Differences Inequalities of General L_p-Mixed Brightness Integrals

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Abstract—**Lutwak introduced the mixed brightness for convex bodies. After, Li and Zhu put forward mixed-brightness integrals. Recently, Yan and Wang defined the general L_p-mixed brightness integrals. In this article, we establish the Brunn-Minkowski, new cycle and Aleksandrov-Fenchel type inequalities for the differences of general L_p-mixed Brightness integrals.**

***Index Terms*—general L_p-mixed brightness integrals, Brunn-Minkowski type inequality, new cycle type inequality, Aleksandrov-Fenchel type inequality.**

I. Introduction

LET $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_{os}^n$, respectively. Let S^{n-1} denote the unit sphere and $V(M)$ denote the n-dimensional volume of the body M. For the centered unit ball B, write $V(B)=\omega_n$.

The projection bodies were introduced by Minkowski at the turn of the previous century. For each $M\in\mathcal{K}^n$, the projection body, ΠM, of M is an origin-symmetric convex body whose support function is defined by (see [3], [18])

$$h(\Pi M,u)=\frac{1}{2}\int_{S^{n-1}}|u\cdot v|dS(M,v),$$

for all $u\in S^{n-1}$. Here $S(M,\cdot)$ denotes the surface area measure of M.

Lutwak first introduced the notion of the mixed brightness of convex bodies in [15]. After, associated with the notion of the projection bodies and the mixed brightness, Li and Zhu [12] introduced the notion of mixed brightness integrals and given the L_p-mixed brightness integrals, moreover, they also established analogous to the Fenchel-Aleksandrov inequality and isoperimetric inequality of the mixed brightness integrals for the mixed volumes. For the mixed brightness integrals, Zhao [24] established the greatest upper bound for the product of the mixed brightness integrals of a convex body and its polar dual. After, Zhou, Wang and Feng [27] obtained some Brunn-Minkowski type inequalities for the mixed brightness integrals. Recently, Li et al. firstly introduced the notion of mixed complex brightness integrals [10] and dual mixed complex brightness integrals [11], they extended the classical concepts of mixed brightness integrals in real vector space to complex cases.

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In 2005, Ludwig ([13]) combined with a function $\varphi_\tau:\mathbf{R}\to[0,+\infty)$ by $\varphi_\tau(t)=|t|+\tau t,\tau\in[-1,1]$, introduced general L_p-projection bodies as follows: For $M\in K_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, the general L_p-projection body $\Pi_p^\tau M\in K_o^n$ is defined by

$$h^p(\Pi_p^\tau M,u)=\alpha_{n,p}(\tau)\int_{S^{n-1}}\varphi_\tau(u\cdot v)^p dS_p(M,v),$$

where

$$\alpha_{n,p}(\tau)=\frac{2\alpha_{n,p}}{(1+\tau)^p+(1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B=B$. Obviously, $\Pi_p^0 M=\Pi_p M$.

Recently, using the general L_p-projection bodies, Yan and Wang [22] defined the general L_p-mixed brightness integrals as follows: For $M_1,\ldots,M_n\in\mathcal{K}_o^n$, $p\geq 1$ and $\tau\in[-1,1]$, the general L_p-mixed brightness integrals, $D_p^{(\tau)}(M_1,\ldots,M_n)$, of $M_1,\ldots,M_n$ is defined by

$$\begin{aligned}&D_p^{(\tau)}(M_1,\ldots,M_n)\\&=\frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(M_1,u)\cdots\delta_p^{(\tau)}(M_n,u)dS(u),\end{aligned}\qquad(1.1)$$

where $\delta_p^{(\tau)}(M,u)=\frac{1}{2}h(\Pi_p^\tau M,u)$ denotes the half general L_p-brightness of $M\in\mathcal{K}_o^n$ in the direction u. Convex bodies $M_1,\ldots,M_n$ are said to have similar general L_p-brightness if there exist constants $\lambda_1,\ldots\lambda_n>0$ such that, for all $u\in\mathcal{S}^{n-1}$,

$$\lambda_1\delta_p^{(\tau)}(M_1,u)=\lambda_2\delta_p^{(\tau)}(M_2,u)=\cdots=\lambda_n\delta_p^{(\tau)}(M_n,u).$$

Obviously, for $\tau=0$ and $p=1$, (1.1) is just the mixed brightness integrals $D(M_1,\ldots,M_n)$.

Let $\underbrace{M_1=\cdots=M_{n-i}}_{n-i}=M$, $\underbrace{M_{n-i+1}=\cdots=M_n}_{i}=N$ $(i=0,1,\ldots,n)$ in (1.1), we write $D_{p,i}^{(\tau)}(M,N)=D_p^{(\tau)}(\underbrace{M,\ldots,M}_{n-i},\underbrace{N,\ldots,N}_{i})$. More general, if allow i is any real, for $M,N\in\mathcal{K}_o^n$, $p\geq 1$, and $\tau\in[-1,1]$, the general L_p-mixed brightness integrals, $D_{p,i}^{(\tau)}(M,N)$, of M and N is defined by

$$D_{p,i}^{(\tau)}(M,N)=\frac{1}{n}\int_{S^{n-1}}\delta_p^{(\tau)}(M,u)^{n-i}\delta_p^{(\tau)}(N,u)^i.\qquad(1.2)$$

For $N=B$ in (1.2), we write $D_{p,i}^{(\tau)}(M,B)=\frac{1}{2^i}D_{p,i}^{(\tau)}(M)$ and notice that $\delta_p^{(\tau)}(B,u)=\frac{1}{2}h(\Pi_p^\tau B,u)=\frac{1}{2}$, for all $u\in S^{n-1}$, which together with (1.2) yields

$$D_{p,i}^{(\tau)}(M)=\frac{1}{2^i\cdot n}\int_{S^{n-1}}\delta_p^{(\tau)}(M,u)^{n-i}dS(u),\qquad(1.3)$$

where $D_{p,i}^{(\tau)}(M)$ is called the i-th general L_p-mixed brightness integrals of M.

For $N = M$ in (1.2), write $D_{p,i}^{(\tau)}(M,M) = D_p^{(\tau)}(M)$, which is called the general L_p-brightness integrals of M. Clearly,

$$D_p^{(\tau)}(M) = \frac{1}{n}\int_{S^{n-1}} \delta_p^{(\tau)}(M,u)^n dS(u), \tag{1.4}$$

From (1.2) and (1.4), we easily obtain

$$D_{p,0}^{(\tau)}(M,N) = D_p^{(\tau)}(M), D_{p,n}^{(\tau)}(M,N) = D_p^{(\tau)}(N). \tag{1.5}$$

For general L_p-mixed brightness integrals, Yan and Wang [22] also established the following cycle, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities.

Theorem 1.A. *If $M, N \in \mathcal{K}_{os}^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i \in \mathbf{R}$, then for $i < n-p$,*

$$D_{p,i}^{(\tau)}(\lambda\circ M \mp_p \mu\circ N)^{\frac{p}{n-i}} \leq \lambda D_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}} + \mu D_{p,i}^{(\tau)}(N)^{\frac{p}{n-i}}; \tag{1.6}$$

for $i > n-p$ and $i \neq n$,

$$D_{p,i}^{(\tau)}(\lambda\circ M \mp_p \mu\circ N)^{\frac{p}{n-i}} \geq \lambda D_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}} + \mu D_{p,i}^{(\tau)}(N)^{\frac{p}{n-i}}, \tag{1.7}$$

in each case, equality holds if and only if M and N have similar general L_p-brightness. For $i = n-p$, equality always holds in (1.6) and (1.7).

Theorem 1.B. *If $M, N \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i,j,k \in \mathbf{R}$ such that $i < j < k$, then*

$$D_{p,j}^{(\tau)}(M,N)^{k-i} \leq D_{p,i}^{(\tau)}(M,N)^{k-j} D_{p,k}^{(\tau)}(M,N)^{j-i}, \tag{1.8}$$

with equality if and only if M and N have similar general L_p-brightness.

Theorem 1.C. *If $M_1,\ldots,M_n \in \mathcal{K}_o^n$, $p \geq 1$ $\tau \in [-1,1]$ and $1 < m \leq n$, then*

$$D_p^{(\tau)}(M_1,\ldots,M_n)^m \tag{1.9}$$

$$\leq \prod_{i=1}^{m} D_p^{(\tau)}(M_1,\ldots,M_{n-m},M_{n-i+1},\ldots,M_{n-i+1}),$$

with equality if and only if $M_{n-m+1},\ldots,M_n$ are all of similar general L_p-brightness.

The general L_p-mixed brightness integrals belong to a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig et al. (see [14], [5], [6], [4]). For the further recent research of asymmetric L_p Brunn-Minkowski theory, also see ([2], [9], [19], [20], [21], [23]).

In 2004, Leng [8] defined the volume differences function of convex bodies D and K, where $D \subseteq K$, by

$$D_v(K,D) = V(K) - V(D).$$

Meanwhile, Leng [8] established the following Brunn-Minkowski type inequality for volume difference functions.

Theorem 1.D. *If K, L and D are compact domains, $D \subseteq K$, $D' \subseteq L$, D' is a homothetic copy of D, then*

$$(V(K+L) - V(D+D'))^{\frac{1}{n}}$$

$$\geq (V(K) - V(D))^{\frac{1}{n}} + (V(L) - V(D'))^{\frac{1}{n}},$$

equality holds if and only if K and L are homothetic and $(V(K),V(D)) = \mu(V(L),V(D'))$, where μ is a constant. Here " $+$ " is Minkowski sum.

Since these seminal paper, inequalities for differences of geometric functionals have become the focus of increased attention (see [16], [25], [26]).

The aim of this paper is to establish the new differences inequalities for general L_p-mixed brightness integrals. First, we establish the following Brunn Minkowski type inequality for differences of general L_p-mixed brightness integrals.

Theorem 1.1. *Let $M,N,K,L \in \mathcal{K}_{os}^n$, $i \in \mathbf{R}$, $\tau \in [-1,1]$ and $\delta_p^{(\tau)}(K,\cdot) < \delta_p^{(\tau)}(M,\cdot)$, $\delta_p^{(\tau)}(L,\cdot) < \delta_p^{(\tau)}(N,\cdot)$, M and N have similar general L_p-brightness, if $p \geq 1$ and $i < n-p$, then*

$$[D_{p,i}^{(\tau)}(M \mp_p N) - D_{p,i}^{(\tau)}(M \mp_p N)]^{\frac{p}{n-i}} \tag{1.10}$$

$$\geq [D_{p,i}^{(\tau)}(M) - C_{p,i}^{(\tau)}(K)]^{\frac{p}{n-i}} + [D_{p,i}^{(\tau)}(N) - D_{p,i}^{(\tau)}(L)]^{\frac{p}{n-i}};$$

if $i > n-p$ and $i \neq n$, then

$$[D_{p,i}^{(\tau)}(M \mp_p N) - D_{p,i}^{(\tau)}(M \mp_p N)]^{\frac{p}{n-i}} \tag{1.11}$$

$$\leq [D_{p,i}^{(\tau)}(M) - C_{p,i}^{(\tau)}(K)]^{\frac{p}{n-i}} + [D_{p,i}^{(\tau)}(N) - D_{p,i}^{(\tau)}(L)]^{\frac{p}{n-i}},$$

in each case, with equality if and only if K and L have similar general L_p-brightness and there exists constant λ such that $(D_{p,i}^{(\tau)}(M), D_{p,i}^{(\tau)}(K)) = \lambda(D_{p,i}^{(\tau)}(N), D_{p,i}^{(\tau)}(L))$.

Next, we give the following new cycle type inequalities for the differences of general L_p-mixed brightness integrals.

Theorem 1.2. *Let $M,N,K,L \in \mathcal{K}_o^n$, $\tau \in [-1,1]$ and $\delta_p^{(\tau)}(K,\cdot) < \delta_p^{(\tau)}(M,\cdot)$, $\delta_p^{(\tau)}(L,\cdot) < \delta_p^{(\tau)}(N,\cdot)$, M and N have similar general L_p-brightness, if $p \geq 1$, $i,j,k \in \mathbf{R}$ and $0 \leq i < j < k$, then*

$$[D_{p,j}^{(\tau)}(M,N) - D_{p,j}^{(\tau)}(K,L)]^{k-i} \tag{1.12}$$

$$\geq [D_{p,i}^{(\tau)}(M,N) - D_{p,i}^{(\tau)}(K,L)]^{k-j} [D_{p,k}^{(\tau)}(M,N) - D_{p,k}^{(\tau)}(K,L)]^{j-i},$$

with equality if and only if K and L have similar general L_p-brightness and there exists constant λ such that $(D_{p,i}^{(\tau)}(M,N), D_{p,i}^{(\tau)}(K,L)) = \lambda(D_{p,k}^{(\tau)}(M,N), D_{p,k}^{(\tau)}(K,L))$.

Finally, we also establish the following Aleksandrov-Fenchel type inequalities for differences of general L_p-mixed brightness integrals.

Theorem 1.3. *Let $p \geq 1$, $\tau \in [-1,1]$, $M_i, N_i \in \mathcal{K}_o^n$ $(i = 1,\ldots,n)$, if $\delta_p^{(\tau)}(N_i,\cdot) < \delta_p^{(\tau)}(M_i,\cdot)$ $(i = 1,\ldots,n)$ and the bodies $M_{n-m+1},\ldots,M_n$ are all of similar general L_p-brightness, then for every $1 < m \leq n$,*

$$[D_p^{(\tau)}(M_1,\ldots,M_n) - C_p^{(\tau)}(N_1,\ldots,N_n)]^m \tag{1.13}$$

$$\geq \prod_{i=1}^{m}[D_p^{(\tau)}(M_1,\ldots,M_{n-m},M_{n-i+1},\ldots,M_{n-i+1})$$

$$-D_p^{(\tau)}(N_1,\ldots,N_{n-m},N_{n-i+1},\ldots,N_{n-i+1})],$$

with equality if and only if $N_{n-m+1},\ldots,N_n$ are all of similar general L_p-brightness and there exists constant λ such that

$$[D_p^{(\tau)}(M_1,\ldots,M_{n-m},M_n,\ldots,M_n),\ldots,D_p^{(\tau)}(M_1,\ldots,M_{n-m}, M_{n-m+1},\ldots,M_{n-m+1})] = \lambda[D_p^{(\tau)}(N_1,\ldots,N_{n-m},N_{n-m+1},\ldots,N_n),\ldots,D_p^{(\tau)}(N_1,\ldots,N_{n-m},N_{n-m+1},\ldots,N_{n-m+1})].$$

II. Preliminaries

A. Support function

For $M \in \mathcal{K}^n$, its support function, (see [3], [18]) $h(M,\cdot): \mathbf{R}^n \to \mathbf{R}$, is defined by

$$h(M,x) = \max\{x\cdot y : y \in M\}, \quad x \in \mathbf{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y.

B. Lp-Blaschke combination

For $M,N \in \mathcal{K}^n_{os}$, $p \geq 1$ and $\lambda,\mu \geq 0$ (not both zero), the L_p-Blaschke combination, $\lambda\circ M \mp_p \mu\circ N \in \mathcal{K}^n_{os}$, of M and N is defined by (see [3], [18])

$$dS_p(\lambda\circ M \mp_p \mu\circ N,\cdot) = \lambda dS_p(M,\cdot) + \mu dS(N,\cdot),$$

where $\lambda\circ M = \lambda^{\frac{1}{n-p}}M$, $\lambda\circ M$ denotes the Blaschke scalar multiplication. If $p=1$, then $\lambda\circ M \mp_p \mu\circ N$ is classical Blaschke combination.

III. Results and Proofs

In this part, we will give the proofs of Theorems 1.1-1.3. First, in order to prove Theorem 1.1, the following lemma is required.

Lemma 3.1 (Bellman's inequality [1]). *Let $\mathbf{a}=\{a_1,\ldots,a_n\}$ and $\mathbf{b}=\{b_1,\ldots,b_n\}$ be two series of positive real numbers. If $a_1^p-\sum_{i=2}^n a_i^p>0$, $b_1^p-\sum_{i=2}^n b_i^p>0$, then for $p>1$,*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}+\left(b_1^p-\sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \leq \left((a_1+b_1)^p-\sum_{i=2}^n(a_i+b_i)^p\right)^{\frac{1}{p}}; \quad (3.1)$$

for $p<0$ or $0<p<1$,

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}+\left(b_1^p-\sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \geq \left((a_1+b_1)^p-\sum_{i=2}^n(a_i+b_i)^p\right)^{\frac{1}{p}}, \quad (3.2)$$

with equality if and only if $\mathbf{a}=c\mathbf{b}$, where c is a constant.

Proof of Theorem 1.1. For $M,N,K,L \in \mathcal{K}^n_{os}$, $p\geq 1$ and $\tau\in[-1,1]$, if $i<n-p$ and $\lambda,\mu=1$, by (1.6), then

$$D^{(\tau)}_{p,i}(K\mp_p L)^{\frac{p}{n-i}} \leq D^{(\tau)}_{p,i}(K)^{\frac{p}{n-i}} + D^{(\tau)}_{p,i}(L)^{\frac{p}{n-i}}, \quad (3.3)$$

with equality if and only if K and L have similar general L_p-brightness. Since M and N have similar general L_p-brightness, thus according to the equality condition of inequality (1.6), we have

$$D^{(\tau)}_{p,i}(M\mp_p N)^{\frac{p}{n-i}} = D^{(\tau)}_{p,i}(M)^{\frac{p}{n-i}} + D^{(\tau)}_{p,i}(N)^{\frac{p}{n-i}}. \quad (3.4)$$

Since $\delta^{(\tau)}_p(K,\cdot)<\delta^{(\tau)}_p(M,\cdot)$, $\delta^{(\tau)}_p(L,\cdot)<\delta^{(\tau)}_p(N,\cdot)$, by (1.3), we obtain

$$D^{(\tau)}_{p,i}(M) > D^{(\tau)}_{p,i}(K), \quad D^{(\tau)}_{p,i}(N) > D^{(\tau)}_{p,i}(L)$$

From these, notice that $n-i>p$ $(i<n-p)$, i.e. $\frac{n-i}{p}$, and from (3.1), (3.3) and (3.4), we obtain

$$\begin{aligned}&(D^{(\tau)}_{p,i}(M\mp_p N)-D^{(\tau)}_{p,i}(K\mp_p L))^{\frac{p}{n-i}}\\ &\geq \left[\left(D^{(\tau)}_{p,i}(M)^{\frac{p}{n-i}}+D^{(\tau)}_{p,i}(N)^{\frac{p}{n-i}}\right)^{\frac{n-i}{p}}\right.\\ &\quad\left.-\left(D^{(\tau)}_{p,i}(K)^{\frac{p}{n-i}}+D^{(\tau)}_{p,i}(L)^{\frac{p}{n-i}}\right)^{\frac{n-i}{p}}\right]^{\frac{p}{n-i}}\\ &\geq \left(D^{(\tau)}_{p,i}(M)-D^{(\tau)}_{p,i}(K)\right)^{\frac{p}{n-i}}+\left(D^{(\tau)}_{p,i}(N)-D^{(\tau)}_{p,i}(L)\right)^{\frac{p}{n-i}}.\end{aligned}$$

This yields inequality (1.10). By the equality conditions of inequality (3.1) and (3.3), we see that equality holds in (1.10) if and only if K and L have similar general L_p-brightness and $(D^{(\tau)}_{p,i}(M),D^{(\tau)}_{p,i}(K))=\lambda(D^{(\tau)}_{p,i}(N),D^{(\tau)}_{p,i}(L))$, where λ is a constant.

Similar to the above method, if $i>n-p$ and $i\neq n$, by (1.3), (1.7) and (3.2), we can prove the inequality (1.11) is true.

Obviously, let $p=1$ and $\tau=0$ in Theorem 1.1, the following result is obtained.

Corollary 3.1. *Let $M,N,K,L\in\mathcal{K}^n_{os}$, $i\in\mathbf{R}$, $\delta(K,\cdot)<\delta(M,\cdot)$, $\delta(L,\cdot)<\delta(N,\cdot)$, M and N have similar brightness, if $i<n-1$, then*

$$\begin{aligned}&[D_i(M\mp N)-D_i(K\mp L)]^{\frac{1}{n-i}}\\ &\geq [D_i(M)-D_i(K)]^{\frac{1}{n-i}}+[D_i(N)-D_i(L)]^{\frac{1}{n-i}};\end{aligned}$$

if $i>n-1$ and $i\neq n$, then

$$\begin{aligned}&[D_i(M\mp N)-D_i(K\mp L)]^{\frac{1}{n-i}}\\ &\leq [D_i(M)-D_i(K)]^{\frac{1}{n-i}}+[D_i(N)-D_i(L)]^{\frac{1}{n-i}},\end{aligned}$$

in each case, with equality if and only if K and L have similar brightness and exists constant λ such that $(D_i(M),D_i(K))=\lambda(D_i(N),D_i(L))$.

Subsequently, we will give the proof of Theorem 1.2, the following lemma is necessary.

Lemma 3.2 (Popviciu's inequality [17]). *Let $p>0$, $q>0$, $\frac{1}{p}+\frac{1}{q}=1$ and $\mathbf{a}=\{a_1,\ldots,a_n\}$ and $\mathbf{b}=\{b_1,\ldots,b_n\}$ be two series of positive real numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$, $b_1^q-\sum_{i=2}^n b_i^q>0$, then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leq a_1b_1-\sum_{i=2}^n a_ib_i, \quad (3.5)$$

with equality if and only if $\mathbf{a}=c\mathbf{b}$, where c is a constant.

Proof of Theorem 1.2. For $M,N,K,L\in\mathcal{K}^n_o$, $\tau\in[-1,1]$, $p\geq 1$, $i,j,k\in\mathbf{R}$ and $0\leq i<j<k$, by (1.8), then

$$D^{(\tau)}_{p,j}(K,L)^{k-i} \leq D^{(\tau)}_{p,i}(K,L)^{k-j}D^{(\tau)}_{p,k}(K,L)^{j-i}, \quad (3.6)$$

with equality if and only if K and L have similar general L_p-brightness. Since M and N have similar general L_p-brightness, thus according to the equality condition of inequality (1.8), we have

$$D^{(\tau)}_{p,j}(M,N)^{k-i} = D^{(\tau)}_{p,i}(M,N)^{k-j}D^{(\tau)}_{p,k}(M,N)^{j-i}. \quad (3.7)$$

Hence, by (3.6) and (3.7), we get

$$\begin{aligned}&D^{(\tau)}_{p,j}(M,N)-D^{(\tau)}_{p,j}(K,L) \quad (3.8)\\ &\geq D^{(\tau)}_{p,i}(M,N)^{\frac{k-j}{k-i}}D^{(\tau)}_{p,k}(M,N)^{\frac{j-i}{k-i}}-D^{(\tau)}_{p,i}(K,L)^{\frac{k-j}{k-i}}D^{(\tau)}_{p,k}(K,L)^{\frac{j-i}{k-i}},\end{aligned}$$

with equality if and only if K and L have similar general L_p-brightness.

Notice that

$$\delta^{(\tau)}_p(K,\cdot)<\delta^{(\tau)}_p(M,\cdot),$$

and

$$\delta^{(\tau)}_p(L,\cdot)<\delta^{(\tau)}_p(N,\cdot),$$

thus, by (1.2), we obtain

$$D_{p,i}^{(\tau)}(M,N) > D_{p,i}^{(\tau)}(M,N), \quad D_{p,k}^{(\tau)}(M,N) \geq D_{p,k}^{(\tau)}(M,N).$$

From these, and notice that $\frac{k-i}{k-j} > 1$, $\frac{k-i}{j-i} > 1$ and $\frac{k-j}{k-i} + \frac{j-i}{k-i} = 1$, thus according to (3.5) and (3.8), we have

$$(D_{p,j}^{(\tau)}(M,N) - D_{p,j}^{(\tau)}(K,L))^{k-i}$$
$$\geq (D_{p,i}^{(\tau)}(M,N) - D_{p,i}^{(\tau)}(K,L))^{k-j}(D_{p,k}^{(\tau)}(M,N) - D_{p,k}^{(\tau)}(K,L))^{j-i}.$$

This is just (1.12).

By the equality conditions of inequalities (3.8) and (3.5), we see that equality holds in (1.12) if and only if K and L have similar general L_p-brightness and there exists constant λ such that $(D_{p,i}^{(\tau)}(M,N), D_{p,i}^{(\tau)}(K,L)) = \lambda(D_{p,k}^{(\tau)}(M,N), D_{p,k}^{(\tau)}(K,L))$.

Furthermore, let $p = 1$ and $\tau = 0$ in Theorem 1.2, the following result is obtained.

Corollary 3.2. *Let $M,N,K,L \in \mathcal{K}_o^n$, $\delta(K,\cdot) < \delta(M,\cdot)$, $\delta(L,\cdot) < \delta(N,\cdot)$, M and N have similar brightness, if $i,j,k \in \mathbf{R}$ and $0 \leq i < j < k$, then*

$$[D_j(M,N) - D_j(K,L)]^{k-i}$$
$$\geq [D_i(M,N) - D_i(K,L)]^{k-j}[D_k(M,N) - D_k(K,L)]^{j-i},$$

with equality if and only if K and L have similar brightness and exists constant λ such that $(D_i(M,N), D_i(K,L)) = \lambda(D_k(M,N), D_k(K,L))$.

In particular, if $N = L = B$ in Theorem 1.2, by (1.3) and (1.12), the following result is obvious.

Corollary 3.3. *Let $M,K \in \mathcal{K}_o^n$, $\tau \in [-1,1]$ and $\delta_p^{(\tau)}(K,\cdot) < \delta_p^{(\tau)}(M,\cdot)$, M have constant general L_p-brightness, if $p > 0$, $i,j,k \in \mathbf{R}$ and $0 \leq i < j < k$, then*

$$[D_{p,j}^{(\tau)}(M) - D_{p,j}^{(\tau)}(K)]^{k-i}$$
$$\geq [D_{p,i}^{(\tau)}(M) - D_{p,i}^{(\tau)}(K)]^{k-j}[D_{p,k}^{(\tau)}(M) - D_{p,k}^{(\tau)}(K)]^{j-i},$$

with equality if and only if K have constant general L_p-brightness.

Specially, if $i = 0$, $k = n$ in (1.12), by (1.5), we have the following cycle Minkowski type inequality for the differences of general L_p-mixed brightness integrals.

Corollary 3.4. *Let $M,N,K,L \in \mathcal{K}_o^n$, $\tau \in [-1,1]$, $\delta_p^{(\tau)}(K,\cdot) < \delta_p^{(\tau)}(M,\cdot)$, $\delta_p^{(\tau)}(L,\cdot) < \delta_p^{(\tau)}(N,\cdot)$, M and N have similar general L_p-brightness, if $p > 0$, $i \in \mathbf{R}$ and $0 < i < n$, then*

$$[D_{p,j}^{(\tau)}(N,M) - D_{p,j}^{(\tau)}(L,K)]^n$$
$$\geq [D_p^{(\tau)}(M) - D_p^{(\tau)}(K)]^{n-j}[D_p^{(\tau)}(N) - D_p^{(\tau)}(L)]^j,$$

with equality if and only if K and L have similar general L_p-brightness and exists constant λ such that $(D_p^{(\tau)}(M), D_p^{(\tau)}(K)) = \lambda(D_p^{(\tau)}(N), D_p^{(\tau)}(L))$.

Finally, according to the following Lemma, we give the proof of Theorem 1.3.

Lemma 3.3 ([7]). *If $c_i > 0,\ b_i > 0,\ c_i > b_i,\ i = 1, \cdots, n$, then*

$$\left(\prod_{i=1}^n (c_i - b_i)\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^n c_i\right)^{\frac{1}{n}} - \left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}}, \tag{3.9}$$

with equality if and only if $\frac{c_1}{b_1} = \frac{c_2}{b_2} = \cdots = \frac{c_n}{b_n}$.

Proof of Theorem 1.3. From (1.9), we have

$$D_p^{(\tau)}(N_1, \ldots, N_n)^m \tag{3.10}$$
$$\leq \prod_{i=1}^m D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-i+1}, \ldots, N_{n-i+1}),$$

with equality if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general L_p-brightness. Since the body $M_{n-m+1}, \ldots, M_n$ are all of similar general L_p-brightness, by (1.9), then

$$B_p^{(\tau)}(M_1, \ldots, M_n)^m \tag{3.11}$$
$$= \prod_{i=1}^m B_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-i+1}).$$

Notice that if $\delta_p^{(\tau)}(N_i,\cdot) < \delta_p^{(\tau)}(M_i,\cdot)$ $(i = 1, \ldots, n)$, by (1.1), we get

$$D_p^{(\tau)}(M_1, \ldots, M_n) > D_p^{(\tau)}(N_1, \ldots, N_n). \tag{3.12}$$

Taking $M_{n-m+1} = \cdots = M_n = M_{n-i+1}$, $N_{n-m+1} = \cdots = N_n = N_{n-i+1}$ in (3.12), we obtain

$$D_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-i+1}) \tag{3.13}$$
$$> D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-i+1}, \ldots, N_{n-i+1}).$$

From (3.10), (3.11) and (3.12), we obtain

$$D_p^{(\tau)}(M_1, \ldots, M_n) - D_p^{(\tau)}(N_1, \ldots, N_n) \tag{3.14}$$
$$\geq \left(\prod_{i=1}^m D_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-i+1})\right)^{\frac{1}{m}}$$
$$-\left(\prod_{i=1}^m D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-i+1}, \ldots, N_{n-i+1})\right)^{\frac{1}{m}}.$$

equality holds if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general L_p-brightness.

By (3.14), (3.13) and (3.9), we obtain

$$[D_p^{(\tau)}(M_1, \ldots, M_n) - D_p^{(\tau)}(N_1, \ldots, N_n)]^m$$
$$\geq \prod_{i=1}^m [D_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-i+1})$$
$$-D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-i+1}, \ldots, N_{n-i+1})].$$

By the equality conditions of inequalities (3.14) and (3.9), we see that equality holds in (1.13) if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general L_p-brightness and there exists constant λ such that

$$[D_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_n, \ldots, M_n), \ldots, D_p^{(\tau)}(M_1, \ldots, M_{n-m}, M_{n-m+1}, \ldots, M_{n-m+1})] = \lambda[D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_n), \ldots, D_p^{(\tau)}(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_{n-m+1})].$$

Obviously, let $p = 1$ and $\tau = 0$ in Theorem 1.3, the following result is obtained.

Corollary 3.5. *Let $M_i, N_i \in \mathcal{K}_o^n$ $(i = 1, \ldots, n)$, if $\delta(N_i,\cdot) < \delta(D_i,\cdot)$ and the bodies $M_{n-m+1}, \ldots, M_n$ are all of similar brightness, then for every $1 < m \leq n$,*

$$[D(M_1, \ldots, M_n) - D(N_1, \ldots, N_n)]^m$$

$$\geq \prod_{i=1}^{m}[D(M_1,\ldots,M_{n-m},M_{n-i+1},\ldots,M_{n-i+1}) \\ -D(N_1,\ldots,N_{n-m},N_{n-i+1},\ldots,N_{n-i+1})],$$

with equality if and only if $N_{n-m+1},\ldots,N_n$ are all of similar brightness and there exists constant λ such that

$$[D(M_1,\ldots,M_{n-m},M_n,\ldots,M_n),\ldots,D(M_1,\ldots,M_{n-m}, M_{n-m+1},\ldots,M_{n-m+1})]=\lambda[D(N_1,\ldots,N_{n-m},N_{n-m+1}, \ldots,N_n),\ldots,D(N_1,\ldots,N_{n-m},N_{n-m+1},\ldots,N_{n-m+1})].$$

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