Bernoulli Polynomials Collocation Method for Multi-term Fractional Differential Equations

Xumei Chen, Siyi Lei, Linjun Wang

***Abstract*—In this paper, the numerical solutions of multi-term fractional differential equations are studied by Bernoulli polynomials collocation method. By using the operational matrix and collocation method, the equations are simplified to a system of algebraic equations with unknown Bernoulli coefficients. Detailed error analysis is also given. Numerical examples are used to verify the efficiency and accuracy of the approach. According to the numerical results, the proposed method can be used as an alternative to obtaining the numerical solutions of this kind of multi-term fractional differential equations.**

***Index Terms*—Multi-term fractional differential equations, Bernoulli polynomials, Operational matrix, Collocation method.**

I. Introduction

FRACTIONAL calculus is a topic with a long history. Due to the lack of applications, it has not received much attention for a long time. With the development of the science and technology, the theory of fractional calculus and its applications have begun to attract the attention of many researchers from different scientific and engineering fields. One of the important applications of fractional calculus is fractional differential equations (FDEs). FDEs can describe non-local models more accurately. For example, FDEs are used in physics [1], heat conduction problems [2], diffusion problems [3], viscoelasticity [4], fluid mechanics [5], electromagnetic waves [6], bioengineering [7] and other fields [8]. Therefore, the study of FDEs has become a hot research topic.

It is not an easy task to get the exact solutions of FDEs. Many scholars have developed numerical methods for approximate solutions of FDEs. In recent years, a number of numerical methods have been proposed. For example, there are finite difference method [9], [10], polynomial method [11], finite element method [12], spectral method [13], wavelet collocation method [14], hybrid collocation method [15], etc.

This paper considers the numerical method of multi-term FDEs, which have the general form:

$$D^{\alpha}y(x) = f\left(x, y(x), D^{\alpha_1}y(x), \ldots, D^{\alpha_M}y(x)\right), x \in [0,1], \quad (1)$$

where $\alpha > \alpha_M > \ldots > \alpha_1 > 0$, $\alpha_i - \alpha_{i-1} \le 1$, $M \in \mathrm{N}^+$ and $\alpha_i \in \mathrm{Q}$ for all i, subjected to the initial conditions

$$y^{(k)}(0) = d_k, \quad k = 0, 1, \ldots, \lceil \alpha \rceil - 1. \quad (2)$$

Here, $\lceil \alpha \rceil$ is used to denote the integer closest to and less than α.

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To date many researches have developed approximate solutions of multi-term FDEs. Shiralashetti et al. [14] applied Haar wavelet collocation method (HWCM) to obtain the numerical solutions of (1). Talaei and Asgari [16] employed Chelyshkov polynomials to solve this type of equations. Mohammadi et al. [17] used Legendre wavelet Tau method to gain the approximate solutions of multi-term FDEs.

Bernoulli polynomials collocation method is a powerful tool for numerically solving integral, differential and integro-differential equations. Recently, Bernoulli polynomials collocation method has been extended to solve nonlinear differential equations [18], stochastic integral equations [19], 1D and 2D fractional optimal control of system [20] and so on. As pointed out in these references, Bernoulli polynomials collocation method can provide accurate approximations of the problem with simple computational procedures.

The purpose of this paper is to solve the multi-term FDEs based on Bernoulli polynomials collocation method. By using Bernoulli operational matrix together with a collocation method, Eqs. (1) and (2) are simplified to a systems of algebraic equations and numerical solutions are obtained by solving the system. We also show error analysis of the proposed method.

The paper is organized as follows. In Section II, the basic definitions of fractional calculus, properties of Bernoulli polynomials and the related Bernoulli operational matrix are given. In Section III, detailed method to solve multi-term FDEs is discussed. An error analysis is investigated in Section IV. Section V provides several examples to illustrate the efficiency and accuracy of the proposed method. Section VI concludes.

II. Preliminary

We first give some necessary definitions and mathematical preliminaries of fractional calculus which will be used further in this paper.

Definition 1. [11] The Riemann-Liouville fractional integral operator J^{α} of order α is given by:

$$J^{\alpha}y(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-s)^{\alpha-1}y(s)ds, & \alpha > 0, \\ y(x), & \alpha = 0, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx.$$

We state the properties of J^{α} as follows [21]:

$$J^{\alpha_1}J^{\alpha_2}y(x) = J^{\alpha_1+\alpha_2}y(x), \quad (3)$$

$$J^{\alpha_1}J^{\alpha_2}y(x) = J^{\alpha_2}J^{\alpha_1}y(x). \quad (4)$$

Definition 2. [11] The Caputo definition of fractional differential operator D^{α} of order α is given by:

$$D^{\alpha}y(x)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^x (x-s)^{n-\alpha-1}\frac{d^n}{ds^n}y(s)ds,\\ n-1<\alpha<n,\\ y^{(n)}(x), \qquad\qquad \alpha=n.\end{cases}$$

The important relationship between J^{α} and D^{α} is [21]

$$J^{\alpha}D^{\alpha}y(x)=y(x)-\sum_{i=0}^{n-1}y^{(i)}(0^{+})\frac{x^i}{i!}, n-1<\alpha<n, n\in \mathrm{N}^{+}. \tag{5}$$

Next, we provide the definition and some properties of Bernoulli polynomials. Bernoulli polynomials $B_n(x)$ of degree n are generated by the following relation [22]

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n, \quad n=0,1,\cdots. \tag{6}$$

We show the first five Bernoulli polynomials as follows

$$\begin{aligned}B_0(x)&=1,\\ B_1(x)&=x-\frac{1}{2},\\ B_2(x)&=x^2-x+\frac{1}{6},\\ B_3(x)&=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\\ B_4(x)&=x^4-2x^3+x^2-\frac{1}{30}.\end{aligned}$$

We collect the following properties of Bernoulli polynomials [23]

$$\begin{aligned}&B_n^{'}(x)=nB_{n-1}(x), \quad n\ge 1,\\ &\int_0^1 B_n(x)dx=0, \quad n\ge 1,\\ &B_n(x+1)-B_n(x)=nx^{n-1}, \quad n\ge 1,\\ &B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k(0)x^{n-k}, \quad n\ge 1.\end{aligned}$$

The Bernoulli vector is defined as

$$B(x)=[B_0(x),B_1(x),\ldots,B_N(x)]^T. \tag{7}$$

Using (6), we can rewrite vector (7) in the following form

$$B(x)=D^{-1}T_N(x), \tag{8}$$

where

$$D=\begin{pmatrix}\binom{1}{0} & 0 & 0 & \cdots & 0\\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 0 & \cdots & 0\\ \frac{1}{3}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{3}\binom{3}{2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N+1}\binom{N+1}{0} & \frac{1}{N+1}\binom{N+1}{1} & \frac{1}{N+1}\binom{N+1}{2} & \cdots & \frac{1}{N+1}\binom{N+1}{N}\end{pmatrix},$$

and

$$T_N(x)=[1,x,x^2,\ldots,x^N]^T. \tag{9}$$

Furthermore, by using the properties of Bernoulli polynomials, if n increase from 0 to N, we get

$$B(x)=\widehat{D}T_N(x), \tag{10}$$

where

$$\widehat{D}=\begin{pmatrix}\binom{0}{0}B_0(0) & 0 & 0 & \cdots & 0\\ \binom{1}{1}B_1(0) & \binom{1}{0}B_0(0) & 0 & \cdots & 0\\ \binom{2}{2}B_2(0) & \binom{2}{1}B_1(0) & \binom{2}{0}B_0(0) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \binom{N}{N}B_N(0) & \binom{N}{N-1}B_{N-1}(0) & \binom{N}{N-2}B_{N-2}(0) & \cdots & \binom{N}{0}B_0(0)\end{pmatrix}.$$

Note that $\widehat{D}=D^{-1}$. By the result in [24], the dual matrix of $B(x)$ is denoted as

$$Q=\int_0^1 B(x)B^T(x)dx=\hat{D}W\hat{D}^T, \tag{11}$$

where

$$\begin{aligned}W&=\int_0^1 T_N(x)T_N^T(x)dx\\ &=\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N+3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1}\end{pmatrix}.\end{aligned}$$

We use $H=L^2([0,1])$ to denote the space of square integrable function with respect to Lebesgue measure on the closed interval $[0,1]$. According to [25], the arbitrary function $y(x)$ in H can be expressed by Bernoulli basis as

$$y(x)\simeq\sum_{k=0}^{N}y_kB_k(x)=B^T(x)Y, \tag{12}$$

where the coefficient vector Y is given by

$$Y=[y_0,y_1,\ldots,y_N]^T.$$

We present results of Bernoulli coefficients y_n as follows.

Lemma 1. [26] Assume that $y(x)\in H$ be an arbitrary function and also is approximated by the truncated Bernoulli series $P_N[y](x)=\sum_{n=0}^{N}y_nB_n(x)$, then the coefficients $y_0,y_1,\ldots,y_N$, can be calculated from the following relation

$$y_n=\frac{1}{n!}\int_0^1 y^{(n)}(x)dx.$$

Lemma 2. [26] Assume that the function $y(x)$ is approximated on the interval $[0,1]$ by Bernoulli polynomials as argued in Lemma 1. Then the coefficients y_n decay as follows

$$y_n\le\frac{Y_n}{n!},$$

where Y_n denotes the maximum of $y^{(n)}(x)$ in the interval $[0,1]$.

We represent the solution of (1) in the form of truncated Bernoulli series (12). It means that $y(x)$ can be written in the following form

$$\begin{aligned}&y(x)\simeq B^T(x)Y,\\ &Y=[y_0,y_1,\ldots,y_N]^T.\end{aligned}$$

By (10), we can write it as

$$y(x)\simeq T_N^T(x)\widehat{D}^TY. \tag{13}$$

Now, taking Caputo fractional derivative D^{α} to both side of Eq. (13), we have

$$D^{\alpha}y(x)\simeq T_N^{(\alpha)}(x)\widehat{D}^TY, \tag{14}$$

where

$$T_N^{(\alpha)}(x) = [D^\alpha 1, D^\alpha x, D^\alpha x^2, \ldots, D^\alpha x^N] = \left[\underbrace{0, 0, \ldots, 0}_{\lceil\alpha\rceil}, \frac{\Gamma(\lceil\alpha\rceil+1)x^{\lceil\alpha\rceil-\alpha}}{\Gamma(\lceil\alpha\rceil+1-\alpha)}, \ldots, \frac{\Gamma(N+1)x^{N-\alpha}}{\Gamma(N+1-\alpha)}\right].$$

At the end of this section, we state the product matrix of Bernoulli polynomials basis from [24]. The result is

$$B(x)B^T(x)Y \simeq \hat{Y}B(x), \quad (15)$$

where $\hat{Y} = \hat{D}\tilde{Y}^T$. For more details, we refer to [24].

III. Solutions of multi-term FDEs

In order to describe the Bernoulli polynomials collocation method in detail, we consider the following form of multi-term FDEs

$$D^\alpha y(x) + D^{\alpha_1}y(x) + D^{\alpha_2}y(x) + f(x)y^n(x) = g(x) \quad (16)$$

with the initial conditions

$$y^{(k)}(0) = d_k,\ k = 0, 1, \ldots, \lceil\alpha\rceil - 1, \quad (17)$$

where n is a positive integer and $f(x)$ is a function smooth enough on $[0, 1]$. By approximating function $y(x)$ and $f(x)$ in the form of Bernoulli polynomials, as described by (12), we have

$$y(x) \simeq B^T(x)Y, \quad (18)$$
$$f(x) \simeq B^T(x)F, \quad (19)$$

where the vectors Y and F are Bernoulli polynomial coefficients of $y(x)$ and $f(x)$, respectively.

Next, we deal with the term $y^n(x)$ in Eq. (16). By (18), (13) together with (15), we get

$$\begin{aligned} y^n(x) &\simeq Y^TB(x)B^T(x)Yy^{n-2}(x) \\ &\simeq Y^T\hat{Y}B(x)y^{n-2}(x) \\ &\simeq Y^T\hat{Y}B(x)B^T(x)Yy^{n-3}(x) \\ &\simeq Y^T\hat{Y}^2B(x)y^{n-3}(x) \\ &\cdots \\ &\simeq Y^T\hat{Y}^{n-1}B(x). \end{aligned} \quad (20)$$

Substituting (14) and (18)-(20) into Eq. (16) yields

$$\begin{aligned} &T_N^{(\alpha)}(x)\hat{D}^TY + T_N^{(\alpha_1)}(x)\hat{D}^TY + T_N^{(\alpha_2)}(x)\hat{D}^TY \\ &+ B^T(x)FY^T\hat{Y}^{n-1}B(x) = g(x). \end{aligned} \quad (21)$$

Now, for initial conditions (17), by (12) and (14), we have

$$y(0) = B^T(0)Y = d_0 \quad (22)$$
$$y^{(k)}(0) = T_N^{(k)}(0)\hat{D}^TY = d_k, k = 1, 2, \ldots, \lceil\alpha\rceil - 1. \quad (23)$$

To obtain more accurate solution, we adopt different strategies to select collocation points.

Case 1

When $n = 1$, we collocate (21) at the $N+1-\lceil\alpha\rceil$ Newton-Cotes nodes as

$$x_l = \frac{2l+1}{2(N-\lceil\alpha\rceil+1)}, \quad l = 0, 1, \ldots, N-\lceil\alpha\rceil,$$

then we get

$$\begin{aligned} &T_N^{(\alpha)}(x_l)\hat{D}^TY + T_N^{(\alpha_1)}(x_l)\hat{D}^TY + T_N^{(\alpha_2)}(x_l)\hat{D}^TY \\ &+ B^T(x_l)FY^T\hat{Y}^{n-1}B(x_l) = g(x_l), \\ &l = 0, 1, \ldots, N-\lceil\alpha\rceil. \end{aligned} \quad (24)$$

It is noted that (22)-(24) is a system of $N+1$ linear algebraic equations with $N+1$ unknown coefficients $y_0, y_1, \ldots, y_N$. After solving it with conventional numerical methods, we can acquire the numerical solution for Eq.(16) by (18).

Case 2

When $n > 1$, we choose the first $N+1-\lceil\alpha\rceil$ roots of shifted Legendre polynomial $P_{N+1}(x)$ as collocation points [27] for (21). Combined with the initial conditions (22) and (23), we can get $N+1$ nonlinear equations. After solving them using Newton's iterative method and putting the values of $y_0, y_1, \ldots, y_N$ into (18), we can obtain the solution of the given problem.

IV. Error analysis

In this section, the error analysis of the proposed method will be discussed. We assume that $\|f(x)\|_\infty = sup_{x\in[0,1]}|f(x)|$. It is noted that the set of Bernoulli polynomials $B_0(x), B_1(x), \ldots, B_N(x) \subset L^2[0,1]$. We suppose

$$Y = span\{B_0(x), B_1(x), \ldots, B_N(x)\},$$

and $h \in L^2[0,1]$ is an arbitrary element. Because Y is a finite dimensional subspace of $L^2[0,1]$, there exists a unique best approximation $\hat{h} \in Y$ for h such that for every $z \in Y$

$$\|h - \hat{h}\| \le \|h - z\|.$$

Moreover, we have

$$h \approx \hat{h} = \sum_{i=0}^{N} h_nB_n(x) = B^T(x)H,$$

where $H = [h_0, h_1, \ldots, h_N]^T$, and $h_0, h_1, \ldots, h_N$ are unique coefficients.

The following theorems will play an important role in our error analysis.

Theorem 1. [26] Suppose $h(x) \in C^\infty[0,1]$ and $P_N[h](x)$ is the approximate polynomial using Bernoulli polynomials. Then the error bound would be obtained as follows

$$E(h) = \|h(x) - P_N[h](x)\|_\infty \le \frac{1}{(N!)}B_NH_N,$$

where B_N and H_N denote the maximum values of $B_N(x)$ and $h^{(N)}(x)$ in the interval $[0,1]$, respectively.

Theorem 2. [21] Let $h : [0,1] \to R$ and $J^\alpha(\cdot)$ denotes the Riemann-Liouville's fractional integration operator. Then,

$$\|J^\alpha(h(x))\|_\infty \le \frac{1}{\Gamma(\alpha+1)}\|h(x)\|_\infty \quad (25)$$

Theorem 3. [28] Suppose $p \in L^2[0,1]$ is approximated by p_N as

$$p(x) \simeq p_N(x) = \sum_{i=0}^{N} p_iB_i(x) = P^TB(x),$$

where

$$B(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T,$$
$$P = [p_1, p_2, \ldots, p_N]^T.$$

Consider

$$L_N(p)=\int_0^1[p(x)-p_N(x)]^2dx,$$

then, we have

$$\lim_{N\to\infty}L_N(p)=0.$$

Next, we give the main results of error analysis.

Theorem 4. Suppose that $y(x)$ and $y_N(x)$ are the exact and numerical solutions of (16) with initial conditions (17). Moreover, we assume
(1) There exist positive numbers ρ_1, ρ_2, ρ_3, such that $\|y_N(x)\|_\infty\le\rho_1$, $\|f(x)\|_\infty\le\rho_2$, $\|y(x)\|_\infty\le\rho_3$,
(2) $1-\varphi-\frac{\rho_4E(f)}{\Gamma(\alpha+1)}>0$.
Then,

$$\|y(x)-y_N(x)\|_\infty\le\frac{\Gamma(\alpha+1)E(p)+\rho_3^nE(f)}{\Gamma(\alpha+1)-\Gamma(\alpha+1)\varphi-\rho_4E(f)},$$

where

$$\begin{aligned}&\varphi=\frac{1}{\Gamma(\alpha-\alpha_1+1)}+\frac{1}{\Gamma(\alpha-\alpha_2+1)}+\frac{\rho_4\rho_2}{\Gamma(\alpha+1)},\\&E(f)=\|error\,(f(x))\|_\infty=\|f(x)-f_N(x)\|_\infty,\\&E(p)=\|p(x)-p_N(x)\|_\infty,\\&\rho_4=\rho_3^{n-1}+\rho_3^{n-2}\rho_1+\cdots+\rho_3\rho_1^{n-2}+\rho_1^{n-1}.\end{aligned}$$

Here,

$$\begin{aligned}p(x)=&\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{y^{(k)}(0)}{k!}x^k\\&+\sum_{k=0}^{\lceil\alpha_1\rceil-1}\frac{y^{(k)}(0)}{\Gamma(k+\alpha-\alpha_1+1)}x^{\alpha-\alpha_1+k}\\&+\sum_{k=0}^{\lceil\alpha_2\rceil-1}\frac{y^{(k)}(0)}{\Gamma(k+\alpha-\alpha_2+1)}x^{\alpha-\alpha_2+k}+J^\alpha g(x).\end{aligned}\tag{26}$$

Proof: Applying the Riemann-Liouville fractional integral operator J^α to both sides of (16) and using the properties of J^α, we have

$$y(x)=p(x)-J^{\alpha-\alpha_1}y(x)-J^{\alpha-\alpha_2}y(x)-J^\alpha\left(f(x)y^n(x)\right),\tag{27}$$

where $p(x)$ is defined as in (26).

Now, suppose that $f(x)$ and $p(x)$ are expanded using Bernoulli polynomials, then the obtained solution is an approximated solution. Here, we want to deduce an upper bound for the associated error between $y(x)$ and $y_N(x)$. Thus, we get

$$\begin{aligned}&\|y(x)-y_N(x)\|_\infty\\&=\|\left(p(x)-p_N(x)\right)-J^{\alpha-\alpha_1}\left(y(x)-y_N(x)\right)\\&-J^{\alpha-\alpha_2}\left(y(x)-y_N(x)\right)\\&-J^\alpha\left(f(x)y^n(x)-f_N(x)y_N^n(x)\right)\|_\infty\\&\le\|p(x)-p_N(x)\|_\infty+\|J^{\alpha-\alpha_1}\left(y(x)-y_N(x)\right)\|_\infty\\&+\|J^{\alpha-\alpha_2}\left(y(x)-y_N(x)\right)\|_\infty\\&+\|J^\alpha\left(f(x)y^n(x)-f_N(x)y_N^n(x)\right)\|_\infty.\end{aligned}\tag{28}$$

By Theorem 3, we have that $E(p)$ decreases to zero as N increases. Besides, by using Theorem 2, it follows

$$\|J^{\alpha-\alpha_1}\left(y(x)-y_N(x)\right)\|_\infty\le\frac{\|y(x)-y_N(x)\|_\infty}{\Gamma(\alpha-\alpha_1+1)}$$

and

$$\|J^{\alpha-\alpha_2}\left(y(x)-y_N(x)\right)\|_\infty\le\frac{\|y(x)-y_N(x)\|_\infty}{\Gamma(\alpha-\alpha_2+1)}.$$

Furthermore, we have

$$\begin{aligned}&\|J^\alpha\left(f(x)y^n(x)-f_N(x)y_N^n(x)\right)\|_\infty\\&\le\|J^\alpha\left(f(x)\left(y^n(x)-y_N^n(x)\right)\right)\|_\infty\\&+\|J^\alpha\left(\left(f(x)-f_N(x)\right)\left(y^n(x)-y_N^n(x)\right)\right)\|_\infty\\&+\|J^\alpha\left(\left(f(x)-f_N(x)\right)\left(y^n(x)\right)\right)\|\end{aligned}$$

By Theorem 2, we can gain

$$\begin{aligned}&\|J^\alpha\left(f(x)y^n(x)-f_N(x)y_N^n(x)\right)\|_\infty\\&\le\frac{1}{\Gamma(\alpha+1)}\|f(x)\|_\infty\|y^n(x)-y_N^n(x)\|_\infty\\&+\frac{1}{\Gamma(\alpha+1)}\|f(x)-f_N(x)\|_\infty\|y^n(x)-y_N^n(x)\|_\infty\\&+\frac{1}{\Gamma(\alpha+1)}\|f(x)-f_N(x)\|_\infty\|y^n(x)\|_\infty\\&\le\frac{1}{\Gamma(\alpha+1)}\Big(\rho_2\|y^n(x)-y_N^n(x)\|_\infty\\&+E(f)\|y^n(x)-y_N^n(x)\|_\infty+\rho_3^nE(f)\Big).\end{aligned}\tag{29}$$

Noting assumptions and the fixed number n , we have

$$\begin{aligned}&\|y^n(x)-y_N^n(x)\|_\infty\\&=\|\left(y(x)-y_N(x)\right)\left(y^{n-1}(x)+y^{n-2}(x)y_N(x)\right.\\&\left.+\cdots+y(x)y_N^{n-2}(x)+y_N^{n-1}(x)\right)\|_\infty\\&\le\|y(x)-y_N(x)\|_\infty\left(\rho_3^{n-1}+\rho_3^{n-2}\rho_1+\cdots+\rho_1^{n-1}\right)\\&=\rho_4\|y(x)-y_N(x)\|_\infty.\end{aligned}\tag{30}$$

In the end, by (28), (29) and (30), we conclude

$$\|y(x)-y_N(x)\|_\infty\le\frac{\Gamma(\alpha+1)E(p)+\rho_3^nE(f)}{\Gamma(\alpha+1)-\Gamma(\alpha+1)\varphi-\rho_4E(f)},$$

and this completes the proof.

V. Numerical examples

In this section, several numerical examples are provided to demonstrate the efficiency and accuracy of the proposed method in Section III.

Example 1. Firstly, we consider a linear non-homogeneous FDE [14]:

$$D^2y(x)+D^{\frac{3}{4}}y(x)+y(x)=x^3+6x+\frac{8.533333333}{\Gamma(0.25)}x^{2.25}$$

with the initial conditions $y(0)=0$, $y'(0)=0$. The exact solution is $y(x)=x^3$.

As shown in Figure 1, the numerical solution obtained by the proposed method is very close to the exact solution when $N=5$. Table I shows the comparison of the results obtained here with the results obtained in [14] for Example 1. In Table II, the absolute error of Bernoulli polynomials collocation method in comparison with HWCM when N takes different values are given. Figure 2 plots the absolute error graph with $N=8$. Table I and Table II demonstrate that Bernoulli polynomials collocation method is able to achieve a higher accuracy than HWCM.

TABLE I
COMPARISON OF EXACT SOLUTION, PRESENT METHOD AND HWCM FOR EXAMPLE 1

$x(=1/16)$	HWCM $(N=8)$	Present method $(N=5)$	Exact solution	Absolute errors $\|E-H\|$	$\|E-P\|$
1	0.000720594592103	0.000244140625000	0.000244140625000	4.7645E-04	2.7657E-21
3	0.007946937149659	0.006591796875000	0.006591796875000	1.3551E-03	2.9108E-20
5	0.032632454777584	0.030517578125000	0.030517578125000	2.1148E-03	1.6347E-19
7	0.086482887501875	0.083740234375000	0.083740234375000	2.7426E-03	6.1592E-19
9	0.181211960646956	0.177978515625000	0.177978515625000	3.2334E-03	1.7185E-18
11	0.328538847506652	0.324951171875000	0.324951171875000	3.5877E-03	3.9167E-18
13	0.540186878967990	0.536376953125000	0.536376953125000	3.8100E-03	7.7603E-18
15	0.827882618666712	0.823974609375000	0.823974609375000	3.9080E-03	1.3895E-17

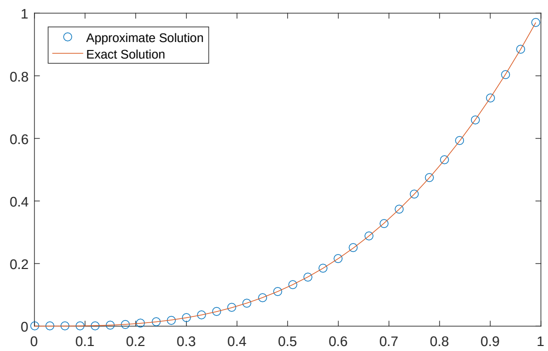

Fig. 1. Comparison of the numerical solution with exact solution for Example 1

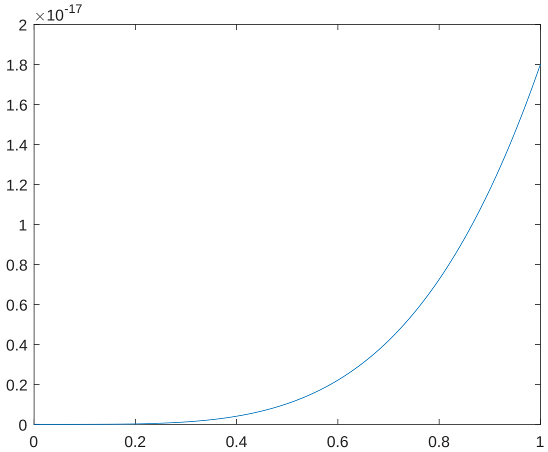

Fig. 2. Graph of absolute error for Example 1 with $N=8$

TABLE II
THE ABSOLUTE ERROR OF BERNOULLI POLYNOMIALS COLLOCATION METHOD IN COMPARISON WITH HWCM FOR EXAMPLE 1

N	HWCM	N	Present method
32	7.5976E-06	4	1.7989E-17
64	9.5202E-07	8	1.8014E-17
128	1.1912E-07	12	1.8012E-17
256	1.4896E-08	16	1.8012E-17

Example 2. We consider another linear non-homogeneous FDE:

$$D^2y(x)+D^{\frac{1}{2}}y(x)+e^xy(x)=e^xx^3+6x+\frac{3.2}{\Gamma(0.5)}x^{2.5},$$

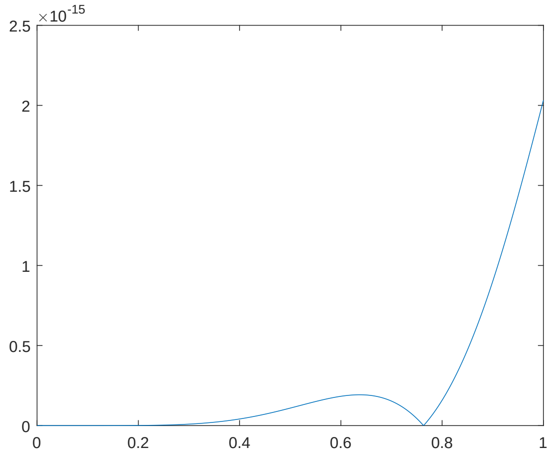

Fig. 3. Graph of absolute error for Example 2 with $N=16$

with initial conditions $y(0)=0$, $y'(0)=0$. The exact solution of this problem is $y(x)=x^3$.

When $N=5$, the numerical solution and exact solution are shown in Table III. Figure 3 plots the absolute error graph with $N=16$. Moreover, we also employ our proposed method with different N. Table IV shows the absolute error of Bernoulli polynomials collocation method with different N.

TABLE III
COMPARISON OF EXACT SOLUTION AND PRESENT SOLUTION WITH $N=5$ FOR EXAMPLE 2

$x(=1/16)$	Present method	Exact solution	Absolute error
1	0.0002441158117	0.0002441406250	0.0000000248133
3	0.0065916736303	0.0065917968750	0.0000001232447
5	0.0305173949514	0.0305175781250	0.0000001831735
7	0.0837400135552	0.0837402343750	0.0000002208198
9	0.1779781894722	0.1779785156250	0.0000003261528
11	0.3249506054657	0.3249511718750	0.0000005664093
13	0.5363760635138	0.5363769531250	0.0000008896112
15	0.8239735812919	0.8239746093750	0.0000010280831

Example 3. Consider a multi-term nonlinear non-

TABLE IV
ABSOLUTE ERROR OF EXAMPLE 2 FOR DIFFERENT N

$x(=1/16)$	$N=4$	$N=6$	$N=8$	$N=10$	$N=16$	$N=32$
1	1.1659E-07	3.0990E-09	1.8867E-11	5.0612E-14	3.8362E-22	1.0317E-21
3	6.9079E-07	1.3286E-08	6.3229E-11	1.9364E-13	6.0475E-19	1.2834E-19
5	1.1458E-06	2.0166E-08	7.0034E-11	9.6096E-13	1.1572E-17	5.0023E-19
7	1.1672E-06	3.0039E-08	2.3125E-11	4.2040E-12	6.2950E-17	1.3674E-18
9	8.6815E-07	4.4408E-08	2.3529E-10	1.0237E-11	1.5982E-16	3.3626E-18
11	7.8933E-07	4.7792E-08	2.3906E-10	1.0829E-11	1.6826E-16	7.9041E-18
13	1.8992E-06	1.1124E-08	7.2159E-10	1.3357E-11	2.2615E-16	1.7891E-17
15	5.5937E-06	8.9265E-08	3.3179E-09	7.8885E-11	1.2925E-15	3.8803E-17

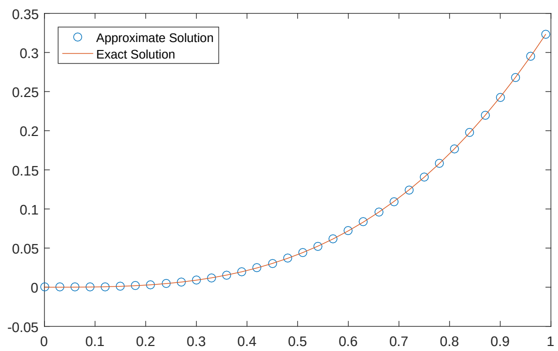

Fig. 4. Comparison of Bernoulli polynomials collocation method solution with exact solution for N=4 for Example 3

homogeneous FDE [14]:

$$\begin{aligned} & D^2y(x)+2D^{0.07621}y(x)+\frac{1}{2}D^{0.00196}y(x)+y^3(x) \\ = \; & 2x+\frac{4}{\Gamma(4-0.07621)}x^{3-0.07621} \\ & +\frac{1}{\Gamma(4-0.00196)}x^{3-0.00196}+\frac{1}{27}x^9, \end{aligned}$$

with the initial conditions $y(0)=0$, $y'(0)=0$, and the exact solution is $y(x)=\frac{x^3}{3}$.

Table V illustrates the comparison results for Example 3. It indicates that the present method $(N=4)$ gives a more accurate solution than HWCM $(N=8)$. From Figure 4, it is easy to see the closeness of the numerical solution and exact solution. The graph of absolute error for numerical solution is plotted in Figure 5 with $N=8$. Table VI shows the absolute error of Bernoulli polynomials collocation method in comparison with HWCM when N takes different values. From Table VI, the results show that the present method can achieve higher accuracy than HWCM. It means that HWCM requires more number of bases to achieve the same accuracy as the present method.

TABLE VI
THE ABSOLUTE ERROR OF BERNOULLI POLYNOMIALS COLLOCATION METHOD IN COMPARISON WITH HWCM FOR EXAMPLE 3

N	HWCM	N	Present method
16	3.9493E-04	3	1.1295E-04
32	9.8846E-05	5	5.8532E-06
64	2.4724E-05	7	8.9226E-08
128	6.1822E-05	9	2.7649E-14

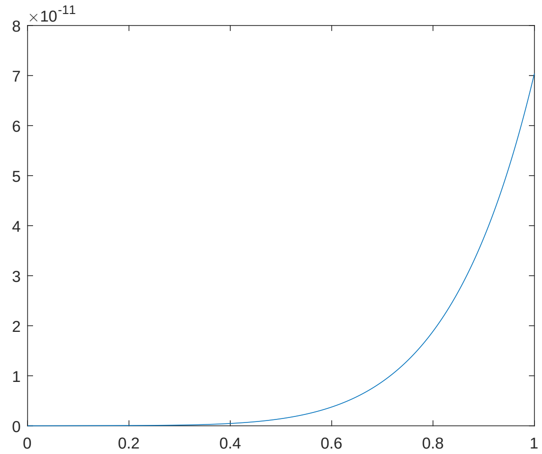

Fig. 5. Graph of absolute error for Example 3 with $N=8$

Example 4. Finally, we consider a multi-term nonlinear higher-order non-homogeneous FDE:

$$\begin{aligned} & D^{2.2}y(x)+D^{1.25}y(x)+D^{0.75}y(x)+sinxy^3(x) \\ = \; & \frac{2}{\Gamma(1.8)}x^{0.8}+\frac{2}{\Gamma(4-1.25)}x^{3-1.25} \\ & +\frac{2}{\Gamma(4-0.75)}x^{3-0.75}+sinx\left(\frac{x^3}{3}\right)^3, \end{aligned}$$

subjected to $y(0)=0$, $y'(0)=0$, $y''(0)=0$. The exact solution is $y(x)=\frac{x^3}{3}$.

The numerical solution and the absolute error are shown in Table VII. The graph of absolute error for numerical solution is plotted in Figure 6. In Table VIII, we give the absolute error of Bernoulli polynomials collocation method when N takes different values. From these figures and tables, we can see that as N increases, the absolute error decreases monotonically.

VI. CONCLUSIONS

This paper proposes Bernoulli polynomials collocation method for numerical solution of multi-term FDEs. The operational matrix and collocation method are used to simplify the problem to a system of algebraic equations. Computational results show that the proposed method is efficient and accurate. Satisfactory results can be obtained only with a few steps. In sum, Bernoulli polynomials collocation method can be used as a alternative to obtaining the numerical solutions of multi-term FDEs.

TABLE V
COMPARISON OF EXACT SOLUTION, PRESENT METHOD AND HWCM FOR EXAMPLE 3

$x(=1/16)$	HWCM $(N=8)$	Present method $(N=4)$	Exact solution	Absolute errors $\|E-H\|$	$\|E-P\|$
1	0.000243	0.000082	0.000081	0.000161	0.000001
3	0.002674	0.002205	0.002197	0.000477	0.000008
5	0.010945	0.010191	0.010172	0.000720	0.000019
7	0.028947	0.027940	0.027913	0.001034	0.000027
9	0.060578	0.059359	0.059326	0.001252	0.000033
11	0.109736	0.108353	0.108317	0.001419	0.000035
13	0.180320	0.178831	0.178792	0.001527	0.000039
15	0.276231	0.274701	0.274658	0.001573	0.000043

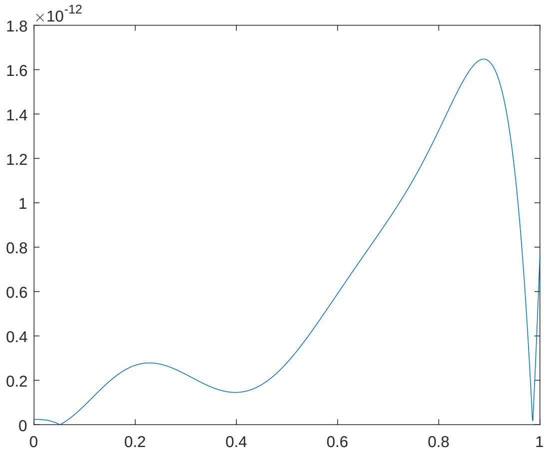

Fig. 6. Graph of absolute error for Example 4 with N=9

TABLE VII
COMPARISON OF EXACT SOLUTION AND PRESENT METHOD FOR EXAMPLE 4

$x(=1/16)$	Present method $(N=4)$	Exact solution	Absolute errors
1	0.000081	0.000081	0.000000
3	0.002197	0.002197	0.000000
5	0.010173	0.010173	0.000000
7	0.027914	0.027913	0.000001
9	0.059327	0.059326	0.000001
11	0.108317	0.108317	0.000000
13	0.178790	0.178792	0.000002
15	0.274651	0.274658	0.000007

TABLE VIII
THE ABSOLUTE ERROR OF BERNOULLI POLYNOMIALS COLLOCATION METHOD FOR EXAMPLE 4

x	$N=3$	$N=5$	$N=7$	$N=9$
0.1	2.9280E-08	1.1496E-09	1.1163E-10	8.7232E-14
0.2	2.3424E-07	1.1994E-08	3.4511E-10	2.6766E-13
0.3	7.9056E-07	4.7895E-08	3.6799E-10	2.2659E-13
0.4	1.8739E-06	1.2630E-07	3.3912E-10	1.4549E-13
0.5	3.6600E-06	2.6225E-07	6.5959E-10	2.7928E-13
0.6	6.3245E-06	4.6385E-07	1.2181E-09	5.9058E-13
0.7	1.0043E-05	7.2782E-07	1.1694E-09	9.2248E-13
0.8	1.4991E-05	1.0349E-06	2.8565E-10	1.3262E-12
0.9	2.1345E-05	1.3455E-06	2.9176E-09	1.6363E-12

REFERENCES

[1] Alkahtani and S. T. Badr, "Atangana–Batogna numerical scheme applied on a linear and non–linear fractional differential equation," *European Physical Journal Plus*, vol. 133, no. 3, p. 111, 2018.

[2] Y. Zhang, "Similarity solutions and the computation formulas of a nonlinear fractional–order generalized heat equation," *Modern Physics Letters B*, 2019.

[3] S. Arshad, W. Bu, J. Huang, Y. Tang, and Y. Zhao, "Finite difference method for time–space linear and nonlinear fractional diffusion equations," *International Journal of Computer Mathematics*, vol. 95, no. 1, pp. 202–217, 2018.

[4] Z. Avazzadeh, M. H. Heydari, and C. Cattani, "Legendre wavelets for fractional partial integro–differential viscoelastic equations with weakly singular kernels," *European Physical Journal Plus*, vol. 134, no. 7, 2019.

[5] S. Guo, L. Mei, Y. Li, and Y. Sun, "The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics," *Physics Letters A*, vol. 376, no. 4, pp. 407–411, 2012.

[6] V. E. Tarasov, "Fractional integro-differential equations for electromagnetic waves in dielectric media," *Theoretical and Mathematical Physics*, vol. 158, no. 3, pp. 355–359, 2009.

[7] M. Jiang and S. Zhong, "Existence of extremal solutions for a nonlinear fractionalq-difference system," *Mediterranean Journal of Mathematics*, vol. 13, no. 1, pp. 279–299, 2016.

[8] L. Li, Z. Wei and Q. Huang, "A numerical method for solving fractional variational problems by the operational matrix based on Chelyshkov polynomials," *Engineering Letters*, vol. 28, no.2, pp. 486-491, 2020.

[9] M. Cui, "Compact finite difference method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 228, no. 20, pp. 7792–7804, 2009.

[10] Q. Feng, "Compact Difference Schemes for a Class of Space-time Fractional Differential Equations," *Engineering Letters*, vol. 27, no.2, pp. 269-277, 2019.

[11] S. Nemati, S. Sedaghat, and I. Mohammadi, "A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels," *Journal of Computational and Applied Mathematics*, vol. 308, pp. 231–242, 2016.

[12] M. Dehghan and M. Abbaszadeh, "A finite element method for the numerical solution of Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivatives," *Engineering with Computers*, vol. 33, no. 3, pp. 587–605, 2017.

[13] S. Javadi, A. Yousefi, and E. Moradi, "Convergence analysis of the chebyshev-legendre spectral method for a class of fredholm fractional integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 358, 2018.

[14] S. C. Shiralashetti and A. B. Deshi, "An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations," *Nonlinear dynamics*, vol. 83, no. 1-2, pp. 293–303, 2016.

[15] L. Wang, F. Wang and Y. Cao, "A hybrid collocation method for fractional initial value problems," *Journal of Integral Equations and Applications*, vol. 31, no. 1, pp. 105–129, 2019.

[16] Y. Talaei and M. Asgari, "An operational matrix based on Chelyshkov polynomials for solving multi-order fractional differential equations," *Neural Computing and Applications*, vol. 30, no. 5, pp. 1369–1376, 2018.

[17] F. Mohammadi and C. Cattani, "A generalized fractional-order Legendre wavelet Tau method for solving fractional differential equations," *Journal of Computational and Applied Mathematics*, vol. 339, pp. 306–316, 2018.

[18] D. Rani and V. Mishra, "Numerical inverse laplace transform based

on bernoulli polynomials operational matrix for solving nonlinear differential equations," *Results in Physics*, vol. 16, p. 102836, 2020.

[19] R. Zeghdane, "Numerical solution of stochastic integral equations by using Bernoulli operational matrix," *Mathematics and Computers in Simulation*, 2019.

[20] K. Rabiei, Y. Ordokhani, and E. Babolian, "Numerical solution of 1D and 2D fractional optimal control of system via Bernoulli polynomials," *International Journal of Applied and Computational Mathematics*, vol. 4, no. 1, p. 7, 2018.

[21] H. D. Azodi and M. R. Yaghouti, "Bernoulli polynomials collocation for weakly singular Volterra integro-differential equations of fractional order," *Filomat*, vol. 32, no. 10, pp. 3623–3635, 2018.

[22] D. H. Lehmer, "A new approach to Bernoulli polynomials," *The American mathematical monthly*, vol. 95, no. 10, pp. 905–911, 1988.

[23] F. A. Costabile and F. Dell'Accio, "Expansion over a rectangle of real functions in Bernoulli polynomials and applications," *BIT Numerical Mathematics*, vol. 41, no. 3, pp. 451–464, 2001.

[24] S. Bazm, "Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations." *Journal of Computational and Applied Mathematics 275*,vol.275, pp. 44–60, 2015.

[25] A. H. Bhrawy, E. Tohidi, and F. Soleymani, "A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 482–497, 2012.

[26] E. Tohidi, A. H. Bhrawy, and K. Erfani, "A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation," *Applied Mathematical Modelling*, vol. 37, no. 6, pp. 4283–4294, 2013.

[27] R. Belgacem, A. Bokhari, and A. Amir, "Bernoulli Operational Matrix of Fractional Derivative for Solution of Fractional Differential Equations," *General Letters in Mathematics*, vol. 5, pp. 32–46, 2018.

[28] P. Rahimkhani, Y. Ordokhani, and E. Babolian, "Numerical solution of fractional pantograph differential equations by using generalized fractional-order bernoulli wavelet," *Journal of Computational and Applied Mathematics*, vol. 309, pp. 493–510, 2017.