On Derivations of State Residuated Lattices

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***Abstract*—In this paper, we introduce the notion of derivations of state residuated lattices (L, τ) and discuss some properties of them. We study the related properties of strong derivations and regular derivations of state residuated lattices (L, τ). Moreover, we propose the notion of (strong) state-morphism residuated lattices and discuss some properties of them. Also, the principal ideal derivation is given and the adjoint of principal ideal derivation is obtained by a Galois connection, and we prove that the set of all principal ideal derivations on state-morphism residuated lattice (L, τ) can form a bounded distributive lattice. Further, a special kind of set $Ima_{(d,\tau)}(L)$ of a derivation on state residuated lattices (L, τ) is introduced and we get that $Ima_{(d,\tau)}(L)$ is a lattice ideal of L, when derivation d is a regular ideal derivation. In particular, if L is a linearly ordered residuated lattice, then $Ima_{(d,\tau)}(L)$ is a prime lattice ideal of L. Finally, by using the set $Ima_{(d,\tau)}(L)$ of principal ideal derivations, we give a characterization of a Heyting algebra.**

***Index Terms*—State residuated lattice; Derivation state residuated lattice; Principal ideal derivation; Heyting algebra**

I. INTRODUCTION

WITH the intent of measuring the average truth-value of propositions in Łukasiewicz logic, the notion of states on MV-algebras were introduced by Mundici [28], which is a generalization of probability measures on Boolean algebras. From then on, states on MV-algebras have been deeply investigated. In 2001, Dvurečenskij [8] investigated states on pseudo MV-algebras. In 2006, Kroupa [21] investigated states on semisimple MV-algebras and the author obtained that every state on semisimple MV-algebra is integral. As a result, the notion of states has been extended to other logical algebras, such as BL-algebras [35], MTL-algebras [23], [24], R_0-algebras [25], residuated lattices [6], [37] and their non-commutative cases and so on. Different approaches to the generalization of states mainly gave two different concepts, namely, Riečan states [35] and Bosbach states [11]. In 2008, Ciungu [6] proved that in any non-commutative residuated lattice, Bosbach states coincide with Riečan states, while the converse is not true. Therefore, the notion of Riečan states is the generalization of Bosbach states.

However, logical algebras with state operators are not universal algebras. To treat state operators in the universal algebraic framework, a new approach to state operators on MV-algebras was introduced by Flaminio and Montagna [9], [10], where they added a unary operation τ (called as an inner state operator) to the language of MV-algebras, which preserves the usual properties of state operators. The resulting algebraic structures were so-called state MV-algebras. Moreover, Flaminio and Montagna [9], [10] presented an algebraizable logic by using a probabilistic approach, and its equivalent algebraic semantics is precisely the variety of state MV-algebras. In [29], [30], states MV-algebras have been deeply investigated. Subsequently, the notion of internal states has also been extended to other algebraic structures. For example, the notion of a state BL-algebra was introduced by Ciungu [4] as an extension of a state MV-algebra. Also, the notion of internal states was extended by Dvurečenskij et al. [7] to Rl-monoids. It is well known that the class of MV-algebras, BL-algebras and Rl-monoids are proper subclass of the class of residuated lattices. As a application of state theories to residuated lattices, in 2015, He et al. [15] introduced the notion of state operators on residuated lattices and investigated some related properties of state operators. Moreover, in 2017, He et al. [14] investigated states and internal states on bounded semihoops.

The notion of derivations, introduced from analytic theory, is helpful to the research of structure and property in algebraic systems. In 1957, Posner [33] proposed the notion of derivations in a prime ring $(R, +, -)$, which is a mapping $d : R \longrightarrow R$ satisfying the two conditions: (i) $d(x+y) = d(x)+d(y)$; (ii) $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$ for all $x, y \in R$. Based on this, some authors further investigated several properties about derivations in other algebras. For example, in 2004, Jun and Xin [17] applied the notion of derivations to BCI-algebras. Based on [17], and as a generalization of the notion of derivations of BCI-algebras, Zhan and Liu [43] introduced the notion of f-derivations of BCI-algebras and some related properties were investigated. Using the idea of regular f-derivations, the authors gave characterizations of a p-semisimple BCI-algebra. Some researchers investigated the properties of derivations of lattices in [42]. In 2010, Alshehri [1] introduced the notion of derivations of MV-algebras, and some related properties are investigated. Using the notion of an isotone derivation, the author gave some characterizations of a derivation of an MV-algebra. In 2013, Torkzadeh and Abbasian [36] defined the notion of derivations of BL-algebras and discussed some related results. In particular, based on the notion of derivations of MV-algebras and BL-algebras [1], [36], in 2016, He et al. [16] introduced the notion of derivations in a residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, which is a mapping: $d : L \longrightarrow L$ satisfying the condition: $d(x \otimes y) = (d(x) \otimes y) \vee (x \otimes d(y))$ for all $x, y \in L$, and then the authors investigated the properties of derivations in residuated lattices and characterized some special types of residuated lattices in terms of derivations. Wang et al. [39] proposed derivations in hyperlatices and derived some basic properties of them. Also, some properties of differential hyperideals and differential hypercongruences are studied. Further, Xiao and Liu [41] introduced the notion of derivations for a quantale. In particular, Rachůnek and

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Šalounová [34] introduced the notion of derivations on non-commutative generalization of MV-algebras, and a complete description of all derivations on any non-commutative generalization of MV-algebras was given. In 2018, Liang et al. [22] introduced the notion of derivations on EQ-algebras and obtained several special types of them. Further, Wang et al. [38] introduced the notion of derivations of commutative multiplicative semilattices, they investigated related properties of some particular derivations and gave some characterizations of regular derivations in commutative multiplicative semilattices. In 2019, Maffeu et al. [26] extended the study of derivations on residuated lattices to residuated multilattices. Special types of derivations (implicative and multiplicative) and their connections with the complemented elements were investigated. In particular, one obtains that the good ideal derivations of a bounded residuated multilattice were completely determined by its complemented elements. Supporting examples of all the notions treated were also included. In 2020, Ciungu [5] defined two types of implicative derivations on pseudo-BCK algebras, he investigated their properties and gave a characterization of isotone implicative derivations. He also introduced and investigated the multiplicative derivations on BCK-algebras with product. At the same time, particular cases of multiplicative derivations were defined and their properties were investigated. Finally, he proved that there exists an order preserving bijection between the fixed points sets of the two operators. Kondo [20] considered some properties of multiplicative derivations and d-filters of commutative residuated lattices.

Based on the above reasons, it is meaningful to give a further discussion on this topic. The present paper aims at providing a framework to combine derivations, state operators and residuated lattices all together, which proposes the concept of derivations on state residuated lattices. In addition, we study some properties of them. It is worth noting that when we apply the derivation theory to state residuated lattices, on the one hand, we can find the impacts of derivations on state residuated lattices, on the other hand, it also reflects the characterizations and properties of derivations in state residuated lattices.

This paper is organized as follows. In Section II, we present some preliminary concepts and results related to residuated lattices, state operators and derivations, which will be used throughout this paper. In Section III, we introduce the notion of derivations on state residuated lattices and discuss some properties of them. In Section IV, we study principal ideal derivations and their adjoint derivations. Also, we discuss the algebraic structure of the set of all principal ideal derivations on a state-morphism residuated lattice. In particular, as an application of principal ideal derivations, we give a characterization of a Heyting algebra.

II. Preliminaries

In this section, we recall some fundamental concepts and definitions which shall be needed in the sequel. At first, we give a brief reminder of the definitions of residuated lattices.

Definition 2.1: [40] A residuated lattice is an algebraic structure $L = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) satisfying the following conditions:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(L, \otimes, 1)$ is a commutative monoid,

(3) $(\otimes, \rightarrow)$ forms an adjoint pair, i.e., $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$.

In what follows, we denote by L a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, unless otherwise specified.

For any $x \in L$ and a natural number n, we define $x' = x \rightarrow 0, x'' = (x')' = (x \rightarrow 0) \rightarrow 0, x^0 = 1$ and $x^n = x^{n-1} \otimes x$ for all $n \geq 1$.

In the following, we list some basic properties of residuated lattices.

Proposition 2.2: [40] For all $x, y, z, w \in L$, the following properties hold.

(1) $1 \rightarrow x = x$, $x \rightarrow 1 = 1$.

(2) $x \leq y$ if and only if $x \rightarrow y = 1$.

(3) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

(4) If $x \leq z$ and $y \leq w$ then $x \otimes y \leq z \otimes w$.

(5) $x \otimes y \leq x \wedge y$, $x \otimes x' = 0$.

(6) $x \rightarrow (y \rightarrow z) = x \otimes y \rightarrow z = y \rightarrow (x \rightarrow z)$.

(7) $0' = 1, 1' = 0, x \leq x'', x''' = x'$.

(8) $x \otimes y = 0$ if and only if $x \leq y'$.

(9) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

(10) $(x \vee y)' = x' \wedge y'$.

(11) $x \vee (y \otimes z) \geq (x \vee y) \otimes (x \vee z)$.

Definition 2.3: [13] L is called to be

(1) divisible if $x \wedge y = x \otimes (x \rightarrow y)$ for all $x, y \in L$;

(2) idempotent if $x \otimes x = x$ for all $x \in L$.

The notion of a state BL-algebra was introduced by Ciungu [4]. In 2015, as a generalization of the notion of a state BL-algebra, He et al. [15] introduced the notion of a state residuated lattice as follows.

Definition 2.4: [15] A mapping: $\tau : L \longrightarrow L$ is called a state operator on L if it satisfies the following conditions: for all $x, y \in L$,

$(SO1)$ $\tau(0) = 0$,

$(SO2)$ $x \rightarrow y = 1$ implies $\tau(x) \rightarrow \tau(y) = 1$,

$(SO3)$ $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \wedge y)$,

$(SO4)$ $\tau(x \otimes y) = \tau(x) \otimes \tau(x \rightarrow (x \otimes y))$,

$(SO5)$ $\tau(\tau(x) \otimes \tau(y)) = \tau(x) \otimes \tau(y)$,

$(SO6)$ $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$,

$(SO7)$ $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$,

$(SO8)$ $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$.

The pair (L, τ) is said to be a state residuated lattice, or more precisely, a residuated lattice with internal state.

Example 2.5: [15] Let $L = \{0, a, b, c, 1\}$ be a chain, where $0 < a < b < c < 1$. Define operations $\otimes$ and $\rightarrow$ as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then it is easy to verify that $L = \{0, a, b, c, 1\}$ is a residuated lattice. Now, we define a mapping $\tau : L \longrightarrow L$ as follows:

$$\tau(x) = \begin{cases} 0, & x = 0, \\ a, & x = a, b, \\ 1, & x = c, 1. \end{cases}$$

One can easily check that τ is a state operator on L. Therefore, (L, τ) is a state residuated lattice.

Next, we recall some properties of state operators on L.

Proposition 2.6: [15] Let (L,τ) be a state residuated lattice. Then, for all $x,y\in L$, the following properties hold.

(1) $\tau(1)=1$, $\tau(x')=\tau(x)'$, $\tau(\tau(x))=\tau(x)$.

(2) If $x\le y$, then $\tau(x)\le\tau(y)$.

(3) $\tau(x\otimes y)\ge\tau(x)\otimes\tau(y)$ and if $x\otimes y=0$, then $\tau(x\otimes y)=\tau(x)\otimes\tau(y)$.

(4) $\tau(x\to y)\le\tau(x)\to\tau(y)$. In particular, if x,y are comparable, then $\tau(x\to y)=\tau(x)\to\tau(y)$.

(5) $\tau(x\otimes y')\ge\tau(x)\otimes\tau(y)'$ and if $x\le y$, then $\tau(x\otimes y')=\tau(x)\otimes\tau(y)'$.

(6) $\tau(L)=\{x\in L|\tau(x)=x\}$ and $\tau(L)$ is a subalgebra of L.

In what follows, we recall the concept of Heyting algebras.

Theorem 2.7: [32] Let $(L,\vee,\wedge,\otimes,\to,0,1)$ be a residuated lattice. Then the following statements are equivalent:

(1) L is a Heyting algebra;

(2) $x\otimes y=x\wedge y=x\otimes(x\to y)$ for all $x,y\in L$.

In what follows, we denote by $B(L)$ the set of all complement elements of the lattice $(L,\wedge,\vee,0,1)$, see [2]. The set $B(L)$ is called the Boolean center of L. For any $t\in L$, $t\in B(L)$ if and only if $t\vee t'=1$ if and only if $t\otimes t=t$ and $t''=t$.

Proposition 2.8: [20] For any $t\in B(L)$ and $x\in L$, $t\otimes x=t\wedge x$.

A mapping $f:L\longrightarrow L$ is called a homomorphism if it satisfies the following conditions:

(1) $f(0)=0$, $f(1)=1$,

(2) $f(x*y)=f(x)*f(y)$, for all $x,y\in L$ and $*\in\{\vee,\wedge,\otimes,\to\}$.

A nonempty subset I of L is called a lattice ideal of L if it satisfies: (1) for all $x,y\in I, x\vee y\in I$; (2) for all $x,y\in L$, if $x\in I$ and $y\le x$, then $y\in I$, i.e., a lattice ideal of L is the notion of ideal in the underlying lattice $(L,\vee,\wedge)$.

A lattice ideal I is called prime if it satisfies for all $x,y\in L$, $x\wedge y\in I$ implies $x\in I$ or $y\in I$.

For a nonempty subset A of L, the smallest lattice ideal containing A is called the lattice ideal generated by A. The lattice ideal generated by A is denoted by $(A]$. In particular, if $A=\{a\}$, we write $(a]$ for $\{(a]\}$, $(a]$ is called a principal lattice ideal of L. It is easy to check that $(a]=\downarrow a=\{x\in L|x\le a\}$, see [12].

In 2008, Xin et al. [42] introduced the notion of a derivation on a lattice $(L,\wedge,\vee)$ as follows:

Definition 2.9: [42] Let L be a lattice. A mapping $d:L\longrightarrow L$ is called a derivation on L if it satisfies the following conditions: for any $x,y\in L$,

$$d(x\otimes y)=(d(x)\wedge y)\vee(x\wedge d(y)).$$

Based on [42], and as a generalization of derivation on a lattice, Çeven and Özturk [3] introduced the notion of an f-derivation on a lattice as follows:

Definition 2.10: [3] Let L be a lattice. A mapping $d:L\longrightarrow L$ is called an f-derivation on L if there exists a mapping $f:L\longrightarrow L$ such that

$$d(x\wedge y)=(d(x)\wedge f(y))\vee(f(x)\wedge d(y))$$

for all $x,y\in L$.

In 2016, He et al. [16] introduced the notion of derivations in a residuated lattice as follow:

Definition 2.11: [16] A mapping $d:L\longrightarrow L$ is called a derivation on L if it satisfies the following conditions: for any $x,y\in L$,

$$d(x\otimes y)=(d(x)\otimes y)\vee(x\otimes d(y)).$$

III. Some derivations based on state residuated lattices

In this section, first, we propose the concept of derivations in a state residuated lattice. And then, we discuss some properties of derivation state residuated lattices, and we also give some equivalent characterizations related to the isotone derivations in state residuated lattices (L,τ). Moreover, we introduce the notion of (weak) state-morphism residuated lattices and study some properties of them.

Definition 3.1: Let (L,τ) be a state residuated lattice and $d:L\longrightarrow L$ be a mapping. Then d is called a multiplicative derivation on (L,τ) if it satisfies the following conditions: for any $x,y\in L$,

$$d(x\otimes y)=(d(x)\otimes\tau(y))\vee(\tau(x)\otimes d(y)).$$

In what follows, unless otherwise stated, a multiplicative derivation on (L,τ) is called a derivation on (L,τ). The pair (L,τ,d) is said to be a derivation state residuated lattice.

Remark 3.2: It is obvious in Definition 3.1 that if d is a derivation on L, see Definition 2.11, then (L,id_L,d) is a derivation state residuated lattice.

Now, we show some examples for derivation state residuated lattice.

Example 3.3: Let (L,τ) be a state residuated lattice and define a mapping $d:L\longrightarrow L$ by $d(x)=0$ for all $x\in L$. One can check that d is a derivation on (L,τ), i.e., (L,τ,d) is a derivation state residuated lattice. Moreover, it is easy to see that (L,id_L,id_L) is also a derivation state residuated lattice.

Example 3.4: Let $L=\{0,a,b,1\}$ be a chain and operations $\otimes$ and $\to$ be defined as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to verify that $L=\{0,a,b,1\}$ is a residuated lattice. Now, we define a mapping $\tau:L\longrightarrow L$ as follows: for all $x\in L$,

$$\tau(x)=\begin{cases}0, & x=0,\\ a, & x=a,\\ 1, & x=b,1.\end{cases}$$

It is easy to see that τ is a state operator on L. Moreover, we define a mapping $d:L\longrightarrow L$ as follows: for all $x\in L$,

$$d(x)=\begin{cases}0, & x=0,a,\\ a, & x=b,1\ .\end{cases}$$

One can check that d is a derivation on (L,τ). Therefore, (L,τ,d) is a derivation state residuated lattice.

Now, we show some properties of derivation state residuated lattices.

Proposition 3.5: Let (L,τ,d) be a derivation state residuated lattice. Then the following statements hold.

(1) $d(0)=0$.

(2) $d(x) \leq \tau(x)''$ and $d(x') \leq \tau(x') \leq (d(x))'$ for all $x \in L$.

(3) $\tau(x) \otimes d(1) \leq d(x) \leq (\tau(x))''$ and $\tau(x) \otimes d(1) \otimes (d(x)') = 0$ for all $x \in L$.

(4) $d(x') \otimes d(y') \leq d(x' \otimes y') \leq d(x') \vee d(y') \leq (\tau(x))' \vee (\tau(y))'$ for all $x, y \in L$.

(5) $(d(x'))^n \leq d((x')^n)$ for all $x \in L, n \geq 1$.

(6) $d(x^n) = d(x) \otimes \tau(x^{n-1})$ for all $x \in L, n \geq 1$.

(7) $d(1) = 1$ if and only if $\tau(x) \leq d(x)$ if and only if $d(x') = \tau(x')$ if and only if $d(x') = (d(x))'$.

(8) If $\tau(x) = \tau(x'')$, then $d(x) \leq \tau(x)$.

(9) Let $A = \{x | x \in L, \tau(x) \in B(L)\}$. If $d(1) = 1$, then for all $x \in A$, $d(x) = \tau(x)$.

Proof. (1) It follows from Definition 3.1 that

$$\begin{aligned} d(0) &= d(0 \otimes 0) \\ &= (d(0) \otimes \tau(0)) \vee (\tau(0) \otimes d(0)) \\ &= d(0) \otimes \tau(0) \\ &= d(0) \otimes 0 \\ &= 0. \end{aligned}$$

(2) Let $x \in L$. Then it follows from Proposition 2.6 (5) that $x \otimes x' = 0$. Hence,

$$\begin{aligned} d(0) &= d(x \otimes x') \\ &= (d(x) \otimes \tau(x')) \vee (\tau(x) \otimes d(x')) \\ &= 0, \end{aligned}$$

which implies $d(x) \otimes \tau(x') = 0$ and $\tau(x) \otimes d(x') = 0$. It follows from Proposition 2.6 (1) and (8) that $d(x) \leq (\tau(x))''$ and $d(x') \leq \tau(x') \leq (d(x))'$.

(3) Let $x \in L$. Then it follows from Proposition 2.6 (1) that

$$\begin{aligned} d(x) &= d(x \otimes 1) \\ &= (d(x) \otimes \tau(1)) \vee (\tau(x) \otimes d(1)) \\ &= d(x) \vee (\tau(x) \otimes d(1)). \end{aligned}$$

Thus, $\tau(x) \otimes d(1) \leq d(x)$. Moreover, it follows from (2) and Proposition 2.6 (1) that $\tau(x) \otimes d(1) \leq d(x) \leq (\tau(x))''$ and $\tau(x) \otimes d(1) \otimes (d(x))' = 0$.

(4) Let $x, y \in L$. Then it follows from (2) that $d(x') \leq \tau(x')$, $d(y') \leq \tau(y')$. Then $d(x') \otimes d(y') \leq d(x') \otimes \tau(y')$, $d(x') \otimes d(y') \leq \tau(x') \otimes d(y')$. It follows from Definition 3.1 that

$$\begin{aligned} d(x') \otimes d(y') &\leq (d(x') \otimes \tau(y')) \vee (d(x') \otimes d(y')) \\ &= d(x' \otimes y') \\ &\leq d(x') \vee d(y') \\ &\leq (\tau(x))' \vee (\tau(y))'. \end{aligned}$$

(5) Let $x \in L$. Then it follows from (2) that $d(x') \otimes d(x') \leq d(x') \otimes \tau(x') = d(x' \otimes x')$ for all $x \in L$. By induction, we can obtain $(d(x'))^n \leq d((x')^n)$ for all $n \geq 1$.

(6) It follows from Definition 3.1 that $d(x \otimes x) = (d(x) \otimes \tau(x)) \vee (\tau(x) \otimes d(x)) = d(x) \otimes \tau(x)$ for all $x \in L$. If $n = 3$, then

$$\begin{aligned} d(x^3) &= (d(x) \otimes \tau(x^2) \vee (d(x^2) \otimes \tau(x))) \\ &= (d(x) \otimes \tau(x^2) \vee (d(x) \otimes \tau(x) \otimes \tau(x))). \end{aligned}$$

It follows from Proposition 2.6 (3) that $\tau(x^2) \geq (\tau(x))^2$, which implies $d(x^3) = d(x) \otimes \tau(x^2)$. By induction, we can obtain $d(x^n) = d(x) \otimes \tau(x^{n-1})$ for all $n \geq 1$.

(7) On the one hand, if $d(1) = 1$, then it follows from (3) that $d(x) = d(x) \vee \tau(x)$, which implies $\tau(x) \leq d(x)$. If $\tau(x) \leq d(x)$, then $\tau(x') \leq d(x')$. It follows from (2) that $d(x') \leq \tau(x')$, which implies that $d(x') = \tau(x')$. If $d(x') = \tau(x')$, then it is easy to know that $d(1) = 1$. On the other hand, if $\tau(x) \leq d(x)$, we have $d(x') = \tau(x')$. It follows from (2) that $\tau(x') \leq (d(x))' \leq (\tau(x))' = \tau(x')$, which implies $d(x') = (d(x))'$. If $d(x') = (d(x))'$, then it is easy to know that $d(1) = 1$.

(8) If $\tau(x) = \tau(x'')$, then it follows from Definition 3.1 that $d(x) \otimes \tau(x') \leq d(x \otimes x') = 0$, which implies $d(x) \leq (\tau(x'))' = \tau(x'') = \tau(x)$.

(9) On the one hand, if $d(1) = 1$, then it follows from (7) that $\tau(x) \leq d(x)$. On the other hand, if $x \in A$, then $\tau(x) \in B(L)$. It follows from (2) that $d(x) \leq (\tau(x))'' = \tau(x)$. Therefore, for all $x \in A$, $d(x) = \tau(x)$. □

Lemma 3.6: Let (L, τ) be a state residuated lattice. For all $x, y \in L$, if $\tau(x \to y) = \tau(x) \to \tau(y)$, then $\tau(x \otimes y) = \tau(x) \otimes \tau(y)$.

Proof.

$$\begin{aligned} \tau(x \otimes y) \to \tau(z) &= \tau(x \otimes y \to z) \\ &= \tau(x \to (y \to z)) \\ &= \tau(x) \to (\tau(y) \to (\tau(z))) \\ &= (\tau(x) \otimes \tau(y)) \to \tau(z). \end{aligned}$$

Hence, if $z = \tau(x) \otimes \tau(y)$, we have

$$\begin{aligned} &\tau(x \otimes y) \to \tau(\tau(x) \otimes \tau(y)) \\ &= (\tau(x) \otimes \tau(y)) \to \tau(\tau(x) \otimes \tau(y)) \\ &= \tau(x) \otimes \tau(y) \to \tau(x) \otimes \tau(y) \\ &= 1, \end{aligned}$$

which implies $\tau(x \otimes y) \to \tau(\tau(x) \otimes \tau(y)) = 1$. It follows from Proposition 2.2 (2) that $\tau(x \otimes y) \leq \tau(x) \otimes \tau(y)$. The converse inequality can be followed from Proposition 2.6 (3). Therefore, $\tau(x \otimes y) = \tau(x) \otimes \tau(y)$. □

In order to give better characterizations of derivations on state residuated lattices. Inspired by the notion of a homomorphism on a residuated lattice, firstly, we give the notion of a (weak) state-morphism residuated lattice as follows.

Definition 3.7: Let $\tau : L \longrightarrow L$ be a mapping.

(1) If it satisfies $(SO1)$, $(SO2)$, $(SO3)$, $(SO5)$, $(SO6)$, $(SO7)$, $(SO8)$ and $(SO9)$ $\tau(x \otimes y) = \tau(x) \otimes \tau(y)$, then (L, τ) is called a strong state-morphism residuated lattice.

(2) If it satisfies $(SO1)$, $(SO2)$, $(SO3)$, $(SO5)$, $(SO6)$, $(SO7)$, $(SO8)$ and $(SO10)$ $\tau(x * y) = \tau(x) * \tau(y)$, $* \in \{\vee, \wedge, \to\}$, then (L, τ) is called a state-morphism residuated lattice.

Example 3.8: Consider the state operator τ on L in Example 3.4. It is easy to verify that (L, τ) is a state-morphism residuated lattice.

Theorem 3.9: Let (L, τ) be a strong state-morphism residuated lattice, $d : L \longrightarrow L$ be a mapping, $d(0) = 0$ and $d(x) = t$, $x \in L - \{0\}$, $t \in L$.

(1) If L is non-trivial and $t \in B(L) - \{0, 1\}$, then d is not a derivation on (L, τ).

(2) If $t \notin B(L)$ and (L, τ, d) is a derivation state residuated lattice, then for any $x \in L - \{0\}$, $t \leq \tau(x)$ or $t \leq \tau(x)'$.

Proof. (1) Assume that d is a derivation on (L, τ). Since $t \otimes t' = 0$, we have

$$\begin{aligned} 0 &= d(t \otimes t') \\ &= (d(t) \otimes \tau(t')) \vee (\tau(t) \otimes d(t')) \\ &= (t \otimes \tau(t')) \vee (\tau(t) \otimes t) \\ &= (t \otimes \tau(t)') \vee (\tau(t) \otimes t) \\ &= t \otimes (\tau(t)' \vee \tau(t)). \end{aligned}$$

Since $t \in B(L)$ and (L, τ) is a weak state-morphism residuated lattice, we have $\tau(t)'' = \tau(t'') = \tau(t)$ and $\tau(t) = \tau(t \otimes t) = \tau(t) \otimes \tau(t)$. Thus $\tau(t) \in B(L)$. Hence, $\tau(t)' \vee \tau(t) = 1$, which implies $t = 0$, contradiction. Therefore, d is not a derivation on (L, τ).

(2) For any $x \in L - \{0\}$, it follows from Definition 3.1 that $d(x \otimes x) = d(x) \otimes \tau(x) = t \otimes \tau(x)$. Now we have two cases:

(i) If $x \otimes x \neq 0$, then $t = t \otimes \tau(x) \leq \tau(x)$.

(ii) If $x \otimes x = 0$, then $t = t \otimes \tau(x) = 0$ and thus $t \leq \tau(x)'$.

Therefore, $t \leq \tau(x)$ or $t \leq \tau(x)'$ for all $x \in L - \{0\}$. □

In what follows, we introduce ideal derivations in a state residuated lattice and investigate some related properties of them.

Definition 3.10: Let (L, τ) be a state residuated lattice and d be a derivation on (L, τ). Then for all $x, y \in L$,

(1) if $x \leq y$ implies $d(x) \leq d(y)$, we call d an isotone derivation on (L, τ),

(2) if $d(x) \leq \tau(x)$, we call d a strong derivation on (L, τ).

In particular, if d is both isotone and strong, we call d an ideal derivation on (L, τ).

Example 3.11: Consider the Example 3.4. One can check that d is an ideal derivation on (L, τ).

Now, some properties of isotone derivations and strong derivations on (L, τ) are investigated, respectively.

Proposition 3.12: Let (L, τ) be a state residuated lattice and d be an isotone derivation on (L, τ). Then the following statements hold.

(1) If $z \leq x \to y$, then $\tau(z) \leq d(x) \to d(y)$ and $\tau(x) \leq d(z) \to d(y)$ for all $x, y, z \in L$.

(2) $\tau(x \to y) \leq d(x) \to d(y)$ and $d(x \to y) \leq \tau(x) \to d(y)$ for all $x, y \in L$.

(3) $\tau(x) \leq d(y) \to d(x)$ and $\tau(y) \leq d(x) \to d(y)$ for all $x, y \in L$.

Proof. (1) Let $x, y, z \in L$ and $z \leq x \to y$. Then $x \otimes z \leq y$. Since d is an isotone derivation on (L, τ), we have $d(x \otimes z) \leq d(y)$. It follows from Definition 3.1 that $d(x \otimes z) = (d(x) \otimes \tau(z)) \vee (\tau(x) \otimes d(z))$. Thus, $(d(x) \otimes \tau(z)) \vee (\tau(x) \otimes d(z)) \leq d(y)$, which implies $d(x) \otimes \tau(z) \leq d(y)$ and $\tau(x) \otimes d(z) \leq d(y)$. Therefore, $\tau(z) \leq d(x) \to d(y)$ and $\tau(x) \leq d(z) \to d(y)$.

(2) Since $x \otimes (x \to y) \leq y$ for all $x, y \in L$, we have $d(x \otimes (x \to y)) \leq d(y)$. It follows from Definition 3.1 that $d(x \otimes (x \to y)) = (d(x) \otimes \tau(x \to y)) \vee (\tau(x) \otimes d(x \to y))$, which implies $d(x) \otimes \tau(x \to y) \leq d(y)$ and $\tau(x) \otimes d(x \to y) \leq d(y)$. Therefore, $\tau(x \to y) \leq d(x) \to d(y)$ and $d(x \to y) \leq \tau(x) \to d(y)$ for all $x, y \in L$.

(3) Since $x \otimes y \leq x$ for all $x, y \in L$, we have $d(x \otimes y) \leq d(x)$. It follows from Definition 3.1 that $d(x \otimes y) = (d(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y))$. Thus, $\tau(x) \leq d(y) \to d(x)$. In a similar way, we have $\tau(y) \leq d(x) \to d(y)$. □

Proposition 3.13: Let (L, τ) be a state residuated lattice and d be a strong derivation on (L, τ). Then the following statements hold.

(1) $d(x) \otimes d(y) \leq d(x \otimes y) \leq d(x) \vee d(y) \leq \tau(x) \vee \tau(y)$ for all $x, y \in L$.

(2) $(d(x))^n \leq d(x^n)$ for all $n \geq 1$.

(3) If d is isotone, then $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to \tau(y)$ for all $x, y \in L$.

(4) $d(1) = 1$ if and only if $d(x) = \tau(x)$ for all $x \in L$.

Proof. (1) On the one hand, since d is a strong derivation on (L, τ), we have $d(x) \otimes d(y) \leq \tau(x) \otimes d(y)$ and $d(x) \otimes d(y) \leq d(x) \otimes \tau(y)$ for all $x, y \in L$. Then

$$\begin{aligned} d(x) \otimes d(y) &\leq (d(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y)) \\ &= d(x \otimes y). \end{aligned}$$

On the other hand, since $d(x) \otimes \tau(y) \leq d(x)$ and $\tau(x) \otimes d(y) \leq d(y)$, we have

$$\begin{aligned} d(x \otimes y) &= (d(x) \otimes \tau(y)) \vee (\tau(x) \otimes d(y)) \\ &\leq d(x) \vee d(y). \end{aligned}$$

Therefore, $d(x) \otimes d(y) \leq d(x \otimes y) \leq d(x) \vee d(y) \leq \tau(x) \vee \tau(y)$.

(2) It follows from (1) that $d(x) \otimes d(x) \leq d(x \otimes x)$. By induction, we can obtain $(d(x))^n \leq d(x^n)$ for all $n \geq 1$.

(3) On the one hand, for all $x, y \in L$, since $x \otimes (x \to y) \leq y$ and d is isotone, we have $d(x \otimes (x \to y)) \leq d(y)$. It follows from the statement (1) that $d(x \to y) \otimes d(x) \leq d(x \otimes (x \to y))$, which implies $d(x \to y) \otimes d(x) \leq d(y)$, i.e., $d(x \to y) \leq d(x) \to d(y)$. On the other hand, since $d(y) \leq \tau(y)$, we have $d(x) \to d(y) \leq d(x) \to \tau(y)$. Therefore, $d(x \to y) \leq d(x) \to d(y) \leq d(x) \to \tau(y)$.

(4) On the one hand, it follows from Proposition 3.5 (3) that $\tau(x) \otimes d(1) \leq d(x)$ for all $x \in L$. If $d(1) = 1$, then we have

$$\begin{aligned} \tau(x) &= \tau(x) \otimes d(1) \\ &\leq d(x) \\ &\leq \tau(x), \end{aligned}$$

which implies $d(x) = \tau(x)$ for all $x \in L$. On the other hand, if $d(x) = \tau(x)$, then $d(1) = \tau(1) = 1$. □

Remark 3.14: It follows from Proposition 3.13 (4) that for any strong derivations on (L, τ), if $d(1) = 1 = \tau(1)$, then d is a state operator on L. In fact, in Example 3.4, let $d = \tau$. Then we obtain (L, τ, τ) is a derivation state residuated lattice.

Theorem 3.15: Let (L, τ, d) be a derivation state residuated lattice. For all $x, y \in L$, if d, τ satisfy $d(x) \to d(y) = d(x) \to \tau(y)$, then d is an ideal derivation on (L, τ).

Proof. Let $d(x) \to d(y) = d(x) \to \tau(y)$ for all $x, y \in L$. On the one hand, since $d(x) \otimes 1 \leq d(x)$, we have $1 \leq d(x) \to d(x) = d(x) \to \tau(x)$. Thus $d(x) \otimes 1 \leq \tau(x)$ for all $x \in L$, which implies d is strong. On the other hand, let $x \leq y, x, y \in L$. Then we have $\tau(x) \leq \tau(y)$. Thus $d(x) \otimes 1 \leq d(x) \leq \tau(x) \leq \tau(y)$, i.e., $d(x) \otimes 1 \leq \tau(y)$, which implies $1 \leq d(x) \to \tau(y) = d(x) \to d(y)$, i.e., $d(x) \otimes 1 \leq d(y)$. Thus, $d(x) \leq d(y)$, i.e., d is isotone. Therefore, d is an ideal derivation on (L, τ). □

Definition 3.16: Let (L, τ) be a state residuated lattice and d be a derivation on (L, τ). If $d(1) \in B(L)$, then d is called a regular derivation on (L, τ), (L, τ, d) is said to be a regular derivation state residuated lattice.

Example 3.17: Consider the Example 3.3. One can check that (L, τ, d) is a regular derivation state residuated lattice.

Theorem 3.18: Let (L, τ) be a state-morphism residuated lattice and d be a regular and strong derivation on (L, τ). Then the following statements are equivalent.

(1) d is an isotone derivation on (L, τ).

(2) $d(x) \leq d(1)$ for all $x \in L$.

(3) $d(x) = d(1) \otimes \tau(x)$ for all $x \in L$.

(4) $d(x \wedge y) = d(x) \wedge d(y)$ for all $x, y \in L$.

(5) $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in L$.
(6) $d(x \otimes y) = d(x) \otimes d(y)$ for all $x, y \in L$.

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Let $d(x) \leq d(1)$ for all $x \in L$. Since d is a regular and strong derivation on state-morphism residuated lattice (L, τ), we have $d(1) \in B(L)$, then it follows from Proposition 2.8 that

$$\begin{aligned} d(x) &= d(1) \wedge d(x) \\ &= d(1) \otimes d(x) \\ &\leq d(1) \otimes \tau(x). \end{aligned}$$

On the other hand, it follows from Proposition 3.5 (3) that $\tau(x) \otimes d(1) \leq d(x)$. Therefore, $d(x) = d(1) \otimes \tau(x)$.

(3) $\Rightarrow$ (4) Let $d(x) = d(1) \otimes \tau(x)$ for all $x \in L$. Then for all $x, y \in L$,

$$\begin{aligned} d(x \wedge y) &= d(1) \otimes \tau(x \wedge y) \\ &= d(1) \otimes (\tau(x) \wedge \tau(y)) \\ &= (d(1) \wedge \tau(x)) \wedge (d(1) \wedge \tau(y)) \\ &= (d(1) \otimes \tau(x)) \wedge (d(1) \otimes \tau(y)) \\ &= d(x) \wedge d(y). \end{aligned}$$

(4) $\Rightarrow$ (1) Let $x \leq y$. It follows from (4) that $d(x) = d(x \wedge y) = d(x) \wedge d(y)$, which implies $d(x) \leq d(y)$ for all $x, y \in L$.

(3) $\Rightarrow$ (5) For all $x, y \in L$, it follows from (3) and Proposition 2.2 (9) that

$$\begin{aligned} d(x \vee y) &= d(1) \otimes \tau(x \vee y) \\ &= d(1) \otimes (\tau(x) \vee \tau(x)) \\ &= d(1) \otimes (\tau(x)) \vee (d(1) \otimes \tau(x)) \\ &= d(x) \vee d(y). \end{aligned}$$

(5) $\Rightarrow$ (1) Let $x \leq y$. It follows from (5) that $d(y) = d(x \vee y) = d(x) \vee d(y)$, which implies $d(x) \leq d(y)$.

(3) $\Rightarrow$ (6) For all $x, y \in L$, since (L, τ) is a state-morphism residuated lattice, it follows from (3) and Lemma 3.6 that

$$\begin{aligned} d(x \otimes y) &= d(1) \otimes \tau(x \otimes y) \\ &= d(1) \otimes (\tau(x) \otimes \tau(y)) \\ &= (d(1) \otimes \tau(x)) \otimes (d(1) \otimes \tau(y)) \\ &= d(x) \otimes d(y). \end{aligned}$$

(6) $\Rightarrow$ (2) For all $x \in L$, it follows from (6) that

$$\begin{aligned} d(x) &= d(x \otimes 1) \\ &= d(x) \otimes d(1) \\ &= d(x) \wedge d(1). \end{aligned}$$

Therefore, $d(x) \leq d(1)$. □

IV. Principal ideal derivations based on state residuated lattices

In this section, we investigate principal ideal derivations. Also, the adjoint of principal ideal derivation is obtained by a Galois connection. In particular, we discuss the algebraic structure of the set of all principal ideal derivations on state-morphism residuated lattice (L, τ). By using the set $Ima_{(d,\tau)}(L)$ of principal ideal derivation, we give a characterization of a Heyting algebra.

In what follows, let (L, τ) be a state-morphism residuated lattice and $a \in L$. We define a mapping $d_a : L \longrightarrow L$ as follows: $d_a(x) = a \otimes \tau(x)$ for all $x \in L$.

Theorem 4.1: Let (L, τ) be a state-morphism residuated lattice and $a \in L$. Then the mapping d_a is an ideal derivation on (L, τ).

Proof. Let $x, y \in L$. Then

$$\begin{aligned} d_a(x \otimes y) &= a \otimes \tau(x \otimes y) \\ &= (a \otimes \tau(x \otimes y)) \vee (a \otimes \tau(x \otimes y)) \\ &= (a \otimes \tau(x) \otimes \tau(y)) \vee (a \otimes \tau(x) \otimes \tau(y)) \\ &= ((a \otimes \tau(x)) \otimes \tau(y)) \vee (\tau(x) \otimes (a \otimes \tau(y))) \\ &= (d_a(x) \otimes \tau(y)) \vee (\tau(x) \otimes d_a(y)). \end{aligned}$$

Then d_a is a derivation on (L, τ).

Now let $x \leq y$. Then $\tau(x) \leq \tau(y)$. Thus $d_a(x) = a \otimes \tau(x) \leq a \otimes \tau(y) = d_a(y)$, which implies d_a is isotone.

Moreover, it is easy to know that $d_a(x) = a \otimes \tau(x) \leq \tau(x)$ for all $x \in L$, which implies d_a is strong.

Therefore, d_a is an ideal derivation on (L, τ). □

Remark 4.2: In Theorem 4.1, d_a is called a principal ideal derivation on (L, τ). We denote by $P(L)$ the set of all principal ideal derivations on (L, τ) for d_a, that is $P(L) = \{d_a | a \in L\}$. Moreover, we denote by $PB(L) = \{d_a | a \in B(L)\}$. For principal ideal derivation $d_a(x) = a \otimes \tau(x)$, if $d_a(x) \leq \tau(y)$, then $\tau(x) \leq a \rightarrow \tau(y)$, we denote by $g_a(y) = a \rightarrow \tau(y)$, $g_a(y)$ is called the adjoint derivation of d_a.

Theorem 4.3: Let (L, τ) be a state-morphism residuated lattice and d_a be a derivation on (L, τ). Then we have

(1) $a \in B(L)$ if and only if d_a is a regular derivation on (L, τ).

(2) If $a \in B(L)$, then d_a is a τ-derivation on lattice L.

Proof. (1) If $a \in B(L)$, then

$$\begin{aligned} d_a(1) \otimes d_a(1) &= (a \otimes \tau(1)) \otimes (a \otimes \tau(1)) \\ &= (a \otimes a) \otimes (\tau(1) \otimes \tau(1)) \\ &= a \otimes \tau(1) \\ &= d_a(1) \end{aligned}$$

and

$$\begin{aligned} (d_a(1))'' &= (a \otimes \tau(1))'' \\ &= a'' \\ &= a \\ &= a \otimes \tau(1) \\ &= d_a(1). \end{aligned}$$

Therefore, it follows from Definition 3.16 that d_a is a regular derivation on (L, τ).

Conversely, if d_a is a regular derivation on (L, τ), then it is easy to check that $a \in B(L)$.

(2) Let $d_a(x) = a \otimes \tau(x)$. Then $d_a(x \wedge y) = a \otimes \tau(x \wedge y)$. Since $a \in B(L)$, it follows from Proposition 2.8 that

$$\begin{aligned} a \otimes \tau(x \wedge y) &= a \wedge \tau(x \wedge y) \\ &= a \wedge \tau(x) \wedge \tau(y) \\ &= (a \wedge \tau(x) \wedge \tau(y)) \vee (a \wedge \tau(x) \wedge \tau(y)) \\ &= (d_a(x) \wedge \tau(y)) \vee (\tau(x) \wedge d_a(x)). \end{aligned}$$

Therefore, $d_a(x \wedge y) = (d_a(x) \wedge \tau(y)) \vee (\tau(x) \wedge d_a(x))$. It follows from Definition 2.10 that d_a is a τ-derivation on a lattice L. □

The following definition generalizes the notions of closure operators and Galois connections on a poset. We introduce the concept of f-closure operators and f-Galois connections, respectively.

Definition 4.4: Let $(L, \leq)$ be a poset. For an isotone mapping $g : L \longrightarrow L$ and a mapping $f : L \longrightarrow L$ satisfy: for all $x \in L$,

(1) $g(x) \geq f(x)$,
(2) $g(g(x)) = g(x)$.
Then g is called an f-closure operators on L.

Moreover, if an isotone mapping $h : L \longrightarrow L$ and a mapping $f : L \longrightarrow L$ satisfy: for all $x \in L$,
(1) $h(x) \leq f(x)$,
(2) $h(h(x)) = h(x)$.
Then h is called an f-dual closure operators on L and the pair (g, h) is called an f-Galois connection.

Using Definition 4.4, we have the following result.

Theorem 4.5: Let (L, τ) be a state-morphism residuated lattice and $\Gamma = \{x \in L | x \otimes \tau(x) = x\}$. If $a \in \Gamma$, then there exists a τ-closure operators g such that (g, d_a) forms a τ-Galois connection.

Proof. Define a mapping: $g : L \longrightarrow L$ as follows: $g(x) = a \to \tau(x)$ for all $x \in L$.

First, if $x \leq y$, then $\tau(x) \leq \tau(y)$, which implies $a \to \tau(x) \leq a \to \tau(y)$, i.e., $g(x) \leq g(y)$.

Next, since $a \leq 1$, we have $g(x) = a \to \tau(x) \geq 1 \to \tau(x) = \tau(x)$.

Finally,

$$\begin{aligned} g(g(x)) &= g(a \to \tau(x)) \\ &= a \to \tau(a \to \tau(x)) \\ &= (a \otimes \tau(a)) \to \tau(x) \\ &= a \to \tau(x) \\ &= g(x). \end{aligned}$$

Combining them, we obtain that g is a closure operator on L. Moreover, it follows from Theorem 4.1 that $d_a(x) = a \otimes \tau(x)$, then $d_a(d_a(x)) = d_a(a \otimes \tau(x)) = a \otimes \tau(a \otimes \tau(x))$. Since (L, τ) is a state-morphism residuated lattice, we have

$$\begin{aligned} a \otimes \tau(a \otimes \tau(x)) &= a \otimes \tau(\tau(x)) \otimes \tau(a) \\ &= a \otimes \tau(x) \otimes \tau(a) \\ &= a \otimes \tau(x) \\ &= d_a(x). \end{aligned}$$

Thus, $d_a(d_a(x)) = d_a(x)$. Further, it follows from Definition 4.4 that d_a is a τ-dual closure operators on L. Therefore, (d_a, g) forms a τ-Galois connection. □

Theorem 4.6: Let (L, τ) be a state-morphism residuated lattice. If L satisfies $x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$ for all $x, y, z \in L$, then $(PB(L), \wedge, \vee, d_0, d_1)$ is a bounded distributive lattice, where $d_a, d_b \in PB(L)$, $(d_a \vee d_b)(x) = d_a(x) \vee d_b(x)$ and $(d_a \wedge d_b)(x) = d_a(x) \wedge d_b(x)$.

Proof. Let $d_a, d_b \in PB(L)$. Then for all $x \in L$, we have

$$\begin{aligned} (d_a \vee d_b)(x) &= d_a(x) \vee d_b(x) \\ &= (a \otimes \tau(x)) \vee (b \otimes \tau(x)) \\ &= (a \vee b) \otimes \tau(x). \end{aligned}$$

For any $a, b \in B(L)$, we prove that $a \vee b \in B(L)$.

In fact, it follows from Proposition 2.2 (10)

$$\begin{aligned} (a \vee b) \vee (a \vee b)' &= (a \vee b) \vee (a' \wedge b') \\ &\geq (a \vee b) \vee (a' \otimes b'). \end{aligned}$$

Moreover, it follows from Proposition 2.2 (11) that

$$\begin{aligned} (a \vee b) \vee (a' \otimes b') &\geq ((a \vee b) \vee a') \otimes ((a \vee b) \vee b') \\ &= (a \vee b \vee a') \otimes (a \vee b \vee b') \\ &= (a \vee a' \vee b) \otimes (a \vee b \vee b') \\ &= ((a \vee a') \vee b) \otimes (a \vee (b \vee b')) \\ &= 1, \end{aligned}$$

which implies $(a \vee b) \vee (a' \otimes b') = 1$. Thus, $(a \vee b) \vee (a \vee b)' = 1$. Since for any $t \in L$, $t \in B(L)$ if and only if $t \vee t' = 1$, we have $a \vee b \in B(L)$. Hence, $(d_a \vee d_b)(x) = d_{a \vee b}(x)$.

Further, since $x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$ for all $x, y, z \in L$, we have

$$\begin{aligned} (d_a \wedge d_b)(x) &= d_a(x) \wedge d_b(x) \\ &= (a \otimes \tau(x)) \wedge (b \otimes \tau(x)) \\ &= (a \wedge b) \otimes \tau(x). \end{aligned}$$

For any $a, b \in B(L)$, we prove that $a \wedge b \in B(L)$.

It is easy to see that $a' \leq (a \wedge b)'$, $b' \leq (a \wedge b)'$, then

$$\begin{aligned} (a \wedge b) \vee (a \wedge b)' &\geq (a \wedge b) \vee (a' \vee b') \\ &\geq (a \otimes b) \vee (a' \vee b') \\ &\geq ((a' \vee b') \vee a) \otimes ((a' \vee b') \vee b) \\ &= 1, \end{aligned}$$

which implies $(a \wedge b) \vee (a \wedge b)' = 1$, i.e., $a \wedge b \in B(L)$. Thus $(d_a \wedge d_b)(x) = d_{a \wedge b}(x)$.

Now, we prove that $(PB(L), \wedge, \vee, d_0, d_1)$ is a bounded distributive lattice.

First, it is easy to see that $d_0, d_1 \in PB(L)$. Let $d_a \in PB(L), x \in L$. Then

$$\begin{aligned} d_0(x) &= 0 \otimes \tau(x) \\ &= 0 \\ &\leq d_a(x) \\ &= a \otimes \tau(x) \\ &\leq \tau(x) \\ &= 1 \otimes \tau(x) \\ &= d_1(x), \end{aligned}$$

which implies $d_0(x) \leq d_a(x) \leq d_1(x)$, i.e., d_0 is the smallest element and d_1 is the greatest element in $PB(L)$.

Next, in order to prove that $(PB(L), \wedge, \vee, d_0, d_1)$ is a bounded distributive lattice, we shall prove that the following equation: for all $a, b, c \in B(L)$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad (1)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (2)$$

In fact,

$$\begin{aligned} a \vee (b \wedge c) &= a \vee (b \otimes c) \\ &\geq (a \vee b) \otimes (a \vee c) \\ &= (a \vee b) \wedge (a \vee c). \end{aligned}$$

The reverse inequality holds in all lattices, then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Moreover,

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= (a \otimes b) \vee (a \otimes c) \\ &= a \otimes (b \vee c) \\ &= a \wedge (b \vee c). \end{aligned}$$

It follows from (1) and (2) that for any $d_a, d_b, d_c \in PB(L), x \in L$, we have

$$\begin{aligned} (d_a \vee (d_b \wedge d_c))(x) &= d_a(x) \vee (d_b \wedge d_c)(x) \\ &= d_a(x) \vee (d_b(x) \wedge d_c(x)) \\ &= d_a(x) \vee d_{b \wedge c}(x) \\ &= d_{a \vee (b \wedge c)}(x) \\ &= d_{(a \vee b) \wedge (a \vee c)}(x) \\ &= d_{(a \vee b)}(x) \wedge d_{(a \vee c)}(x) \\ &= ((d_a \vee d_b) \wedge (d_a \vee d_c))(x). \end{aligned}$$

Thus, $d_a \vee (d_b \wedge d_c) = (d_a \vee d_b) \wedge (d_a \vee d_c)$.

In a similar way, we have $(d_a \wedge (d_b \vee d_c)) = (d_a \wedge d_b) \vee (d_a \wedge d_c)$.

Therefore, we obtain that $(PB(L), \wedge, \vee, d_0, d_1)$ is a bounded distributive lattice. $\square$

Remark 4.7: Let (L, τ) be a state-morphism residuated lattice and $d : L \longrightarrow L$ be a derivation on (L, τ). We denote by a special kind of set $Ima_{(d,\tau)}(L)$ as follows: $Ima_{(d,\tau)}(L) = \{x \in L | d(x) = \tau(x)\}$.

Using Remark 4.7, we have the following result.

Theorem 4.8: Let d be a regular ideal derivation on state-morphism residuated lattice (L, τ). For all $x, y \in L$, if L satisfies the condition $x \wedge y = x \otimes (x \to y)$, then $Ima_{(d,\tau)}(L)$ is a lattice ideal of L.

Proof. Let $x \in Ima_{(d,\tau)}(L)$, $y \in L$ and $y \leq x$. Then $\tau(y) \leq \tau(x)$. Thus

$$\begin{aligned} d(y) &= d(x \wedge y) \\ &= d(x \otimes (x \to y)) \\ &= (d(x) \otimes \tau(x \to y)) \vee (d(x \to y) \otimes \tau(x)) \\ &= (d(x) \otimes \tau(x \to y)) \vee (d(x \to y) \otimes d(x)) \\ &= d(x) \otimes \tau(x \to y). \end{aligned}$$

Since x, y are comparable, it follows from Proposition 2.6 (4) that $\tau(x \to y) = \tau(x) \to \tau(y)$. Thus

$$\begin{aligned} d(x) \otimes \tau(x \to y) &= d(x) \otimes (\tau(x) \to \tau(y)) \\ &= \tau(x) \otimes (\tau(x) \to \tau(y)) \\ &= \tau(x) \wedge \tau(y) \\ &= \tau(y), \end{aligned}$$

which implies $d(x) = \tau(x)$, i.e., $y \in Ima_{(d,\tau)}(L)$.

Next, for any $x, y \in Ima_{(d,\tau)}(L)$, it follows from Theorem 3.18 that

$$\begin{aligned} d(x \vee y) &= d(x) \vee d(y) \\ &= \tau(x) \vee \tau(y) \\ &= \tau(x \vee y), \end{aligned}$$

which implies $d(x \vee y) = \tau(x \vee y)$, i.e., $x \vee y \in Ima_{(d,\tau)}(L)$.

Therefore, $Ima_{(d,\tau)}(L)$ is a lattice ideal of L. $\square$

Since every ideal is prime in a linearly ordered lattice, combining Theorem 4.8, we can obtain the following result.

Corollary 4.9: Let L be a linearly ordered residuated lattice and d be a regular ideal derivation on state-morphism residuated lattice (L, τ). For all $x, y \in L$, if L satisfies the condition $x \wedge y = x \otimes (x \to y)$, then $Ima_{(d,\tau)}(L)$ is a prime lattice ideal of L.

Finally, we characterize a Heyting algebra in terms of a principal ideal derivation.

Theorem 4.10: Let $a \in B(L)$ and $\tau(x) \leq x$. If L is a Heyting algebra, then $(a] \subseteq Ima_{(d,\tau)}(L)$.

Proof. Suppose that L is a Heyting algebra, we have $x \otimes y = x \wedge y = x \otimes (x \to y)$ for all $x, y \in L$. Taking $y = \tau(x)$, we have $x \otimes \tau(x) = x \wedge \tau(x) = \tau(x)$ for all $x \in L$. Since $d_a(x) = a \otimes \tau(x)$, we have $d_a(a) = a \otimes \tau(a) = \tau(a)$ for all $a \in L$, which implies $a \in Ima_{(d,\tau)}(L)$. Since L is a Heyting algebra, which satisfies $x \wedge y = x \otimes (x \to y)$, it follows from Theorem 4.8 that $Ima_{(d,\tau)}(L)$ is a lattice ideal of L, i.e., for all $x \in L$, if $x \leq a$, we have $x \in Ima_{(d,\tau)}(L)$, which implies $(a] \subseteq Ima_{(d,\tau)}(L)$. $\square$

V. Conclusions

The notion of derivations is helpful for studying structures and properties in algebraic systems. In this paper, by combining derivations, state operators and residuated lattices all together, we introduced the concept of derivations on state residuated lattices. Some properties of particular derivations are discussed. The main conclusions in this paper and the further work to do are listed as follows.

(1) We investigated the properties of isotone, regular and strong derivations on state residuated lattices (L, τ). Also, we introduced the notion of (strong) state-morphism residuated lattices and studied some properties of them.

(2) We obtained that the adjoint of principal ideal derivation by a Galois connection and we discuss the algebraic structure of the set of all principal ideal derivations on state-morphism residuated lattice (L, τ), i.e., we proved that the set of all principal ideal derivations on state-morphism residuated lattice (L, τ) can form a bounded distributive lattice.

(3) We introduced a special kind of set $Ima_{(d,\tau)}(L)$ of derivations on state residuated lattices (L, τ) and we get that $Ima_{(d,\tau)}(L)$ is a lattice ideal of L, when derivation d is regular ideal derivation. Further, if L is a linearly ordered residuated lattice, then we obtained that $Ima_{(d,\tau)}(L)$ is a prime lattice ideal of L. In particular, by using the set $Ima_{(d,\tau)}(L)$ of principal ideal derivations, we characterized a Heyting algebras.

As an extension of this work, the following topics may be considered:

(1) Constructing derivation theory to other algebras with s-tate operators, such as state semihoops, state skew residuated latices, state-R_0-algebras and so on.

(2) Studying generalized derivations in state residuated latices.

(3) Investigating generalized derivations in generalized state residuated latices.

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