# Conditions of Interval-Valued Optimality Problems under Subdifferentiability

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***Abstract*—This paper addresses interval-valued optimality problems with the sub-derivative. By using the new concept of differentiability, we prove optimality conditions more operational and less restrictive than before in interval-valued nonconstrained problems.**

***Index Terms*—sub-derivative; Interval-valued optimality problem; Optimality condition.**

## I. Introduction

UNCERTAINTY arises widely in many practical engineering and economic fields [1], [4], [7], [10], [17], [25], [32], [34]. In most cases, due to the increasingly complex environment and the inherent subjectivity of human thinking, decision information is often uncertain. Therefore, it is a challenge to quantify their opinions accurately with crisp numbers [11], [13]–[15], [21], [25], [30]. Fuzzy time series have been widely used to deal with forecasting problems [12]. In view of the low prediction accuracy of the filtering method used in parameter learning of adaptive neural fuzzy inference system, a training method based on improved square root unscented Kalman filter and noise statistical estimator is proposed [30]. In [33], optimization of convex and generalized convex fuzzy mappings are derived and studied the fuzzy differential equations in the quotient space of fuzzy numbers. Hence, interval analysis is introduced to deal with the uncertainty in many deterministic phenomena in the real world [27]–[29], [31]. Interval-valued optimization problems can provide a more useful choice for evaluating uncertainty in optimization problems [9], [13]–[15].

Interval-valued optimization is an optimization problem which objective function is an interval-valued function. In the practical application, it often has trouble determining the probability distribution function of random parameters and membership function of fuzzy parameters, nevertheless it is comparatively easy to obtain the range of parameters [8], [16], [19], [26], [33]. Therefore, interval planning can better solve the optimization problem of uncertain systems [15]. Derivative is an important concept for interval-valued optimization problems. The derivative describes the changing trend of the function [3], [22]–[24]. In this paper, the theory of sub-derivative is introduced, which is more applicable than generalized Hukuhara derivative [22]. The limitation of necessary and sufficient condition for the existence of generalized Hukuhara derivative shows in [5]. By using the sub-derivative, this paper establishes the optimization conditions of interval-valued function. Compared with other methods found in the literature, these methods are more operational and less restrictive [5], [6]. If the one-sided derivatives of the lower and upper endpoint functions of interval-valued functions exist, the sub-derivative of interval-valued functions exists. At the same time, the relationship between the local minimum point and the global minimum point of interval-valued function is explained, and the conclusion that all the global minimum points of interval-valued functions constitute an interval is given. Section 2 presents the basic definitions and conclusions that will be used later. In Section 3, the optimization conditions of interval-valued optimization problems are given. Examples are given to illustrate the applicability of the conditions. Section 4 is a summary.

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## II. Preliminaries

The definitions and results which will be used throughout the paper are introduced in this section.

The $\mathbb{R}$ denotes the family of all real numbers, $U(c,\delta) = (c-\delta, c+\delta)$ denotes the neighborhood of $c \in \mathbb{R}$, $U^-(c,\delta) = (c-\delta, c]$ and $U^+(c,\delta) = [c, c+\delta)$ denote the left and right neighborhood of $c \in \mathbb{R}$, respectively. Let $\mathcal{K}_c$ be the bounded and closed intervals of $\mathbb{R}$, i.e.,

$$\mathcal{K}_c = \{[\underline{a}, \overline{a}] | \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \le \overline{a}\}.$$

For any $A = [\underline{a}, \overline{a}]$, $B = [\underline{b}, \overline{b}]$ and $\lambda \in \mathbb{R}$, the sum and scalar multiplication are defined by

$$A + B = [\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{1}$$

$$\lambda \cdot A = \begin{cases} [\lambda\underline{a}, \lambda\overline{a}], & if \quad \lambda \ge 0, \\ [\lambda\overline{a}, \lambda\underline{a}], & if \quad \lambda < 0. \end{cases} \tag{2}$$

Stefanini and Bede [22] introduced the $gH$-difference of two intervals.

*Definition 1:* [22] The generalized Hukuhara difference of two intervals, $A$ and $B$, ($gH$-difference for short) is an interval $C$ such that

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (i) \quad A = B + C \\ or \quad (ii) \quad B = A - C \end{cases}.$$

The $gH$-difference of two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ always exists and equals to

$$A \ominus_{gH} B = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}].$$

We suppose that $M$ is an open and nonempty subset of $\mathbb{R}^n$ and $F$: $M \to \mathcal{K}_c$ is an interval-valued function. Then we obtain $F(x) = \left[\underline{F}(x), \overline{F}(x)\right]$, where $\underline{F}(x) \le \overline{F}(x)$ for $x \in M$. $\underline{F}(x)$ and $\overline{F}(x)$ are called the lower and upper endpoint functions of $F(x)$. Based on the $gH$-difference, Stefanini and Bede [22] introduced the following derivative for interval-valued functions.

*Definition 2:* [22] Let $x_0 \in M \subset \mathbb{R}$ and $h$ be such that $x_0 + h \in M$, then the $gH$-derivative of an interval-valued function $F$ at $x_0$ is defined as

$$F'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h}\left[F(x_0+h) \ominus_{gH} F(x_0)\right]. \quad (3)$$

If $F'_{gH}(x_0) \in \mathcal{K}_c$ satisfying Equation (3) exists, we say that $F(x)$ is generalized Hukuhara differentiable ($gH$-differentiable for short) at $x_0$.

The limitation in Definition 2 is taken in the metric space $(\mathcal{K}_c, H)$, where $H$ is defined by

$$H(A,B) = \max\{\max_{a\in A} d(a,B), \max_{b\in B} d(b,A)\},$$

with $d(a,B) = \min_{b\in B}|a-b|$. The necessary and sufficient condition for the existence of $gH$-derivative of interval-valued functions is given in [5].

*Theorem 1:* [5] $F(x)$ is $gH$-differentiable at $x_0 \in M$ if and only if one of the following cases holds:
(a) $\underline{F}(x)$ and $\overline{F}(x)$ are differentiable at $x_0$ and $F'_{gH}(x_0)$ is equal to

$$\left[\min\{(\underline{F})'(x_0), (\overline{F})'(x_0)\}, \max\{(\underline{F})'(x_0), (\overline{F})'(x_0)\}\right].$$

(b) $(\underline{F})'_-(x_0)$, $(\underline{F})'_+(x_0)$, $(\overline{F})'_-(x_0)$ and $(\overline{F})'_+(x_0)$ exist and satisfy $(\underline{F})'_-(x_0) = (\overline{F})'_+(x_0)$ and $(\underline{F})'_+(x_0) = (\overline{F})'_-(x_0)$. Moreover $F'_{gH}(x_0)$ is equal to

$$\begin{aligned}&[\min\{(\underline{F})'_+(x_0), (\overline{F})'_+(x_0)\}, \max\{(\underline{F})'_+(x_0), (\overline{F})'_+(x_0)\}]\\ &= [\min\{(\underline{F})'_-(x_0), (\overline{F})'_-(x_0)\}, \max\{(\underline{F})'_-(x_0), (\overline{F})'_-(x_0)\}].\end{aligned}$$

Obviously, real-valued functions are special interval-valued functions. For the function $f(x) = |x|$, it gets the minimum at point 0 but are not $gH$-differentiable at 0. In order to enlarging the class of differentiable interval-valued functions, following concept is introduced.

*Definition 3:* Let $F$: $M \subset \mathbb{R} \to \mathcal{K}_c$ be an interval-valued function and $x_0 \in M$. We define that $F(x)$ is sub-differentiable at $x_0$, if the one-sided derivatives $(\underline{F})'_-(x_0)$, $(\underline{F})'_+(x_0)$, $(\overline{F})'_-(x_0)$ and $(\overline{F})'_+(x_0)$ exist, thus the sub-derivative of $F$: $M \to \mathcal{K}_c$ at $x_0$, $\partial F(x_0)$ is defined as

$$\begin{aligned}&[\min\left\{\begin{array}{l}(\underline{F})'_-(x_0), (\underline{F})'_+(x_0),\\ (\overline{F})'_-(x_0), (\overline{F})'_+(x_0)\end{array}\right\},\\ &\max\left\{\begin{array}{l}(\underline{F})'_-(x_0), (\underline{F})'_+(x_0),\\ (\overline{F})'_-(x_0), (\overline{F})'_+(x_0)\end{array}\right\}].\end{aligned} \quad (4)$$

It is straightforward that when interval-valued function satisfies Theorem 1, the sub-derivative of interval-valued function is consistent with $gH$-derivative. Definition 3 can also be appropriate for real-valued functions. And the sub-derivative of $f(x) = |x|$ at point $x_0 = 0$ is

$$\partial f(0) = [\min\{f'_-(0), f'_+(0)\}, \max\{f'_-(0), f'_+(0)\}] = [-1, 1].$$

*Example 1:* The interval-valued function $F(x) = [\underline{F}(x), \overline{F}(x)]$ is defined by

$$\overline{F}(x) = \begin{cases} -x, & x \le 0 \\ 2x, & x > 0, \end{cases}$$

$$\underline{F}(x) = x - 10, x \in \mathbb{R}.$$

The right and left derivative of $\underline{F}(x)$ and $\overline{F}(x)$ are $(\underline{F})'_-(0) = (\underline{F})'_+(0) = 1$, but $(\overline{F})'_-(0) = -1 \neq (\overline{F})'_+(0) = 2$. Thus $F(x)$ is not $gH$-differentiable at $x_0 = 0$. However, by Definition 3, we can obtain

$$\begin{aligned}\partial F(0) &= [\min\{\underline{F}'_-(0), \underline{F}'_+(0), \overline{F}'_-(0), \overline{F}'_+(0)\},\\ &\quad \max\{\underline{F}'_-(0), \underline{F}'_+(0), \overline{F}'_-(0), \overline{F}'_+(0)\}]\\ &= [\min\{1,1,-1,2\}, \max\{1,1,-1,2\}]\\ &= [-1, 2].\end{aligned}$$

For any two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, we say $A \le B$ if $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$, and $A < B$ if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$. If the relationship between $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ can not be judged, we can have the result that $A$ and $B$ are not comparable. Thus, the definition of minimum point of interval-valued function is shown as follows.

*Definition 4:* [9] Let $F$: $M \subset \mathbb{R}^n \to \mathcal{K}_c$ and $x_0 \in M$. $x_0$ is a global minimum point of $F(x)$ if there exists no $x \in M$ such that $F(x) < F(x_0)$. Correspondingly, $x_0$ is a local minimum point of $F(x)$ if there exists a neighborhood $U(x,\delta)$ such that no $x \in U(x,\delta)$ satisfying $F(x) < F(x_0)$.

*Remark 1:* Note that if $x_0$ is a local minimum point for one of the endpoint functions of $F(x)$, then $x_0$ is the local minimum point of $F(x)$. Without loss of generality, suppose that $x_0$ is a local minimum point of $\underline{F}(x)$. Thus there is not any $x \in U(x_0,\delta)$ such that $\underline{F}(x) < \underline{F}(x_0)$ for neighborhood $U(x,\delta)$. Further derivation, there is not any $x \in U(x_0,\delta)$ such that $F(x) < F(x_0)$. By Definition 4, $x_0$ is the local minimum point of $F(x)$.

*Remark 2:* If $F(x)$ is not comparable with any $x \in U(x_0,\delta)$ at point $x_0$, there is not any $x \in U(x_0,\delta)$ such that $F(x) < F(x_0)$. According to Definition 4, $x_0$ is a local minimum point of $F(x)$.

*Definition 5:* Let $F(x)$ be an interval-valued function defined on $M \subset \mathbb{R}^n$ and $x_0 = (x_1, ..., x_n)$ be an element of $M$. Considering the interval-valued function $h(x_i) = F(x_1, ..., x_i, .., x_n)$, if $h(x_i)$ exists the sub-derivative at $x_i$, $F(x)$ has the $i$th partial sub-derivative at $x_0$ and is defined as

$$\partial_{x_i} F(x_0) = \partial h(x_i).$$

*Definition 6:* Let $F$: $M \subset \mathbb{R}^n \to \mathcal{K}_c$. If all the partial sub-derivatives of function $F(x)$ exist at $x_0 = (x_1, x_2, \ldots, x_n)$ and the $n$-dimensional interval-valued vector is defined as

$$\widetilde{\nabla} F(x_0) = (\partial_{x_1} F(x_0), \partial_{x_2} F(x_0), \ldots, \partial_{x_n} F(x_0)).$$

We define $\widetilde{\nabla} F(x_0)$ as the sub-gradient of $F(x)$ at $x_0$. For any $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, we have

$$d^T \widetilde{\nabla} F(x_0) = (\partial_{x_1} F(x_0) d_1, \partial_{x_2} F(x_0) d_2, \ldots, \partial_{x_n} F(x_0) d_n).$$

*Definition 7:* [18] Let $F$ be an interval-valued function. The $F(x)$ is convex at $x_0$ if

$$F(\lambda x_0 + (1-\lambda)x) \le \lambda F(x_0) + (1-\lambda) F(x),$$

for all $\lambda \in [0,1]$ and each $x \in M$.

*Remark 3:* Let $F(x)$ be an interval-valued function. It is easy to obtain that $F(x)$ is convex at $x_0$ if and only if all the endpoint functions of $F(x)$ are convex at $x_0$.

*Lemma 1:* Let $f(x): \mathbb{R} \to \mathbb{R}$. If $f(x)$ is sub-differentiable and convex at $x$, then $f(y) - f(x) \ge k \cdot (y - x)$ for each $k \in \partial f(x)$ and $y \in M$.

**Proof.** Because of the convexity of $f(x)$, it follows $f'_-(x) \le f'_+(x)$ and $\frac{f(y)-f(x)}{y-x}$ is a monotone nondecreasing function about $y$. Thus, whenever $y-x<0$, we get

$$\frac{f(y)-f(x)}{y-x} \le f'_-(x).$$

For $k \in [f'_-(x), f'_+(x)]$, we have

$$f(y)-f(x) \ge f'_-(x)(y-x) \ge k(y-x).$$

For the same reason, whenever $y-x>0$,

$$\frac{f(y)-f(x)}{y-x} \ge f'_+(x).$$

Thus for $k \in [f'_-(x), f'_+(x)]$, we obtain

$$f(y)-f(x) \ge f'_+(x)(y-x) \ge k(y-x).$$

It is clear that $f(y)-f(x) \ge k\cdot(y-x)$ for any $k \in \partial f(x) = [f'_-(x), f'_+(x)]$. □

## III. Optimization conditions of interval-valued functions

In this section, we will give a series of optimization conditions of interval-valued functions based on sub-derivative.

*Theorem 2:* Let $F: M \subset \mathbb{R} \to \mathcal{K}_c$ be an interval-valued function and be sub-differentiable on $M$. If $x_0$ is a local minimum point of $F(x)$, then we have

$$0 \in \partial F(x_0). \tag{5}$$

**Proof.** Suppose Equation (5) dose not hold and $\partial F(x_0) > [0,0]$. By Definition 3, we obtain

$$\left[\min\left\{\begin{array}{l}(\underline{F})'_+(x_0), (\underline{F})'_-(x_0),\\ (\overline{F})'_+(x_0), (\overline{F})'_-(x_0)\end{array}\right\},\right. \left.\max\left\{\begin{array}{l}(\underline{F})'_+(x_0), (\underline{F})'_-(x_0),\\ (\overline{F})'_+(x_0), (\overline{F})'_-(x_0)\end{array}\right\}\right] > [0,0]. \tag{6}$$

From Equation (6), we know that $(\overline{F})'_-(x_0) > 0$ which implies

$$\lim_{h\to 0^-} \frac{\overline{F}(x_0+h)-\overline{F}(x_0)}{h} > 0. \tag{7}$$

From Equation (7), there exists $h<0$ such that $\overline{F}(x_0+h) < \overline{F}(x_0)$, which implies there exists $U^-(x_0,\delta_1)$ such that $x_0+h \in U^-(x_0,\delta_1)$,

$$\overline{F}(x_0+h) < \overline{F}(x_0).$$

By Equation (6), we also get $(\underline{F})'_-(x_0) > 0$ which implies

$$\lim_{h\to 0^-} \frac{\underline{F}(x_0+h)-\underline{F}(x_0)}{h} > 0. \tag{8}$$

It follows that there exists $U^-(x_0,\delta_1)$ such that whenever $x_0+h \in U^-(x_0,\delta_2)$,

$$\underline{F}(x_0+h) < \underline{F}(x_0).$$

Let $\delta_0 = \min\{\delta_1,\delta_2\}$. For $x_0+h \in U^-(x_0,\delta_0)$, we have

$$F(x_0+h) < F(x_0).$$

It would be a contradiction in fact that $x_0$ is a local minimum point of $F(x)$. □

*Theorem 3:* Let $F: M \subset \mathbb{R} \to \mathcal{K}_c$ be an interval-valued function. If $F(x)$ is sub-differentiable and convex in $U(x_0,\delta)$, $0 \in \partial F(x_0)$, then $x_0$ is a local minimum point of $F(x)$.

**Proof.** Considering Definition 3, we divide $0 \in \partial F(x_0)$ into the following two cases

$$(1)(\underline{F})'_-(x_0)\cdot(\underline{F})'_+(x_0)\cdot(\overline{F})'_-(x_0)\cdot(\overline{F})'_+(x_0) = 0,$$

$$(2)(\underline{F})'_-(x_0)\cdot(\underline{F})'_+(x_0)\cdot(\overline{F})'_-(x_0)\cdot(\overline{F})'_+(x_0) \ne 0.$$

For Case (1), since $(\underline{F})'_-(x_0)\cdot(\underline{F})'_+(x_0)\cdot(\overline{F})'_-(x_0)\cdot(\overline{F})'_+(x_0) = 0$, we obtain that one of $(\underline{F})'_-(x_0)$, $(\underline{F})'_+(x_0)$, $(\overline{F})'_-(x_0)$ and $(\overline{F})'_+(x_0)$ equals to 0. We assume $(\underline{F})'_-(x_0) = 0$. Taking into account the convexity of $F(x)$, $\underline{F}(x)$ is convex and $(\underline{F})'_+(x_0) \ge 0$. From the first sufficient condition of extremum [2], $x_0$ is a local minimum point of $\underline{F}(x)$. By Remark 1, $x_0$ is also a local minimum point of $F(x)$.

In Case (2), it can be divided into the following cases:

$$(a)(\underline{F})'_+(x_0) > (\underline{F})'_-(x_0) > 0,\quad (\overline{F})'_-(x_0) < (\overline{F})'_+(x_0) < 0;$$

$$(b)(\underline{F})'_-(x_0) < (\underline{F})'_+(x_0) < 0,\quad (\overline{F})'_+(x_0) > (\overline{F})'_-(x_0) > 0;$$

$$(c)(\underline{F})'_-(x_0) < 0, (\underline{F})'_+(x_0) > 0,\quad (\overline{F})'_-(x_0) < (\overline{F})'_+(x_0) < 0;$$

$$(d)(\underline{F})'_-(x_0) < 0, (\underline{F})'_+(x_0) > 0,\quad (\overline{F})'_+(x_0) > (\overline{F})'_-(x_0) > 0;$$

$$(e)(\overline{F})'_-(x_0) < 0, (\overline{F})'_+(x_0) > 0,\quad (\underline{F})'_-(x_0) < (\underline{F})'_+(x_0) < 0;$$

$$(f)(\overline{F})'_-(x_0) < 0, (\overline{F})'_+(x_0) > 0,\quad (\underline{F})'_+(x_0) > (\underline{F})'_-(x_0) > 0;$$

$$(g)(\underline{F})'_+(x_0) > 0, (\overline{F})'_+(x_0) > 0,\quad (\underline{F})'_-(x_0) < 0, (\overline{F})'_-(x_0) < 0.$$

For Case $(a)$ and Case $(b)$, take Case $(a)$ as an example. We obtain that $F(x)$ is not comparable with any $x \in U(x_0,\delta)$ at point $x_0$. From Remark 2, we get that $x_0$ is a local minimum point of $F(x)$. From Case $(c)$ to Case $(g)$, we first prove Case $(c)$. In Case $(c)$, because of $(\underline{F})'_-(x_0) < 0, (\underline{F})'_+(x_0) > 0$, we get that $x_0$ is a local minimum point of $\underline{F}(x)$ [2]. According to Remark 1, we know that $x_0$ is a local minimum point of $F(x)$. Similar to Case $(c)$, we can get the proof from Case $(d)$ to Case $(g)$.

In summary, if $0 \in \partial F(x_0)$, then $x_0$ is a local minimum point of $F(x)$. □

*Theorem 4:* Let $F: M \subset \mathbb{R} \to \mathcal{K}_c$ be an interval-valued function and be convex on $M$. If $x_0$ is a local minimum point of $F(x)$, then $x_0$ is also a global minimum point of $F(x)$.

**Proof.** Suppose that $x_0$ is a local minimum point of $F(x)$, that is, there exists a neighborhood $U(x_0,\delta)$ such that $F(x)$ is not less than $F(x_0)$ for all $x \in U(x_0,\delta)$. Suppose that there exists $x_1 \in M$ satisfying

$$F(x_1) < F(x_0). \tag{9}$$

The Equation (9) implies

$$\begin{cases}\underline{F}(x_1) < \underline{F}(x_0)\\ \overline{F}(x_1) < \overline{F}(x_0).\end{cases}$$

By the convexity of $\underline{F}(x)$, we get

$$\underline{F}(\lambda x_0 + (1-\lambda)x_1) \leq \lambda \underline{F}(x_0) + (1-\lambda)\underline{F}(x_1),$$

for any $\lambda \in [0,1]$. On account of $\underline{F}(x_1) < \underline{F}(x_0)$, we have

$$\underline{F}(\lambda x_0 + (1-\lambda)x_1) < \underline{F}(x_0).$$

Similarly, we obtain

$$\overline{F}(\lambda x_0 + (1-\lambda)x_1) < \overline{F}(x_0).$$

On the one hand, if $1-\lambda$ is small enough, then we get $|\lambda x_0 + (1-\lambda)x_1 - x_0| = (1-\lambda)|x_1 - x_0| < \delta$. That means there exists $\lambda x_0 + (1-\lambda)x_1 \in U(x_0, \delta)$ such that $F(\lambda x_0 + (1-\lambda)x_1) < F(x_0)$. This is in contradiction with the fact that $x_0$ is a local minimum point of $F(x)$. □

*Theorem 5:* Let $F(x)$ be a one-variable interval-valued function. If $F(x)$ is convex, then the set of all the global minimum points of $F(x)$ is an interval.

**Proof.** If there exists only one global minimum point of $F(x)$, the conclusion is obvious. Suppose that $x_1$ and $x_2$ are global minimum points of $F(x)$ and $x_1 < x_2$. If $F(x_1)$ and $F(x_2)$ are comparable, then $F(x_1) = F(x_2)$. For any $x_0$ satisfying $x_1 < x_0 < x_2$, according to the definition of convex function, we know

$$F(x_0) \leq F(x_1).$$

If there exists $x_3$ such that $F(x_3) < F(x_0)$, then $F(x_3) < F(x_1)$. It contradicts that $x_1$ is a global minimum point of $F(x)$. Thus there is no such $x_3$, that is to say, $x_0$ is a global minimum point of $F(x)$.

If $F(x_1)$ and $F(x_2)$ are not comparable, suppose that $\underline{F}(x_2) < \underline{F}(x_1)$ and $\overline{F}(x_1) < \overline{F}(x_2)$. For any $x_0$ such that $x_1 < x_0 < x_2$, if $\underline{F}(x_0) < \underline{F}(x_2)$, then $\overline{F}(x_0) \geq \overline{F}(x_2)$. On the other hand, by the convexity of $\overline{F}(x)$, we know $\overline{F}(x_0) < \overline{F}(x_2)$. Therefore, $\underline{F}(x_1) > \underline{F}(x_0) \geq \underline{F}(x_2)$ and $\overline{F}(x_1) \leq \overline{F}(x_0) < \overline{F}(x_2)$.

If there exists $x_3$ satisfying $F(x_3) < F(x_0)$ and $x_3 < x_1$, then $\underline{F}(x_3) < \underline{F}(x_0) < \underline{F}(x_1)$. There is a contradiction that $\underline{F}(x)$ is convex at $[x_3, x_0]$. If $x_1 < x_3 < x_0$, then $\underline{F}(x_0) > \underline{F}(x_3)$ and $\underline{F}(x_0) \geq \underline{F}(x_2)$. It contradicts that $\underline{F}(x)$ is convex at $[x_3, x_2]$. Similarly, it can be proved that $x_3$ can not be greater than $x_0$. Therefore there does not exist $x_3$ such that $F(x_3) < F(x_0)$ and $x_0$ is a global minimum point of $F(x)$. □

*Example 2:* Let $F : \mathbb{R} \to \mathcal{K}_c$ be

$$\underline{F}(x) = \begin{cases} x - 10, & x \leq 0 \\ 2x - 10, & x > 0, \end{cases}$$

$$\overline{F}(x) = x^2, x \in \mathbb{R}.$$

By Definition 3, the sub-derivative of $F(x)$ is

$$\partial F(x) = \begin{cases} [2x, 1], & x < 0 \\ [0, 2], & x = 0 \\ [2x, 2], & 0 < x \leq 1 \\ [2, 2x], & x > 1. \end{cases}$$

We obtain that $0 \in \partial F(x)$ for any $x \leq 0$. According to Theorem 3 and 4, all $x \leq 0$ are global minimum points of $F(x)$.

In the following part, we give the optimal conditions for global minimum points of multi-variable interval-valued functions.

*Theorem 6:* Let $F : M \to \mathcal{K}_c$ be a multi-variable interval-valued function and be sub-differential at $x_0$. If $x_0 = \{x_1, ..., x_n\}$ is a global minimum point of $F(x)$, then the next inequalities system does not have a solution for any $y \in \mathbb{R}^n$

$$y\widetilde{\nabla} F(x_0) < [0, 0]^n.$$

**Proof.** Arguing by contradiction, suppose that for each $i = 1, ..., n$, there exists $y_i \in \mathbb{R}$ such that

$$\begin{aligned} &y_i[\min\{(\underline{F})'_-(x_i), (\underline{F})'_+(x_i), \\ &(\overline{F})'_-(x_i), (\overline{F})'_+(x_i)\} \\ &\quad \max\{(\underline{F})'_-(x_i), (\underline{F})'_+(x_i), \\ &(\overline{F})'_-(x_i), (\overline{F})'_+(x_i)\}] < [0, 0]. \end{aligned} \tag{10}$$

If $y_i < 0$, then from the Equation (10), we know

$$y_i \lim_{h_i \to 0^+} \frac{\underline{F}(x_1, ..., x_i + y_i h_i, ..., x_n) - \underline{F}(x_0)}{y_i h_i} < 0.$$

Therefore, we get

$$\lim_{h_i \to 0^+} \frac{\underline{F}(x_1, ..., x_i + y_i h_i, ..., x_n) - \underline{F}(x_0)}{h_i} < 0.$$

It follows that there exists $U^-(x_0, \delta_1)$ such that if $(x_1, ..., x_i + y_i h_i, ..., x_n) \in U^-(x_0, \delta_1)$, then we have

$$\underline{F}(x_1, ..., x_i + y_i h_i, ..., x_n) < \underline{F}(x_0).$$

Similarly, by

$$y_i \lim_{h_i \to 0^+} \frac{\overline{F}(x_1, ..., x_i + y_i h_i, ..., x_n) - \overline{F}(x_0)}{y_i h_i} < 0,$$

there exists $U^-(x_0, \delta_2)$ such that if $(x_1, ..., x_i + y_i h_i, ..., x_n) \in U^-(x_0, \delta_2)$, then

$$\overline{F}(x_1, ..., x_i + y_i h_i, ..., x_n) < \overline{F}(x_0).$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$. For any $(x_1, ..., x_i + y_i h_i, ..., x_n) \in U^-(x_0, \delta_0)$, we obtain that

$$F(x_1, ..., x_i + y_i h_i, ..., x_n) < F(x_0).$$

In summary, there is a contradiction that $x_0$ is a global minimum point of $F(x)$. □

*Theorem 7:* Let $F : M \to \mathcal{K}_c$ be sub-differentiable and convex at $x_0$. If the sub-gradients of $\underline{F}(x)$ and $\overline{F}(x)$ satisfying $0^n \in \widetilde{\nabla}\underline{F}(x_0)$ or $0^n \in \widetilde{\nabla}\overline{F}(x_0)$, then $x_0$ is a global minimum point of $F(x)$.

**Proof.** Because of $0^n \in \widetilde{\nabla}\underline{F}(x_0)$, then

$$0^n \in \widetilde{\nabla}\underline{F}(x_0) = \left(\partial_{x_1}\underline{F}(x_0), \partial_{x_2}\underline{F}(x_0), \ldots, \partial_{x_n}\underline{F}(x_0)\right). \tag{11}$$

Equation (11) implies $0 \in \partial_{x_i}\underline{F}(x_0)$, $(i = 1, ..., n)$. It is easy to get from Remark 3 that $\underline{F}(x)$ is convex at $x_0$. According to Lemma 1, we know that $x_0$ is a global minimum point of $\underline{F}(x)$. Thus, from Remark 1, $x_0$ is a global minimum point of $F(x)$. When $0^n \in \widetilde{\nabla}\overline{F}(x_0)$, the proof is similar to $0^n \in \widetilde{\nabla}\underline{F}(x_0)$. □

*Example 3:* The interval-valued function $F : \mathbb{R}^n \to \mathcal{K}_c$ is defined by

$$\underline{F}(x) = x_2 - 100, x \in \mathbb{R}^2,$$

$$\overline{F}(x) = x_1^2 + x_2^2, x \in \mathbb{R}^2.$$

By Definition 6, we get

$$\widetilde{\nabla}\underline{F}(x) = ([0, 0], [1, 1]), x \in \mathbb{R}^2,$$

$$\widetilde{\nabla}\overline{F}(x) = ([2x_1, 2x_1], [2x_2, 2x_2]), x \in \mathbb{R}^2,$$

$$\widetilde{\nabla}F(x) = \begin{cases} ([2x_1, 0], [2x_2, 1]), & x_1 \leq 0 \ \ and \ \ x_2 \leq \frac{1}{2} \\ ([2x_1, 0], [1, 2x_2]), & x_1 \leq 0 \ \ and \ \ x_2 > \frac{1}{2} \\ ([0, 2x_1], [2x_2, 1]), & x_1 > 0 \ \ and \ \ x_2 \leq \frac{1}{2} \\ ([0, 2x_1], [1, 2x_2]), & x_1 > 0 \ \ and \ \ x_2 > \frac{1}{2}. \end{cases}$$

We obtain $0^n \in \widetilde{\nabla}\overline{F}(x)$ and $0^n \in \widetilde{\nabla}F(x)$ at $x = (0, 0)$. According to Theorem 6 and 7, we can find that $x = (0, 0)$ is a global minimum point of $F(x)$.

## IV. Conclusion

The generalized Hukuhara derivative is a general tool for dealing with interval-valued optimization problems [22]. In order to extend the application of generalized Hukuhara derivative, we first introduce the concept of sub-derivative. The new concept of sub-derivative unifies and extends others appeared in the recent literature [5], [6]. Thanks to a characterization result of sub-derivative, that we have provided on sub-differentiability, an interesting interpretation of the sub-derivative in terms of condition has been introduced. Based on the sub-derivative of interval-valued functions, this paper studies the conditions of interval-valued optimization problems, and gives examples to illustrate the applicability of the optimization conditions.

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