Linearizability of Nonlinear Second-Order Ordinary Differential Equations by Using a Generalized Linearizing Transformation

Prakrong Voraka, Supaporn Suksern, and Nontakan Donjiwprai

Abstract—In this paper, we have proposed the linearization problem of second-order ordinary differential equation under the generalized linearizing transformation. We found the necessary form for reducing the second-order ordinary differential equation to simple linear equation. We also obtained sufficient condition for making the above form to be linear. Further, the procedure of linear transformation within the study is demonstrated in the explicit form. Moreover, we apply the obtained linearization criteria to the interesting problems of nonlinear ordinary differential equations and nonlinear partial differential equations, for examples the parachute equation, the Painlevé - Gambier XI equation, the equation for the variable frequency oscillator, the one-dimensional non-polynomial oscillator, the equation that can be linearizable by point and Sundman transformations, the modified generalized Vakhnenko equation.

Index Terms—linearization problem, generalized linearizing transformation, nonlinear second-order ordinary differential equation.

I. Introduction

THE linearization problem is one of the important branches in differential equation field. A number of mathematicians has been studying this branch continuously until the present time. To discover theory for finding new knowledge has shown to be a great benefit for academic world and country development. It is known that theories and new knowledge obtained from research not only offer benefits to improve existing knowledge within the branch itself, but also they can be applied to other branches or fields and can be key fundamental to develop basic science which is basic research to build many other new knowledge. This would be a fundamental step to develop the country.

The linearization problem is a branch of study that can be applied widely in particular to the study involving solving the equations. Most important physical problems are in the form of nonlinear differential equations which are normally difficult to solve and there are relatively few method to find their exact solutions. Numerical method therefore is often used to solve these nonlinear differential equations but the obtained solutions are just the approximate solutions. However, the exact solution is claimed to be more interesting because it can be used to analyze the properties of the studied equations. One of the methods used to determine the exact solutions is to linearize the interested equation and find solutions directly by fundamental method. The solutions obtained from such linear equation are yet still solutions of initial equation. By mentioned above, we are required to seek for transformation in order to transform initial equation to be linear equation.

There are a number of interesting transformations. For example, in the case that the transformation consists of derivative, we call it as tangent transformation, in the case that the transformation depends only on independent and dependent variables, we call it as point transformation and we will call the tangent transformation which the independent and dependent variables can be changed and involves the first derivative as contact transformation. In addition, another type of transformation which its transformation set is different from any mentioned above since there is a nonlocal term $T = \int G(t,x)\,dt$, such transformation is called generalized Sundman transformation. In this paper, we use the generalized linearizing transformation which is an extended transformation from generalized Sundman transformation where the selected G function is $G(t,x,x')$.

Up to the present time, all researchers who study the linearization of second-order ordinary differential equations via generalized linearizing transformation have not covered all cases yet. Therefore, in this paper we focus on the remaining cases that have not yet been studied, which we also find that those cases can be applied to solve several nonlinear equations in real-world phenomenon.

A. Historical Review

From above facts as mentioned, the researcher would like to give a brief background of this study. Since 19th century the linearization problem of ordinary differential equation has attracted some interests from various well-known mathematicians e.g. S. Lie and E. Cartan etc. The first person who could solve the linearization problem of ordinary differential equation is Lie [1]. Lie could discover the standard form of every second-order ordinary differential equation which could be reduced the form to become linear equation via changing the

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S. Suksern and N. Donjiwprai (corresponding author e-mail: supapornsu@nu.ac.th) are with the Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

S. Suksern is with the Research center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok 65000, Thailand.

P. Voraka is with the Faculty of Science and Engineering, Kasetsart University, Chalermphrakiat Sakon Nakhon Province Campus, Sakon Nakhon 47000, Thailand.

independent and dependent variables (or can be called point transformation). Later, Liouville [2] and Tresse [3] used the relative invariants of equivalence group under point transformation to solve the equivalence of second-order ordinary differential equations which can be reduced from second-order nonlinear ordinary differential equations to second-order linear ordinary differential equations. Moreover, Lie discovered that every second-order ordinary differential equation can be reduced to second-order linear ordinary differential equation without any conditions via contact transformation.

Having mentioned some methods above, there are yet still other methods to solve linearization problem of second-order ordinary differential equation. For example, the method of Cartan [4], the reducing order method, the differential substitution method etc.

Another transformation that is very interesting and has not been mentioned yet is the generalized Sundman transformation

$$X = F(t,x),\ dT = G(t,x)dt. \quad (1)$$

Duarte, Moreira and Santos [5] used generalized Sundman transformation to determine the conditions for linearizing the second-order ordinary differential equation to be simple linear equation. In [6] Nakpim and Meleshko demonstrated that the general linear equation in the canonical form of Laguerre was not sufficient for solving linearization problem via generalized Sundman transformation. The canonical form of Laguerre could only particularly be applied with point and contact transformations. Therefore, in [6] they found the conditions for linearizing the second-order ordinary differential equation to be general linear equation.

In this paper, we extend the generalized Sundman transformation which was studied before as shown in [7]-[9], where they called such a transformation in this form as generalized linearizing transformation

$$X = F(t,x),\ dT = G(t,x,x')dt. \quad (2)$$

They demonstrated that this transformation can be used to linearize a more extensive class of nonlinear standard differential equations including some equations that can't be linearized by the non-point and invertible point transformations. In the case that the function G in (2) does not depend on the variable x' , then it can be turned into a non-point transformation. If G is a differentiable function, then it turns into an invertible point transformation. In this way, (2) is a unified transformation as it incorporates non-point and invertible point transformations as extraordinary cases. A case of an equation that can be linearized by a change of the structure (2) is given in [8].

In [7], Chandrasekar, Senthilvelan and Lakshmanan applied a particular class of transformations (2), where the function $G(t,x,x')$ is linear with respect to x'.

They payed attention to the case where G is a polynomial function in x' and in particular where it is linear in x' with coefficients which are arbitrary functions of t and x. To be specific, they focused here on the case

$$X = F(t,x),\ dT = (G_1(t,x)x' + G_2(t,x))dt.$$

Notice that for the case $G_1 = 0$, the generalized linearizing transformation becomes a generalized Sundman transformation, so that they assumed $G_1 \neq 0$.

The authors of [7] obtained that any second-order linearizable ordinary differential equation which can be mapped into the equation $X'' = 0$ via a generalized linearizing transformation has to be of the form

$$\begin{aligned} x'' + A_3(t,x)x'^3 + A_2(t,x)x'^2 + A_1(t,x)x' \\ + A_0(t,x) = 0, \end{aligned} \quad (3)$$

and the functions A_i's $(i = 0,1,2,3)$ are connected to the transform functions F and G through the relations

$$\begin{aligned} A_3 =&\ (G_1F_{xx} - F_xG_{1x})/M, \\ A_2 =&\ (G_2F_{xx} + 2G_1F_{xt} - F_xG_{2x} - F_tG_{1x} \\ &-F_xG_{1t})/M, \\ A_1 =&\ (2G_2F_{xt} + G_1F_{tt} - F_xG_{2t} - F_tG_{2x} \\ &-F_tG_{1t})/M, \\ A_0 =&\ (G_2F_{tt} - F_tG_{2t})/M \end{aligned} \quad (4)$$

with $M = F_xG_2 - F_tG_1 \neq 0$.

They have analyzed a particular case of equation (3), namely, $A_3 = 0$ and $A_2 = 0$ in equation (4). Complete analysis of the compatibility of arising equations is given for the case $F_x \neq 0$.

Therefore, in this paper we will apply the generalized linearizing transformation with second-order ordinary differential equation to complete the remaining cases $(F_x = 0)$ which are different from the work by Chandrasekar and Lakshmanan [7].

II. Formulation of the Linearization Theorems

A. Obtaining Necessary Condition of Linearization

We begin with investigating the necessary conditions for linearization. We consider the second-order ordinary differential equation

$$x'' = F(t,x,x') \quad (5)$$

which can be transformed to a simplest linear equation

$$X'' = 0 \quad (6)$$

under the generalized linearizing transformation

$$\begin{aligned} X =&\ F(t,x), \\ dT =&\ [G_1(t,x)x' + G_2(t,x)]\,dt, \end{aligned} \quad (7)$$

where $G_1 \neq 0$. So, we arrive at the following theorem.

Theorem 2.1: Any second-order ordinary differential equations (5) obtained from a linear equation (6) by a generalized linearizing transformation (7) has to be the form

$$\begin{aligned} x'' + A_3(t,x)x'^3 + A_2(t,x)x'^2 + A_1(t,x)x' \\ + A_0(t,x) = 0, \end{aligned} \quad (8)$$

where

$$A_3 = (-F_{xx}G_1 + F_xG_{1x})/(F_tG_1 - F_xG_2), \quad (9)$$

$$\begin{aligned} A_2 =&(-2F_{tx}G_1 + F_tG_{1x} - F_{xx}G_2 + F_xG_{1t} \\ &+ F_xG_{2x})/(F_tG_1 - F_xG_2), \end{aligned} \quad (10)$$

$$\begin{aligned} A_1 =&(-2F_{tx}G_2 - F_{tt}G_1 + F_tG_{1t} + F_tG_{2x} \\ &+ F_xG_{2t})/(F_tG_1 - F_xG_2), \end{aligned} \quad (11)$$

$$A_0 = (-F_{tt}G_2 + F_tG_{2t})/(F_tG_1 - F_xG_2). \quad (12)$$

Proof. Applying a generalized linearizing transformation (7), one obtains the following transformations

$$\begin{aligned} X'(T) =& \frac{D_t F}{D_t \int [G_1 x' + G_2] dt} \\ =& \frac{F_t + x' F_x}{G_1 x' + G_2} \\ =& P(t, x, x'), \\ X''(T) =& \frac{D_t P}{D_t \int [G_1 x' + G_2] dt} \\ =& \frac{P_t + P_x x' + P_{x'} x''}{G_1 x' + G_2}, \end{aligned}$$

where

$$\begin{aligned} P_t =& \frac{F_{tt}(G_1 x' + G_2) - F_t (G_1 x'' + G_{1t} x' + G_{2t})}{(G_1 x' + G_2)^2}, \\ P_x =& \frac{F_{tx}(G_1 x' + G_2) - F_t (G_{1x} x' + G_{2x})}{(G_1 x' + G_2)^2}, \\ P_{x'} =& -\frac{F_t G_1 x''}{(G_1 x' + G_2)^2}, \end{aligned}$$

and $D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + x'' \frac{\partial}{\partial x'} + ...$ is a total derivative. Substituting the resulting expression into the linear equation (6) we arrive at the necessary form (8), where A_0, A_1, A_2 and A_3 are some functions of t and x as defined in system of equations (9)-(12).

B. Obtaining Sufficient Conditions of Linearization and Linearizing Transformation

For obtaining sufficient conditions of linearizability of equation (8), one has to solve the compatibility problem of the system of equations (9)-(12), considering it as overdetermined system of partial differential equations for the functions F, G_1 and G_2 with given coefficients A_i of equation (8).

For convenience of calculations, we set

$$G_3 = \frac{G_2}{G_1}.$$

So that system of equations (9)-(12) become

$$\begin{aligned} A_3 =& (-F_{xx} G_1 + F_x G_{1x}) / (G_1 (F_t - F_x G_3)), && (13) \\ A_2 =& (-2F_{tx} G_1 + F_t G_{1x} - F_{xx} G_1 G_3 + F_x G_{1t} \\ & + F_x G_{1x} G_3 + F_x G_{3x} G_1) / (G_1 (F_t - F_x G_3)), && (14) \\ A_1 =& (-2F_{tx} G_1 G_3 - F_{tt} G_1 + F_t G_{1t} + F_t G_{1x} G_3 \\ & + F_t G_{3x} G_1 + F_x G_{1t} G_3 \\ & + F_x G_{3t} G_1) / (G_1 (F_t - F_x G_3)), && (15) \\ A_0 =& (-F_{tt} G_1 G_3 + F_t G_{1t} G_3 \\ & + F_t G_{3t} G_1) / (G_1 (F_t - F_x G_3)). && (16) \end{aligned}$$

According to the notation $K = G_1(F_x G_3 - F_t)$, we define the derivative F_t as

$$F_t = (F_x G_1 G_3 - K) / G_1. \quad (17)$$

Solving equations (13)-(16) with respect to F_{xx}, K_x, K_t and G_{3t}, one finds

$$\begin{aligned} F_{xx} =& (F_x G_{1x} + A_3 K) / G_1, && (18) \\ K_x =& (-F_x G_{1t} G_1 + F_x G_{1x} G_1 G_3 + F_x G_{3x} G_1^2 \\ & + 3G_{1x} K - A_2 G_1 K + 3A_3 G_1 G_3 K) / (2G_1), && (19) \\ K_t =& (-F_x G_{1t} G_1 G_3 + F_x G_{1x} G_1 G_3^2 + F_x G_{3x} G_1^2 G_3 \\ & + 4G_{1t} K - G_{1x} G_3 K + 2G_{3x} G_1 K - 2A_1 G_1 K \\ & + 3A_2 G_1 G_3 K - 3A_3 G_1 G_3^2 K) / (2G_1), && (20) \\ G_{3t} =& G_{3x} G_3 + A_0 - A_1 G_3 + A_2 G_3^2 - A_3 G_3^3. && (21) \end{aligned}$$

Comparing the mixed derivative $(K_x)_t = (K_t)_x$, one obtains

$$\begin{aligned} G_{3xx} =& (2A_{0x} F_x G_1^3 - 2A_{1x} F_x G_1^3 G_3 + 4A_{1x} G_1^2 K \\ & - 2A_{2t} G_1^2 K + 2A_{2x} F_x G_1^3 G_3^2 - 6A_{2x} G_1^2 G_3 K \\ & + 6A_{3t} G_1^2 G_3 K - 2A_{3x} F_x G_1^3 G_3^3 + 6A_{3x} G_1^2 G_3^2 K \\ & + 4F_x G_{1tx} G_1^2 G_3 - 2F_x G_{1tt} G_1^2 + 3F_x G_{1t}^2 G_1 \\ & - 6F_x G_{1t} G_{1x} G_1 G_3 + 2F_x G_{1t} G_{3x} G_1^2 \\ & - 2F_x G_{1t} A_1 G_1^2 + 4F_x G_{1t} A_2 G_1^2 G_3 \\ & - 6F_x G_{1t} A_3 G_1^2 G_3^2 - 2F_x G_{1xx} G_1^2 G_3^2 \\ & + 3F_x G_{1x}^2 G_1 G_3^2 - 2F_x G_{1x} G_{3x} G_1^2 G_3 \\ & + 2F_x G_{1x} A_0 G_1^2 - 2F_x G_{1x} A_2 G_1^2 G_3^2 \\ & + 4F_x G_{1x} A_3 G_1^2 G_3^3 - F_x G_{3x}^2 G_1^3 - 2G_{1tx} G_1 K \\ & + 3G_{1t} G_{1x} K - G_{1t} A_2 G_1 K + 3G_{1t} A_3 G_1 G_3 K \\ & + 2G_{1xx} G_1 G_3 K - 3G_{1x}^2 G_3 K + G_{1x} G_{3x} G_1 K \\ & + G_{1x} A_2 G_1 G_3 K - 3G_{1x} A_3 G_1 G_3^2 K \\ & - 5G_{3x} A_2 G_1^2 K + 15G_{3x} A_3 G_1^2 G_3 K \\ & + 6A_0 A_3 G_1^2 K - 6A_1 A_3 G_1^2 G_3 K \\ & + 6A_2 A_3 G_1^2 G_3^2 K - 6A_3^2 G_1^2 G_3^3 K) / (4G_1^2 K). && (22) \end{aligned}$$

The compatibility analysis depends on the value of F_x. A complete study of all cases is cumbersome. Here a complete solution is given for the case where $F_x = 0$.

Case $F_x = 0$

Since $F_x = 0$, then substituting it into F_{xx} in equation (18), one gets the condition

$$A_3 = 0. \quad (23)$$

Comparing the mixed derivative $(F_t)_x = (F_x)_t$, one obtains the derivative

$$G_{1x} = A_2 G_1 - 3A_3 G_1 G_3 \quad (24)$$

and this satisfies equation $(F_{xx})_t = (F_t)_{xx}$. Setting

$$\begin{aligned} \lambda_1 =& -A_{1x} + 2A_{2t}, \\ \lambda_2 =& -A_{0xx} - A_{0x} A_2 + A_{2tt} + A_{2t} A_1 - A_{2x} A_0 \\ & - \lambda_{1t} - A_1 \lambda_1 \end{aligned}$$

then, equation $(G_{3xx})_t = (G_{3t})_{xx}$ becomes

$$G_{3x} \lambda_1 + G_3 A_2 \lambda_1 + \lambda_2 = 0. \quad (25)$$

The compatibility analysis depends on the value of λ_1. A complete study of all cases is given here.

3.3.1. Case $\lambda_1 = 0$
From equation (25), one finds the condition

$$\lambda_2 = 0. \tag{26}$$

3.3.2. Case $\lambda_1 \neq 0$
Equation (25) provides the derivative

$$G_{3x} = -(G_3 A_2 \lambda_1 + \lambda_2)/\lambda_1. \tag{27}$$

Subtituting G_{3x} into G_{3xx} in equation (22), one arrives at the condition

$$\lambda_{2x} = (-A_{2t}\lambda_1^2 + \lambda_{1x}\lambda_2 + \lambda_1^3)/\lambda_1. \tag{28}$$

Comparing the mixed derivaties $(G_{3x})_t = (G_{3t})_x$, one gets the condition

$$\begin{aligned}\lambda_{2t} = -&(A_{0x}\lambda_1^2 - \lambda_{1t}\lambda_2 + A_0 A_2 \lambda_1^2 \\ &+ A_1\lambda_1\lambda_2 + \lambda_2^2)/\lambda_1.\end{aligned} \tag{29}$$

Combining all derived results in the case $F_x = 0$ the following theorems are proven.

Theorem 2.2: Sufficient conditions for equation (8) to be equivalent to a linear equation (6) via generalized linearizing transformation (7) with the function $F = F(t)$ is the equation (23) and the additional conditions are as follows.

(a) If $\lambda_1 = 0$, then the condition is equation (26).

(b) If $\lambda_1 \neq 0$, then the conditions are equations (28) and (29).

Theorem 2.3: Provided that the sufficient conditions in Theorem 2.2 are satisfied, the transformation (7) with the function $F = F(t)$ mapping equation (8) to a linear equation (6) is obtained by solving the compatible system of equations :

(a) (17), (19), (20), (21), (22), and (24).

(b) (17), (19), (20), (21), (24), and (27).

III. Some Applications

In this section we focus on finding some applications which satisfy Theorem 2.1, Theorem 2.2 and Theorem 2.3. The obtained results are as follows.

A. Parachute Equation

An application to this equation can be applied to a model of motion for a parachutist by using Newton's law II which is $\sum F = ma$. The movement of skydiver when the coefficient of air opposition changes between free-fall and the last consistent state drop with the parachute is slowly conveyed.

Consider the parachute equation [10], in the form

$$x'' - kx'^2 + g = 0, \tag{30}$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$.
Here $k = \frac{\pi\rho C_d D^2}{8m}$, where

- m is the mass of the body and parachute,
- ρ is the density of the fluid in which the body moves,
- C_d is the drag coefficient for the parachute (1.5 for parabolic profile and 0.75 for flat),
- D is the effective diameter of the parachute.

Equation (30) is an equation of the form (8) in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = k,\ A_1 = 0,\ A_0 = g,\ \lambda_1 = 0,\ \lambda_2 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. case (a). Thus, equation (30) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Theorem 2.3 case (a), which become

$$\begin{gathered}F_t = -\tfrac{K}{G_1},\ \ K_x = kK,\ \ K_t = \tfrac{K(2G_{1t}+G_1G_3k)}{G_1}, \\ G_{3t} = g + G_3^2 k,\ \ G_{3xx} = 0,\ \ G_{1x} = G_1 k.\end{gathered} \tag{31}$$

One can find the particular solution for equations in (31) as

$$\begin{gathered}G_1 = e^{kx},\ \ G_3 = \sqrt{\tfrac{g}{k}}i,\ \ G_2 = \sqrt{\tfrac{g}{k}}ie^{kx}, \\ K = e^{kx+\sqrt{kg}it},\ \ F = \tfrac{i}{\sqrt{kg}}e^{\sqrt{kg}it}.\end{gathered}$$

So that, one obtains the linearizing transformation

$$X = \frac{i}{\sqrt{kg}}e^{\sqrt{kg}it},\ dT = (e^{kx}x' + \sqrt{\frac{g}{k}}ie^{kx})dt. \tag{32}$$

Hence, equation (30) is mapped by the transformation (32) into the linear equation

$$X'' = 0. \tag{33}$$

The general solution of equation (33) is

$$X = c_1 T + c_2, \tag{34}$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (32) to equation (34), we obtain that the general solution of equation (30) is

$$\frac{i}{\sqrt{kg}}e^{\sqrt{kg}it} = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = (x' + \sqrt{\frac{g}{k}}i)e^{kx}.$$

B. Painlevé - Gambier XI Equation

In [11], Koudahoun, Akande, Adjai, Kpomahou and Monsia considered the Painlevé - Gambier XI equation

$$x'' + \frac{x'^2}{x} = 0. \tag{35}$$

To investigate the exact classical and quantum mechanical solutions, they offered a generalized singular differential equation of quadratic Lienard type.

By using our obtained theorems, we get the results as follow. Equation (35) is an equation of the form (8) in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = \frac{1}{x},\ A_1 = 0,\ A_0 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. case (a). Thus, equation (35) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Theorem 2.3 case (a), which become

$$\begin{gathered}F_t = -\tfrac{K}{G_1},\ \ K_x = \tfrac{K}{x},\ \ K_t = \tfrac{K(2G_{1t}x+G_1G_3)}{G_1 x}, \\ G_{3t} = \tfrac{G_3^2}{x},\ \ G_{3xx} = \tfrac{G_3}{x^2},\ \ G_{1x} = \tfrac{G_1}{x}.\end{gathered} \tag{36}$$

One can find the particular solution for equations in (36) as

$$G_1 = x,\ \ G_3 = 0,\ \ G_2 = 0,\ \ K = x,\ \ F = -t.$$

So that, one obtains the linearizing transformation

$$X = -t,\ dT = xx'dt. \quad (37)$$

Hence, equation (35) is mapped by the transformation (37) into the linear equation

$$X^{''} = 0. \quad (38)$$

The general solution of equation (38) is

$$X = c_1T + c_2, \quad (39)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (37) to equation (39), we obtain that the general solution of equation (35) is

$$-t = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = xx'.$$

C. Equation for the Variable Frequency Oscillator

In 2013, Mastafa, Al-Dueik and Mara'beh [12] considered the ordinary differential for the variable frequency oscillator

$$x'' + xx'^2 = 0. \quad (40)$$

They showed that this equation can be linearizable by generalized Sundman transformation.

By using our obtained theorems, we get the results as follow. Equation (40) is an equation of the form (8) in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = x,\ A_1 = 0,\ A_0 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2.2. case (a). Thus, equation (40) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Theorem 2.3 case (a), which become

$$\begin{gathered} F_t = -\frac{K}{G_1},\ \ K_x = xK,\ \ K_t = \frac{K(2G_{1t}+G_1G_3x)}{G_1}, \\ G_{3t} = G_3^2x,\ \ G_{3xx} = -G_3,\ \ G_{1x} = G_1x. \end{gathered} \quad (41)$$

One can find the particular solution for equations in (41) as

$$G_1 = e^{\frac{x^2}{2}},\ \ G_3 = 0,\ \ G_2 = 0,\ \ K = e^{\frac{x^2}{2}},\ \ F = -t.$$

So that, one obtains the linearizing transformation

$$X = -t,\ dT = e^{\frac{x^2}{2}}x'dt. \quad (42)$$

Hence, equation (40) is mapped by the transformation (42) into the linear equation

$$X^{''} = 0. \quad (43)$$

The general solution of equation (43) is

$$X = c_1T + c_2, \quad (44)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (42) to equation (44), we obtain that the general solution of equation (40) is

$$-t = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = e^{\frac{x^2}{2}}x'.$$

D. The One-Dimensional Non-Polynomial Oscillator

In the note [13], Mathew and Lakshmanan presented a remarkable nonlinear system that all its bounded periodic motions are simple harmonic. The system is a particle obeying the highly nonlinear equation of motion

$$(1 + \lambda x^2)x'' + (\alpha - \lambda x'^2)x = 0, \quad (45)$$

where λ and α are arbitrary parameters.

By using our obtained theorems, we get the results as follow. Equation (45) is an equation of the form (8) in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = -\frac{\lambda x}{(\lambda x^2+1)},\ A_1 = 0,\ A_0 = \frac{\alpha x}{(\lambda x^2+1)},$$

$$\lambda_1 = 0,\ \lambda_2 = \alpha\lambda x(-\lambda x^2 + 2).$$

One can check that the condition (23) in Theorem 2.2. case (a) are satisfied. Now, the condition (26) is satisfied when the following condition holds, that is,

$$\alpha\lambda x(-\lambda x^2 + 2) = 0.$$

Two cases arise, that are $\alpha = 0$ and $\lambda = \frac{2}{x^2}$. (Note that for $\lambda = 0$ equation (45) is linear equation.)

Here we consider only case $\alpha = 0$. In this case, the equation (45) takes the form

$$(1 + \lambda x^2)x''\lambda xx'^2 = 0. \quad (46)$$

The linearizing transformation is found by solving equations in Theorem 2.3 case (a), which become

$$\begin{gathered} F_t = -\frac{K}{G_1},\ \ K_x = -\frac{\lambda xK}{(1+\lambda x^2)}, \\ K_t = \frac{K(2G_{1t}\lambda x^2+2G_{1t}-\lambda xG_1G_3)}{G_1(1+\lambda x^2)}, \\ G_{3t} = -\frac{\lambda x^2G_3^2}{(1+\lambda x^2)},\ \ G_{3xx} = \frac{\lambda G_3(-\lambda x^2+1)}{(1+\lambda x^2)^2}, \\ G_{1x} = -\frac{\lambda xG_1}{(1+\lambda x^2)}. \end{gathered} \quad (47)$$

One can find the particular solution for equations in (47) as

$$\begin{gathered} G_1 = \frac{1}{(1+\lambda x^2)^{\frac{1}{2}}},\ \ G_3 = 0,\ \ G_2 = 0, \\ K = \frac{1}{(1+\lambda x^2)^{\frac{1}{2}}},\ \ F = -t. \end{gathered}$$

So that, one obtains the linearizing transformation

$$X = -t,\ dT = \frac{1}{(1+\lambda x^2)^{\frac{1}{2}}}x'dt. \quad (48)$$

Hence, equation (46) is mapped by the transformation (48) into the linear equation

$$X^{''} = 0. \quad (49)$$

The general solution of equation (49) is

$$X = c_1T + c_2, \quad (50)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (48) to equation (50), we obtain that the general solution of equation (46) is

$$-t = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{1}{(1+\lambda x^2)^{\frac{1}{2}}}x'.$$

E. Equation That Can Be Linearizable by Point and Sundman Transformations

Consider the nonlinear second-order ordinary differential equation

$$x'' + \mu_3 x^{k_3} x'^2 + \mu_2 x^{k_2} x' + \mu_1 x^{k_1} = 0, \quad (51)$$

where $k_3, k_2, k_1, \mu_1, \mu_2$ and $\mu_3 \neq 0$ are arbitrary constants. The Lie criteria [1], showed that the nonlinear equation (51) is linearizable by a point transformation if and only if $\mu_1 = 0$ and $\mu_2 = 0$. In [6], Nakpim and Meleshko showed that the nonlinear equation (51) is linearizable by a generalized Sundman transformation if and only if $\mu_2 \neq 0$ and $\mu_1 = 0$.

By using our obtained theorems, we get the results as follow. Equation (51) is an equation of the form (8) in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = \mu_3 x^{k_3},\ A_1 = \mu_2 x^{k_2},$$
$$A_0 = \mu_1 x^{k_1},\ \lambda_1 = k_2 \mu_2 x^{k_2},$$

$$\lambda_2 = x^{(k_1+k_3)}\mu_1\mu_3 k_1 x + x^{(k_1+k_3)}\mu_1\mu_3 k_3 x + x^{k_1}\mu_1 k_1^2 + x^{k_1}\mu_1 k_1 - x^{2k_2}\mu_2^2 k_2 x.$$

Now, the conditions in Theorem 2.2. case (a) is satisfied when the following conditions holds, that are,

$$k_2 \mu_2 x^{k_2} = 0,$$

$$x^{(k_1+k_3)}\mu_1\mu_3 k_1 x + x^{(k_1+k_3)}\mu_1\mu_3 k_3 x + x^{k_1}\mu_1 k_1^2 + x^{k_1}\mu_1 k_1 - x^{2k_2}\mu_2^2 k_2 x = 0.$$

Two cases arise.

Case 1: $\mu_2 = 0$ and $\mu_1 = 0$
In this case, the equation (51) takes the form

$$x'' + \mu_3 x^{k_3} x'^2 = 0. \quad (52)$$

The linearizing transformation is found by solving equations in Theorem 2.3 case (a), which become

$$F_t = -\frac{K}{G_1},\ K_x = \mu_3 x^{k_3} K,$$
$$K_t = \frac{K(2G_{1t} + \mu_3 x^{k_3} G_1 G_3)}{G_1},\ G_{3t} = \mu_3 x^{k_3} G_3^2, \quad (53)$$
$$G_{3xx} = -\frac{\mu_3 k_3 x^{k_3} G_3}{x},\ G_{1x} = \mu_3 x^{k_3} G_1.$$

One can find the particular solution for equations in (53) as

$$G_1 = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}},\ G_3 = 0,\ G_2 = 0,$$
$$K = e^{\frac{\mu_3 x^{k_3}}{k_3+1}},\ F = -t.$$

So that, one obtains the linearizing transformation

$$X = -t,\ dT = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}} x' dt. \quad (54)$$

Hence, equation (52) is mapped by the transformation (54) into the linear equation

$$X'' = 0. \quad (55)$$

The general solution of equation (55) is

$$X = c_1 T + c_2, \quad (56)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (54) to equation (56), we obtain that the general solution of equation (52) is

$$-t = c_1 \phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}} x',$$

where $k_3 \neq -1$.

For $k_3 = -1$, one can find the particular solution for equations in (53) as

$$G_1 = x^{\mu_3},\ G_3 = 0,\ G_2 = 0,\ K = x^{\mu_3},\ F = -t.$$

So that, one obtains the linearizing transformation

$$X = -t,\ dT = x^{\mu_3} x' dt. \quad (57)$$

Hence, equation (51) is mapped by the transformation (57) into the linear equation

$$X'' = 0. \quad (58)$$

The general solution of equation (58) is

$$X = c_1 T + c_2, \quad (59)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (57) to equation (59), we obtain that the general solution of equation (51) is

$$-t = c_1 \phi(t) + c_2,$$

where the funtion $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = x^{\mu_3} x'.$$

Case 2: $k_2 = 0$ and $\mu_1 = 0$
In this case, the equation (51) takes the form

$$x'' + \mu_3 x^{k_3} x'^2 + \mu_2 x' = 0. \quad (60)$$

The linearizing transformation is found by solving equations in Theorem 2.3 case (a), which become

$$F_t = -\frac{K}{G_1},\ K_x = K\mu_3 x^{k_3},$$
$$K_t = \frac{K(2G_{1t} + \mu_3 x^{k_3} G_1 G_3 - \mu_2 G_1)}{G_1},$$
$$G_{3t} = G_3(x^{k_3} G_3 \mu_3 - \mu_2),\ G_{3xx} = -\frac{G_3 \mu_3 k_3 x^{k_3}}{x}, \quad (61)$$
$$G_{1x} = G_1 \mu_3 x^{k_3}.$$

One can find the particular solution for equations in (61) as

$$G_1 = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}},\ G_3 = 0,\ G_2 = 0,$$
$$K = e^{\frac{\mu_3 x^{k_3}}{k_3+1} - \mu_2 t},\ F = \frac{e^{\mu_2 t}}{\mu_2}.$$

So that, one obtains the linearizing transformation

$$X = \frac{e^{\mu_2 t}}{\mu_2},\ dT = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}} x' dt. \quad (62)$$

Hence, equation (60) is mapped by the transformation (62) into the linear equation

$$X'' = 0. \quad (63)$$

The general solution of equation (63) is

$$X = c_1 T + c_2, \quad (64)$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (62) to equation (64), we obtain that the general solution of equation (60) is

$$\frac{e^{\mu_2 t}}{\mu_2} = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = e^{\frac{\mu_3 x^{k_3+1}}{k_3+1}} x',$$

where $k_3 \neq -1$.

For $k_3 = -1$, one can find the particular solution for equations in (61) as

$$G_1 = x^{\mu_3},\ \ G_3 = 0,\ \ G_2 = 0, \\ K = x^{\mu_3} e^{-\mu_2 t},\ \ F = \tfrac{e^{\mu_2 t}}{\mu_2}.$$

So that, one obtains the linearizing transformation

$$X = \frac{e^{\mu_2 t}}{\mu_2},\ dT = x^{\mu_3} x' dt. \tag{65}$$

Hence, equation (60) is mapped by the transformation (65) into the linear equation

$$X'' = 0. \tag{66}$$

The general solution of equation (66) is

$$X = c_1 T + c_2, \tag{67}$$

where c_1 and c_2 are arbitrary constants. Applying the generalized linearizing transformation (65) to equation (67), we obtain that the general solution of equation (60) is

$$\frac{e^{\mu_2 t}}{\mu_2} = c_1\phi(t) + c_2,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = x^{\mu_3} x',$$

Remark 3.1: The conditions in Theorem 2.2. case (b) are satisfied if only if $\mu_1 = 0$.

F. Modified Generalized Vakhnenko Equation

In 2009, Ma, Li and Wang [14] focus on a modified generalized Vakhnenko equation (mGVE),

$$\frac{\partial}{\partial x}(L^2 u + \frac{1}{2} p u^2 + \beta u) + qLu = 0,\ L = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{68}$$

where ρ, q, β are arbitrary non-zero constants.

To develop the specific solutions for mGVE is exceedingly significant. For models, when $\rho = \beta = 0$ and $q = 1$, equation (68) is reduced to notable Vakhnenko equation (VE), which oversees the nonlinear engendering of high-recurrence wave in a loosening up medium [15]-[17]. The VE has solition solutions [17]. When $\rho = q = 1$ and β an arbitrary non-zero constant, equation (68) is become as the generalized VE (GVE), in [18] it was indicated that GVE has N-soliton solution. When $\rho = 2q$ and β is an arbitrary non-zero constant, equation (68) has a loop-like, hump-like and cusp-like soliton solutions [19]. In [20], it was appeared that equation (68) has travelling wave solution and single-soliton solution.

Consider a modified generalized Vakhnenko equation (68), we can rewrite it in the form

$$2u_t u_{tx} + 2[uu_x u_{tx} + u_t(uu_{xx} + u_x^2)] + 2u^2 u_{xx} \\ +2uu_x^3 + \rho uu_x + \beta u_x + q(u_t + uu_x) = 0. \tag{69}$$

Of particular interest among solutions of equation (69) are travelling wave solutions:

$$u(t, x) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (69), one finds

$$2D^2 H'H'' - 2DH'(2HH'' + H'^2) \\ +2H^2 H'H'' + 2HH'^3 + \rho HH' + \beta H' \\ +q(-DH' + HH') = 0. \tag{70}$$

By using the obtained theorems, we get the results as follow. Equation (70) is an equation of the form in Theorem 2.1 with the coefficients

$$A_3 = 0,\ A_2 = -\tfrac{1}{(D-H)},\ A_1 = 0, \\ A_0 = \tfrac{\rho H + \beta - qD + qH}{2(D^2 - 2DH + H^2)},\ \lambda_1 = 0,\ \lambda_2 = \rho D + \beta.$$

From Theorem 2.2. case (a), equation (70) is linearizable if only if $\rho D + \beta = 0$.

G. Burgers' Equation

Burgers' equation is acquired because of the relationship between nonlinear wave movement and linear diffusion. It is the model for the investigation of consolidated impact of nonlinear advection and diffusion. The presence of the viscous term covers the wave-breaking, smooth out stun discontinuities, and thus we wish to get a tide and smooth solution. Also, as the dispersion term turns out to be vanishingly small, the smooth viscous solutions converge non-uniformly to the appropriate discontinuous shock wave, causing to another system for examining traditionalist nonlinear dynamical processes.

Consider the nonlinear convection-diffusion equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = 0, v > 0, \tag{71}$$

which is known as Burgers' equation. This equation balances between time advancement, nonlinearity, and dissemination. This is the nonlinear model equation for diffusive waves in fluid dynamics. Burgers (1948) first built up this equation basically to illuminate disturbance depicted by the collaboration of two inverse impacts of convection and dissemination.

The term uu_x will have a stunning up impact that will make waves break and the term vu_{xx} is a diffusion term like the one appearing in the heat equation.

Of particular interest among solutions of equation (71) are travelling wave solutions:

$$u(t, x) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (71), one finds

$$-DH' + HH' - vH'' = 0. \tag{72}$$

By using the obtained theorems, we get the results as follow. Equation (72) is an equation of the form in Theorem 2.1 with the coefficients

$$A_3 = 0, A_2 = 0, A_1 = \frac{D-H}{v}, A_0 = 0,$$
$$\lambda_1 = \frac{1}{v}, \lambda_2 = \frac{-D+H}{v^2}.$$

One can check that these coefficients obey the condition in Theorem 2.2. case (b). Thus, equation (72) is linearizable via a generalized linearizing transformation.

IV. Conclusion

In this paper, the necessary condition which guarantee that the second-order ordinary differential equation can be linearized by generalized linearizing transformation is found in Theorem 2.1. Theorem 2.2 case (a) and case (b) are sufficient conditions for the linearization problem, they are selected by the value of λ_1. A new algorithm for finding linearizing transformation is summarized in Theorem 2.3. Finally, some applications are provided to demonstrate our procedure.

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Prakrong Voraka received her B.S. degree in mathematics from Khon Kaen University, Khon Kaen, Thailand, in 2001 and her M.S. degree in mathematics from Kasetsart University, Bangkok, Thailand, in 2003. She received Ph.D. degree in applied mathematics in 2008 from Suranaree University of Technology, Nakhonratchasima, Thailand. Presently, she is a professor with the Department of General Science, Faculty of Science and Engineering, Kasetsart University Chalermphrakiat Sakon Nakhon Province Campus, Sakon Nakhon, Thailand. Her research interests are Lie group analysis, linearization problems and differential equations.

Supaporn Suksern received her B.S. degree in mathematics from Chiangmai University, Chiangmai, Thailand, in 2001 and her M.S. degree in applied mathematics from Chiangmai University, Chiangmai, Thailand, in 2004. She received Ph.D. degree in applied mathematics in 2009 from Suranaree University of Technology, Nakhonratchasima, Thailand. Presently, she is a professor with the Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, Thailand. Her research interests are linearization problems and differential equations.

Nontakan Donjiwprai received his B.S. degree in mathematics from Naresuan University, Phitsanulok, Thailand, in 2020.