Rightarrow$ applied to Proposition 2.1(2) and $(\alpha_k \& (\alpha_k \Rightarrow \alpha_i) \Rightarrow \alpha_i$.

If $\alpha_i = \Box\alpha_k$ comes using Necessitation Rules from α_k, then from $\Box\alpha \Rightarrow \alpha_k$, we may derive $\Box\alpha \Rightarrow \Box\alpha_k$ using Proposition 2.1(7).

Conversely, to the derivation given by the hypothesis add a step with α. In the next step put $\Box\alpha$, which follows from the previous formula using Necessitation Rule. Then derive β using Modus Ponens. ∎

III. Monadic filters in monadic BL-algebras

Algebraizability also gives us the notion of filters in monadic BL-algebras, which will have natural interpretation as the sets of all provable formulas in the logic $\mathbf{S5}'(\mathbf{BL})$. In this section, we focus on lattice structures of the set of all monadic filters in monadic BL-algebras.

Given a monadic BL-algebra $(L, \forall, \exists)$, a filter F is called a monadic filter of $(L, \forall, \exists)$ if it closed under $\forall$. For any nonempty subset X of L, we denote by $\langle X \rangle_\forall$ the monadic filter of $(L, \forall, \exists)$ generated by X, that is, $\langle X \rangle_\forall$ is the smallest monadic filter of $(L, \forall, \exists)$ containing X. Indeed,

$$\langle X \rangle_\forall = \{x \in L | x \geq \forall x_1 \odot \forall x_2 \odot \cdots \odot \forall x_n, x_i \in X, n \geq 1\}.$$

In particular, $\langle a \rangle_\forall = \{x \in L | x \geq (\forall a)^n, n \geq 1\}$. Also, if F is a monadic filter of $(L, \forall, \exists)$ and $x \notin F$, then we put

$$F \vee [\forall x) := \langle F \cup \{x\} \rangle_\forall = \{y \in L | x \geq f \odot (\forall x)^n, f \in F\}.$$

We denote the set of all monadic filters of $(L, \forall, \exists)$ by $MF[L, \forall, \exists]$ and easily prove that $MF[L, \forall, \exists]$ is a complete lattice with respect to inclusion.

The following example shows that monadic filters exist in monadic BL-algebras. Also, it indicates that the concept of monadic filters in monadic BL-algebras is not the same as that of filters in BL-algebras.

***Example** 3.1:* Let $L = \{0, a, b, c, 1\}$, where $0 \leq a \leq b, c \leq 1$. Define operations $\odot$ and $\to$ are as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	1
c	0	a	a	c	1
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	c	1	1
b	0	a	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Table 1. The operations $\odot$ and $\to$ of Example 3.1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Define $\forall$ and $\exists$ are as follows:

$$\forall x = \begin{cases} 0_1, & x = 0, a, b \\ c, & x = c \\ 1, & x = 1 \end{cases} \qquad \exists x = \begin{cases} 0, & x = 0 \\ c, & x = a, c. \\ 1, & x = b, 1 \end{cases}$$

Then $(L, \forall, \exists)$ is a monadic BL-algebra. It is easily checked that monadic filters of $(L, \forall, \exists)$ are $\{c, 1\}$, $\{1\}$ and L. However, $\{a, b, c, 1\}$ is a filter of L, but not a monadic filter of $(L, \forall, \exists)$, since $\forall a = \forall b = 0 \notin \{a, b, c, 1\}$.

***Theorem** 3.2:* Let $(L, \forall, \exists)$ be a monadic BL-algebra. Define operations $\wedge, \vee, \mapsto$ on $MF[L, \forall, \exists]$ are as follows: for all $F_1, F_2 \in MF[L, \forall, \exists]$,

$F_1 \wedge F_2 = F_1 \cap F_2$,
$F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle_\forall$,
$F_1 \mapsto F_2 = \{x \in L | \forall x \vee f_1 \in F_2$ for any $f_1 \in F_1\}$.

Then $(MF[L, \forall, \exists], \wedge, \vee, \mapsto, 1, L)$ is a complete Heyting algebra.

Proof: Let $\{F_i\}_{i\in I}$ be a family of monadic filters of $(L,\forall,\exists)$. Then easily obtain that the infimum and supermum are

$$\{F_i\}_{i\in I} = \cap_{i\in I}F_i,\ \vee_{i\in I}F_i = \cup_{i\in I F_i}.$$

Hence $(MF[L,\forall,\exists],\wedge,\vee,1,L)$ is a complete lattice under the inclusion order $\subseteq$.

Then we define

$$F_1 \mapsto F_2 = \{x \in L | \forall x \vee f_1 \in F_2 \text{ for any } f_1 \in F_1\}$$

for any $F_1, F_2 \in MF[L,\forall,\exists]$, and shall prove that

$$F_1 \wedge F_2 \le F_3 \text{ if and only if } F_2 \le F_1 \mapsto F_3$$

for all $F_1, F_2, F_3 \in MF[L,\forall,\exists]$.

We show that $F_1 \mapsto F_2$ is a monadic filter of $(L,\forall,\exists)$. Clearly $1 \in F_1 \mapsto F_2$. If $x \in F_1 \mapsto F_2$ and $x \le y$, then for any $f_1 \in F_1$ such that $\forall x \vee f_1 \in F_2$. Since $\forall x \vee f_1 \le \forall y \vee f_1 \in F_2$, we have $y \in F_1 \mapsto F_2$. If $x, y \in F_1 \mapsto F_2$, then for any $f_1 \in F_1$, $\forall x \vee f_1, \forall y \vee f_1 \in F_2$, and hence $\forall(x \odot y) \vee f_1 \in F_2$. So $x \odot y \in F_1 \mapsto F_2$. Obviously, if $x \in F_1 \mapsto F_2$, then $\forall x \in F_1 \mapsto F_2$. Thus $F_1 \mapsto F_2$ is a monadic filter of $(L,\forall,\exists)$.

Then we will prove that $F_1 \wedge F_2 \le F_3$ if and only if $F_1 \le F_2 \mapsto F_3$. Assume that $F_1 \wedge F_2 \le F_3$. Let $f_1 \in F_1$. Then $\forall f_1 \in F_1$, and for any $f_2 \in F_2$, we have $f_2 \vee \forall f_1 \ge \forall f_1$, $f_2 \vee \forall f_1 \ge f_2$, and hence $f_2 \vee \forall f_1 \in F_1 \wedge F_2 \le F_3$. Thus $f_1 \in F_2 \mapsto F_3$. Conversely, assume that $F_1 \le F_2 \mapsto F_3$. Let $x \in F_1 \wedge F_2$, then $x \in F_2 \mapsto F_3$. For any $y \in F_2$, we have $\forall x \vee y \in F_3$. Taking $y = x \in F_2$, we have $x \vee \forall x = x \in F_3$. Thus $F_1 \le F_2 \mapsto F_3$. ■

***Corollary** 3.3:* Let $(L,\forall,\exists)$ be a monadic BL-algebra. Then $(MF[L,\forall,\exists],\subseteq)$ is a pseudocomplemented lattice. Moreover, for any $F \in MF[L,\forall,\exists]$, the pseudocomplemented of F is

$$F \mapsto 1 = \{x \in L | \forall x \vee f = 1 \text{ for any } f \in F\}.$$

Let $(L,\forall,\exists)$ be a monadic BL-algebra and F be a monadic filter of $(L,\forall,\exists)$. Given a nonempty subset X of L, we put

$$X_F^{\perp\forall} = \{a \in L | \forall a \vee x \in F, \text{ for any } x \in X\},$$

which is called a *monadic co-annihilator* of X with respect to F. Clearly, $1 \in X_F^{\perp\forall}$. In particular, it is easy to prove that $X_F^{\perp\forall}$ is a monadic filter and $F \subseteq X_F^{\perp\forall}$, which will generalized the related results in [13].

Then we give some properties of monadic co-annihilators in monadic BL-algebras.

***Proposition** 3.4:* Let $(L,\forall,\exists)$ be a monadic BL-algebra, F and G be monadic filters of $(L,\forall,\exists)$. Given nonempty sets X, Y of L, we have:

1) $F \subseteq G$ implies $X_F^{\perp\forall} \subseteq X_G^{\perp\forall}$,
2) $X \subseteq Y$ implies $Y_F^{\perp\forall} \subseteq X_F^{\perp\forall}$,
3) $((X_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall} = X_F^{\perp\forall}$,
4) if $F \subseteq G$, then $G \cap G_F^{\perp\forall} = F$,
5) $(X_F^{\perp\forall})_F^{\perp\forall} \cap X_F^{\perp\forall} = F$,
6) $X_F^{\perp\forall} = L$ if and only if $X \subseteq F$,
7) $(\cup_{i\in I}X_i)_F^{\perp\forall} = \cap_{i\in I}(X_i)_F^{\perp\forall}$.

Proof: (1) If $F \subseteq G$ and $a \in X_F^{\perp\forall}$, then $\forall a \vee x \in F$ for any $x \in X$, and hence $\forall a \vee x \in G$ for any $x \in X$, that is, $a \in X_G^{\perp\forall}$. Thus, $X_F^{\perp\forall} \subseteq X_G^{\perp\forall}$.

(2) If $X \subseteq Y$ and $a \in Y_F^{\perp\forall}$, then $\forall a \vee y \in F$ for any $y \in Y$, and hence $\forall a \vee x \in G$ for any $x \in X$, that is, $a \in X_F^{\perp\forall}$. Thus, $Y_F^{\perp\forall} \subseteq X_F^{\perp\forall}$.

(3) If $x \in X$, then $\forall a \vee x \in F$ for any $a \in X_F^{\perp\forall}$, which implies $x \in (X_F^{\perp\forall})_F^{\perp\forall}$, and hence $X \subseteq (X_F^{\perp\forall})_F^{\perp\forall}$. Applying (2), we have $((X_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall} \subseteq X_F^{\perp\forall}$. On the other hand, taking $X = X_F^{\perp\forall}$ in $X \subseteq ((X_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall}$, we obtain $X_F^{\perp\forall} \subseteq ((X_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall}$. Thus $((X_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall} = X_F^{\perp\forall}$.

(4) If $x \in G \cap G_F^{\perp\forall}$, then $x \in G$ and $x \in G_F^{\perp\forall}$. It follows that $\forall a \vee x \in F$ for any $a \in G$. In particular, taking $x = a$ in $\forall a \vee x \in F$, we have $\forall x \vee x = x \in F$, that is, $x \in F$, which implies $G \cap G_F^{\perp\forall} \subseteq F$. On the other hand, since $F \subseteq G$, we have $F = F \cap G = G_F^{\perp\forall} \cap G$. Thus, $G \cap G_F^{\perp\forall} = F$.

(5) Since $X_F^{\perp\forall}$ is a monadic filter of $(L,\forall,\exists)$ and $F \subseteq X_F^{\perp\forall}$, we can obtain $(X_F^{\perp\forall})_F^{\perp\forall} \cap X_F^{\perp\forall} = F$ by (4).

(6) If $X_F^{\perp\forall} = L$ and $a \in X \subseteq L$, then $a \in X_F^{\perp\forall}$, and hence $\forall a \vee a = a \in F$. Thus $X \subseteq F$. Conversely, if $X \subseteq F$ and $x \in L$, we have $x \in X_F^{\perp\forall}$. Therefore, $X \subseteq F$.

(7) It follows from (2) that $(\cup_{i\in I}X_i)_F^{\perp\forall} \subseteq (X_i)_F^{\perp\forall}$ for any $i \in I$. We deduce that $(\cup_{i\in I}X_i)_F^{\perp\forall} = \cap_{i\in I}(X_i)_F^{\perp\forall}$. Conversely, let $a \in \cap_{i\in I}(X_i)_F^{\perp\forall}$, we have $a \in (X_i)_F^{\perp\forall}$ for any $i \in I$. Hence $\forall a \vee x \in F$ for any $x_i \in X$ and $i \in I$, which implies $a \in (\cup_{i\in I}X_i)_F^{\perp\forall}$. Therefore, $(\cup_{i\in I}X_i)_F^{\perp\forall} = \cap_{i\in I}(X_i)_F^{\perp\forall}$. ■

We will study the lattice structures of two special types of monadic filters in monadic BL-algebras via monadic co-annihilators.

***Theorem** 3.5:* Let $(L,\forall,\exists)$ be a monadic BL-algebra and F be a monadic filter of $(L,\forall,\exists)$. Given a nonempty set X of L, we have

$$\langle X\rangle_\forall \mapsto F = X_F^{\perp\forall}$$

in the Heyting algebra $(MF[L,\forall,\exists],\wedge,\vee,\mapsto,1,L)$.

Proof: If $a \in \langle X\rangle_\forall \mapsto F$, then $\langle a\rangle_\forall \subseteq \langle X\rangle_\forall \mapsto F$, and hence $\langle a\rangle_\forall \cap \langle X\rangle_\forall \subseteq F$. For any $x \in X$, we have $\forall a \vee x \in \langle \forall a \vee x\rangle_\forall \subseteq \langle a\rangle_\forall \cap \langle X\rangle_\forall \subseteq F$ for any $x \in X$, and hence $\forall a \vee x \in F$, which implies $a \in X_F^{\perp\forall}$. Thus, $\langle X\rangle_\forall \mapsto F \subseteq X_F^{\perp\forall}$.

Conversely, if $a \in X_F^{\perp\forall}$, then $\forall a \vee x \in F$, which implies $a \in \langle X\rangle_\forall \mapsto F$. Thus $X_F^{\perp\forall} \subseteq \langle X\rangle_\forall \mapsto F$. Therefore, $\langle X\rangle_\forall \mapsto F = X_F^{\perp\forall}$. ■

***Corollary** 3.6:* Let $(L,\forall,\exists)$ be a monadic BL-algebra and F be a monadic filter of $(L,\forall,\exists)$. Given a monadic filter G of L, we have

$$G \mapsto F = G_F^{\perp\forall}$$

in the Heyting algebra $(MF[L,\forall,\exists],\wedge,\vee,\mapsto,1,L)$.

In what follows, using monadic co-annihilators, we introduce stable monadic filters relative a nonempty set X in monadic BL-algebras.

***Definition** 3.7:* Let $(L,\forall,\exists)$ be a monadic BL-algebra and F be a monadic filter of $(L,\forall,\exists)$. Then F is called *a stable monadic filter relative* X if $X_F^{\perp\forall} = F$, that is, $\langle X\rangle_\forall \mapsto F = F$ in the complete Heyting algebra $(MF[L,\forall,\exists],\wedge,\vee,\mapsto,1,L)$.

We will denote the set of all stable monadic filters relative X of $(L,\forall,\exists)$ by $S_X MF[L,\forall,\exists]$.

***Example** 3.8:* Let $(L,\forall,\exists)$ be a monadic BL-algebra and $F = \{c, 1\}$ be a moandic filter in Example 3.1. If $X = \{a\}$, then $X_F^{\perp\forall} = \{c, 1\} = F$, that is, F is stable monadic filter relative X.

***Theorem** 3.9:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and X be a nonempty set of L. Then a stable filter F is stable relative X if and only if has a form $\langle X\rangle_\forall \mapsto F$ in the complete Heyting algebra $(MF[L, \forall, \exists], \wedge, \vee, \mapsto, 1, L)$.

Proof: If F is a stable state filter relative X of $(L, \forall, \exists)$, then follows from Definition 3.7 that $F = \langle X\rangle_\forall \mapsto F$ for a nonempty set X of L.

Conversely, if $G = \langle X\rangle_\forall \mapsto F$, then $\langle X\rangle_\forall \to G = \langle X\rangle_\forall \mapsto (\langle X\rangle_\forall \mapsto F) = (\langle X\rangle_\forall \wedge \langle X\rangle_\forall) \mapsto F = \langle X\rangle_\forall \mapsto F = G$. Thus, G is a stable filter relative X of $(L, \forall, \exists)$. ∎

Applying Theorem 3.9, we have

$$S_X MF[L, \forall] = \{\langle X\rangle_\forall \mapsto F | F \in MF[L, \forall]\}.$$

The next result shows that $(S_X MF[L, \forall, \exists], \subseteq)$ forms a complete Heyting algebra.

***Theorem** 3.10:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and X be a nonempty set of L. Define $\wedge$ and $\sqcup$ on $S_X MF[L, \forall, \exists]$ are as follows: for any $\langle X\rangle_\forall \mapsto F_1, \langle X\rangle_\forall \mapsto F_2$,

$(\langle X\rangle_\forall \mapsto F_1) \wedge (\langle X\rangle_\forall \mapsto F_2) = \langle X\rangle_\forall \mapsto (F_1 \wedge F_2) = \langle X\rangle_\forall \mapsto (F_1 \cap F_2)$,

$(\langle X\rangle_\forall \mapsto F_1) \sqcup (\langle X\rangle_\forall \mapsto F_2) = \langle X\rangle_\forall \mapsto (F_1 \vee F_2) = \langle X\rangle_\forall \mapsto \langle F_1 \cup F_2\}_\forall$.

Then $(S_X MF[L, \forall, \exists], \wedge, \sqcup, \mapsto)$ is a complete Heyting algebra.

Proof: If $\{\langle X\rangle_\forall \mapsto F_i\}_{i\in I} \subseteq S_X MF[L, \forall, \exists]$, then

$$\wedge_{i\in I}\{\langle X\rangle_\forall \mapsto F_i\}_{i\in I} = \langle X\rangle_\forall \mapsto \wedge_{i\in I} F_i.$$

Thus $(S_X MF[L, \forall, \exists], \subseteq)$ is a complete lattice.

Next, we show that $\langle X\rangle_\forall \mapsto (F_1 \vee F_2)$ is the supremum of $\{\langle X\rangle_\forall \mapsto F_1, \langle X\rangle_\forall \mapsto F_2\}$. Notice that

$$\langle X\rangle_\forall \mapsto F_1 \subseteq \langle X\rangle_\forall \mapsto (F_1 \vee F_2),$$
$$\langle X\rangle_\forall \mapsto F_2 \subseteq \langle X\rangle_\forall \mapsto (F_1 \vee F_2).$$

Conversely, if $\langle X\rangle_\forall \mapsto F$ is any stable monadic filter relative X of $(L, \forall, \exists)$ and

$$\langle X\rangle_\forall \mapsto F_1, \langle X\rangle_\forall \mapsto F_2 \subseteq \langle X\rangle_\forall \mapsto F,$$

then $\langle X\rangle_\forall \wedge (\langle X\rangle_\forall \mapsto F_1 \subseteq F$ and $\langle X\rangle_\forall \wedge (\langle X\rangle_\forall \mapsto F_2 \subseteq F$, which implies $\langle X\rangle_\forall \wedge F_1 \subseteq F$ and $\langle X\rangle_\forall \wedge F_2 \subseteq F$. Since Heyting algebra is a distributive lattice, we have

$$\langle X\rangle_\forall \wedge (F_1 \vee F_2) = (\langle X\rangle_\forall \wedge F_1) \vee (\langle X\rangle_\forall \wedge F_2) \subseteq F,$$

and hence $\langle X\rangle_\forall \mapsto (F_1 \vee F_2) = (\langle X\rangle_\forall \mapsto (\langle X\rangle_\forall \wedge (F_1 \vee F_2)) \subseteq \langle X\rangle_\forall \mapsto F$. Thus, $(\langle X\rangle_\forall \mapsto F_1) \sqcup (\langle X\rangle_\forall \mapsto F_2) = \langle X\rangle_\forall \mapsto (F_1 \vee F_2)$ is the supremum of $\{\langle X\rangle_\forall \mapsto F_1, \langle X\rangle_\forall \mapsto F_2\}$.

Finally, we prove that

$$F \wedge G \subseteq K \text{ if and only if } F \subseteq G \mapsto K$$

for any $F, G, K \in S_X MF[L, \forall, \exists]$. If $F, G, K \in S_X MF[L, \forall, \exists]$, then there exists F_1, F_2, F_3 such that $F = \langle X\rangle_\forall \mapsto F_1$, $G = \langle X\rangle_\forall \mapsto F_2$, $K = \langle X\rangle_\forall \mapsto F_3$. Clearly, $G \mapsto K \in S_X MF[L, \forall, \exists]$. Indeed, $G \mapsto K = (\langle X\rangle_\forall \mapsto F_2) \mapsto (\langle X\rangle_\forall \mapsto F_3) = (\langle X\rangle_\forall \wedge (\langle X\rangle_\forall \mapsto F_2) \mapsto F_3)$. Since $F_2 \mapsto F_3 \in MF[L, \forall, \exists]$, we have $G \mapsto K \in S_X MF[L, \forall, \exists]$. Also, $F \wedge G \subseteq K$ if and only if $((\langle X\rangle_\forall \mapsto F_1) \wedge ((\langle X\rangle_\forall \mapsto F_2) \subseteq \langle X\rangle_\forall \mapsto F_3$ if and only if $\langle X\rangle_\forall \mapsto (F_2 \wedge F_2) \subseteq \langle X\rangle_\forall \mapsto F_3$ if and only if $\langle X\rangle_\forall \wedge ((\langle X\rangle_\forall \mapsto F_1 \wedge F_2) \subseteq F_3$ if and only if $\langle X\rangle_\forall \wedge (F_1 \wedge F_2) \subseteq F_3$ if and only if $\langle X\rangle_\forall \wedge F_1 \subseteq F_2 \mapsto F_3$ if and only if $\langle X\rangle_\forall \wedge ((\langle X\rangle_\forall \mapsto F_1) \subseteq F_2 \mapsto F_3$ if and only if $\langle X\rangle_\forall \mapsto F_1 \subseteq \langle X\rangle_\forall \mapsto (F_2 \mapsto F_3)$ if and only if $F \subseteq G \mapsto K$ for any $F, G, K \in S_X MF[L, \forall, \exists]$. ∎

By using monadic co-annihilators, we introduced involutory monadic filters in monadic BL-algebras.

***Definition** 3.11:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and F be a monadic filter of $(L, \forall, \exists)$. A monadic filter G of $(L, \forall, \exists)$ is called an *involutory monadic filter relative F* if $(G_F^{\perp\forall})_F^{\perp\forall} = G$.

We will denote the set of all stable monadic filters relative X of $(L, \forall, \exists)$ by $I_F MF[L, \forall, \exists]$.

***Example** 3.12:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and $F = \{c, 1\}$ be a monadic filter in Example 3.1. If $F = \{c, 1\}$ and $G = \{1\}$, then $(G_F^{\perp\forall})_F^{\perp\forall} = G$, that is, G is stable monadic filter relative F.

***Proposition** 3.13:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and F be a monadic filter of $(L, \forall, \exists)$. For any $G, H \in I_F MF[L, \forall, \exists]$, we have that $G \cap H \subseteq F$ if and only if $H \subseteq G_F^{\perp\forall}$.

Proof: In the Heyting algebra $(MF[L, \forall, \exists], \wedge, \vee, \to)$, $G \cap H \subseteq F$ if and only if $H \subseteq G \mapsto F$ for all $G, H \in MF[L, \forall, \exists]$. From Corollary 3.6, we have $G \mapsto F = G_F^{\perp\forall}$. Thus, $G \cap H \subseteq F$ if and only if $H \subseteq G_F^{\perp\forall}$ for all $F, G, H \in MF[L, \forall, \exists]$. ∎

***Theorem** 3.14:* Let $(L, \forall, \exists)$ be a monadic BL-algebra and F be a monadic filter of $(L, \forall, \exists)$. Then $(I_F MF[L, \forall, \exists], \subseteq)$ is a complete Boolean algebra, where $\vee_{i\in I} G_i = (\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall} = (\cap_{i\in I}(G_i)_F^{\perp\forall})_F^{\perp\forall}$, $\wedge_{i\in I} G_i = \cap_{i\in I} G_i$, $G_{iF}^{\perp\forall}$ is the complement of G, for all G_i and $G \in I_F MF[L, \forall, \exists]$ and F, L are the bottom and top elements in $I_F MF[L, \forall, \exists]$, respectively.

Proof: Assume that $\{G_i | i \in I\} \subseteq I_F MF[L, \forall, \exists]$. Then we have $G_i = ((G_i)_F^{\perp\forall})_F^{\perp\forall}$. It follows from Proposition 3.4(7) that

$$\cap_{i\in I} G_i = \cap_{i\in I}((G_i)_F^{\perp\forall})_F^{\perp\forall} = (\cup_{i\in I}((G_i)_F^{\perp\forall})_F^{\perp\forall} = \cap_{i\in I} G_i.$$

Hence $\cap_{i\in I} G_i$ is an involutory monadic filter relative F. It is easy to check that $\cap_{i\in I} G_i$ is the infimum of $\{G_i | i \in I\}$ in $I_F MF[L, \forall, \exists]$. Clearly, $(\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall}$ is an involutory monadic filter relative F of $(L, \forall, \exists)$. Since $G_i \subseteq \cup_{i\in I} G_i \subseteq (\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall}$ for any $i \in I$, we obtain that $(\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall}$ is the upper bound of $\{G_i | i \in I\}$ in $I_F MF[L, \forall, \exists]$. Thus $(I_F MF[L, \forall, \exists], \subseteq)$ is a lattice.

Next, let G be any involutory monadic filter relative F and $G_i \subseteq G$ for all $i \in I$. Then $G_F^{\perp\forall} \subseteq (G_i)_F^{\perp\forall}$ for all $i \in I$. Hence $G_{iF}^{\perp\forall} \subseteq \cap_{i\in I}(G_i)_F^{\perp\forall}$. It follows that

$$(\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall} = (\cap_{i\in I}(G_i)_F^{\perp\forall})_F^{\perp\forall} \subseteq G_F^{\perp\forall})_F^{\perp\forall} = G.$$

Hence $(\cup_{i\in I} G_i)_F^{\perp\forall})_F^{\perp\forall}$ is the supremum of $\{G_i | i \in I\}$ in $I_F MF[L, \forall, \exists]$. Thus $(I_F MF[L, \forall, \exists], \subseteq)$ is a complete lattice.

Moreover, by Propositions 3.4(4) and (6), we have $F_F^{\perp\forall} = L$ and $L_F^{\perp\forall} = F$. Hence we can obtain that $(F_F^{\perp\forall})_F^{\perp\forall} = F$ and $(L_F^{\perp\forall})_F^{\perp\forall} = L$, which implies $F, L \in I_F MF[L, \forall, \exists]$.

For all $G \in I_F MF[L,\forall,\exists]$, we have $F = (G_F^{\perp\forall})_F^{\perp\forall} \cap G_F^{\perp\forall} = G \cap G_F^{\perp\forall} \subseteq G$ by Proposition 3.4(5). Thus, F and L are the bottom and top elements in $I_F MF[L,\forall,\exists]$, respectively. Also, by Proposition 3.4(4), we can obtain $G \cap G_F^{\perp\forall} = F$. For any $G \in I_F MF[L,\forall,\exists]$, we have that $G \vee G_F^{\perp\forall} = ((G \cup G_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall} = (G_F^{\perp\forall} \cap G_F^{\perp\forall})_F^{\perp\forall})_F^{\perp\forall} = (G_F^{\perp\forall} \cap G)_F^{\perp\forall}) = F_F^{\perp\forall} = L$, that is, $G_F^{\perp\forall}$ is the complement of G. Thus, $(I_F MF[L,\forall,\exists],\subseteq)$ is a complemented lattice.

Finally, we shall prove that $(I_F MF[L,\forall,\exists],\subseteq)$ is also a distributive lattice. For any $G_1,G_2,G_3 \in I_F MF[L,\forall,\exists]$, let $G = (G_1 \cap G_2) \vee (G_1 \cap G_3)$. Then $G_1 \cap G_2 \subseteq G$ and $G_1 \cap G_3 \subseteq G$. It follows that $G_1 \cap G_2 \cap G_F^{\perp\forall} \subseteq G \cap G_F^{\perp\forall} = F$ and $G_1 \cap G_3 \cap G_F^{\perp\forall} \subseteq G \cap G_F^{\perp\forall} = F$, that is, $G_2 \cap (G_1 \cap G_F^{\perp\forall}) \subseteq F$ and $G_3 \cap (G_1 \cap G_F^{\perp\forall}) \subseteq F$. Then $(G_2 \vee G_3) \subseteq (G_1 \cap G_F^{\perp\forall})_F^{\perp\forall} \subseteq F$. From Proposition 3.13, we obtain that $G_2 \subseteq (G_1 \cap G_F^{\perp\forall}) \subseteq F$, that is, $(G_1 \cap (G_2 \vee G_3)) \cap G_F^{\perp\forall} \subseteq F$. Thus, $G_1 \cap (G_2 \vee G_3) \subseteq (G_F^{\perp\forall})_F^{\perp\forall} = G = (G_1 \cap G_2) \vee (G_1 \cap G_3)$. Conversely, $G_1 \cap G_2) \vee (G_1 \cap G_3) \subseteq G_1 \cap (G_2 \vee G_3)$ is obvious. Hence, we obtain $G_1 \cap G_2) \vee (G_1 \cap G_3) = G_1 \cap (G_2 \vee G_3)$, that is, $(I_F MF[L,\forall,\exists],\subseteq)$ is a distributive lattice.

Therefore, $(I_F MF[L,\forall,\exists],\subseteq)$ is a complete Boolean algebra. ■

IV. Conclusion

Motivated by previous research on monadic BL-algebras, we discussed the lattice structure of monadic filters in monadic BL-algebras. Since the above topics are of current interest, we suggest further directions of research:

1) Introducing and studying polyadic BL-algebras, namely generalizations of monadic BL-algebras given by polyadic structures.
2) Focusing on varieties of monadic BL-algebras. In particular, we can investigate locally finite, finitely approximated and splitting varieties of monadic BL-algebras as well as varieties with the disjunction and existence properties.

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