Existence of Positive Solutions for a State-dependent Hybrid Functional Differential Equation

EL-Sayed A.M and *Ebead H.R

Abstract—**In this paper, we study the existence of positive solutions for an initial value problem (IVP) of a state-dependent hybrid functional differential equation. The continuous dependence of the unique solution will be proved. Some special cases and examples will be given.**

Index Terms—**Hybrid differential equations, state-dependent, quadratic integral equation, existence of positive solutions, continuous dependence, fixed point theorem.**

I. Introduction

IN the last years, quadratic perturbations of nonlinear equations have received a lot of attention from specialists. This type of equations called hybrid equations. Many specialists work on the theory of hybrid equations, see for example [9], [22], [30].

Usually, the equations of deviating arguments are with deviation depends only on the time, however, when the deviation of the arguments depend upon both the state variable ξ and also the time t is incredibly important theoretically and practically. This type of equations is named self-reference or state-dependent equations.

Equations which have state-dependent delays attract the attention of specialists since it have plenty of application models, like the two-body problem of classical electrodynamics, also have many applications in the class of problems that have past memories, for example in hereditary phenomena, see [15-18]. Several papers have devoted to this category of equations, see for instance [1], [7]-[16], [19], [20] and [21]-[29].

One of the first studies in this field was introduced by Eder [11], the author studied the problem

$$\xi'(t) = \xi\big(\xi(t)\big) \quad t \in A \subset R.$$

with the condition $\xi(t_0) = \xi_o$.

A generalization of Eder's results was introduced by Fečkan [19], the author worked on the problem

$$\xi'(t) = \psi\big(\xi(\xi(t))\big), \quad t \in A \subset R$$

where $\psi \in C^1(R)$.

The problem

$$\begin{aligned} \xi'(t) &= \psi\ (t, \xi(\xi(t))), \quad t \in [a,b] \\ \xi(0) &= \xi_o \end{aligned}$$

was studied by Buicá [8] where ψ satisfied Lipshitz condition.

EL-Sayed and Ebead [16] relaxed the assumptions of Buicá and generalized their results, they studied the delay-refereed problem

$$\xi(t) = \psi\ (t, \int_0^t \eta\big(s, \xi(\xi(s))\big)\ ds), \quad t \in [0,T]$$

where η satisfies Caratheodory condition.

El-Sayed and Ebead [14] and [15] studied the delay-refereed problem

$$\begin{aligned} \frac{d}{dt}\,\xi(t) &= \psi\ (t, \xi(\eta(t, \xi(t)))), \quad a.e.\ \ t \in (0,T], \\ \xi(0) &= \xi_0 \end{aligned}$$

under the two cases:

(i) $\eta(t,\xi) \le t$.

(ii) $\eta(t,\xi) \le \xi$.

Here we shall study the IVP of a state-dependent hybrid functional differential equation

$$\frac{d}{dt}\left\{\frac{\xi(t) - \lambda(t)}{\eta(t, \xi(\xi(\vartheta(t))))}\right\} = \psi\ (t, \xi(\xi(\vartheta(t))), \quad a.e.\ \ t \in [0,T] \tag{1}$$

$$\xi(0) = A(0). \tag{2}$$

Our aim in this work is to prove the existence of positive solution of (1)-(2). The continuous dependence of the unique solution on the functions λ, ψ will be proved. To illustrate our results some examples will be given.

This paper consists of six sections. Section 1 is the introduction and a brief survey of the topic. In Section 2, we introduce the main theorem (existence). We prove the uniqueness theorem in Section 3. Continuous dependence of the solution has been proved in Section 4. Some examples was introduced by Section 5. Finally, in Section 6; as an application to our work, we give a numerical example, we use successive approximation method to give an estimation for the solution of a problem of this type.

II. Existence of solutions

First, We provide proof of the existence of positive solutions $\xi \in C[0,T]$ for the state-dependent equation

$$\begin{aligned} \xi(t) \;&=\; \lambda(t) + \eta(t, \xi(\xi(\vartheta(t))) \int_0^t \psi\ (s, \xi(\xi(\vartheta(s)))\ ds, \\ & t \in [0,T], \end{aligned} \tag{3}$$

under the following assumptions:

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EL-Sayed A.M.A is with the Department of Mathematics, Faculty of Sciences, Alexandria University, Egypt, Emails:amasayed@alexu.edu.eg.

Ebead H.R is with the Department of Mathematics, Faculty of Sciences, Alexandria University, Egypt, Emails:HanaaRezqalla@alexu.edu.eg.

(1) $\lambda : [0,T] \longrightarrow R^+$ is continuous and there exists a positive constant c such that:
$$|\lambda(t_2) - \lambda(t_1)| \leq c\ |t_2 - t_1|.$$

(2) $\eta : [0,T] \times [0,T] \longrightarrow R^+$ is continuous, non zero on $[0,T]$ and
$$|\eta(t_2, \xi) - \eta(t_1, y)| \leq k_1|t_2 - t_1| + k_2|\xi - y|,$$
$|\eta(t,0)| \leq A_2$

where $k_1,\ k_2$ and A_2 are positive constants.

(3) $\psi : [0,T] \times [0,T] \longrightarrow R^+$ satisfies Carathéodory condition.

(4) There exists a function $m : [0,T] \longrightarrow R^+$ where m is measurable and bounded with $m(t) \leq A_1,\ t \in [0,T]$ and a constant $k_3 \geq 0$ such that:
$$\psi\ (t,\xi) \leq m(t) + k_3\ \xi.$$

(5) $\vartheta : [0,T] \longrightarrow [0,T],\ \vartheta(0) = 0$ and
$$|\vartheta(t) - \vartheta(s)| \leq |t - s|.$$
This assumption implies that $\vartheta(t)\ \leq\ t.$

(6) There exists a real solution $L \in (0,1)$ for the equation
$$k_2 M_1 T L^2 - L + (c + M_1 M_2 + k_1 M_1 T) = 0$$
where $M_1 = A_1 + k_3 T,\ \ M_2 = A_2 + k_2 T.$

(7) $L\ T + |\xi(0)| \leq T.$

Theorem 2.1. Let the assumptions (1) – (7) be satisfied, then the state-dependent equation (3) has a positive solution $\xi \in C[0,T]$.

Proof. Define the set S_L by
$$S_L = \{\xi \in C[0,T]\ :\ |\xi(t_2) - \xi(t_1)|\ \leq\ L\,|t_2 - t_1|\} \subset C[0,T].$$
Now define the operator H associated with (3) by:
$$H\ \xi(t)\ =\ \lambda(t) + \eta(t, \xi(\xi(\vartheta(t)))\ \int_0^t\ \psi\ (s, \xi(\xi(\vartheta(s)))ds, \quad t \in [0,T].$$
First, we show that $\{H\ \xi\}$ is uniformly bounded on S_L. Let $\xi \in C[0,T]$, then
$$|H\ \xi(t)|\ \leq\ |\lambda(t)| + |\eta\big(t, \xi(\xi(\vartheta(t)))\big)| \int_0^t |\psi\big(s, \xi(\xi(\vartheta(s)))\big)|ds.$$
Using Assumptions (2) we can get
$$\begin{aligned} & |\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) - \eta\big(t_1, \xi(\xi(\vartheta(t_1)))\big)| \\ \leq\ & k_1\ |t_2 - t_1| + k_2\ |\xi(\xi(\vartheta(t_2))) - \xi(\xi(\vartheta(t_1)))| \\ \leq\ & k_1\ |t_2 - t_1| + k_2\ L\ |\xi(\vartheta(t_2)) - \xi(\vartheta(t_1))| \\ \leq\ & k_1\ |t_2 - t_1| + k_2\ L^2\ |\vartheta(t_2) - \vartheta(t_1)| \\ \leq\ & k_1\ |t_2 - t_1| + k_2\ L^2\ |t_2 - t_1|, \end{aligned} \tag{4}$$
and
$$\begin{aligned} & |\eta\big(t, \xi(\xi(\vartheta(t)))\big)| \leq k_2|\xi(\xi(\vartheta(t)))| + |\eta(t,0)| \\ \leq\ & k_2|\xi(\xi(\vartheta(t)))| + A_2 \end{aligned} \tag{5}$$
and from Assumptions (4) we can deduce that
$$|\psi\big(t, \xi(\xi(\vartheta(t)))\big)| \leq k_3|\xi(\xi(\vartheta(t)))| + m(t) \tag{6}$$
$$\leq\ k_3|\xi(\xi(\vartheta(t)))| + A_1. \tag{7}$$
Also $x \in S_L$ implies that
$$\begin{aligned} & |\xi(\xi(\vartheta(t)))| \leq |\xi(\xi(\vartheta(t))) - \xi(0)|\ +\ |\xi(0)| \\ \leq\ & L\ |\xi(\vartheta(t))| + |\xi(0)| \\ \leq\ & L\ t + |\xi(0)| \\ \leq\ & L\ T + |\xi(0)| \leq T. \end{aligned} \tag{8}$$
From (5)-(8), we obtain
$$\begin{aligned} |H\ \xi(t)|\ &\leq\ |\lambda\ (t)| + (k_2 T + A_2\big) \int_0^t \big(k_3 T + A_1\big)\ ds \\ &\leq\ |\lambda(t)| + M_1\ M_2\ T. \end{aligned}$$
Assumption (1) implies that
$$\begin{aligned} |\lambda(t)|\ &\leq\ |\lambda(t) - \lambda(0)| + |\lambda(0)| \\ &\leq\ c\ t + |\xi(0)| \\ &\leq\ c\ T + |\xi(0)|. \end{aligned}$$
Hence
$$\begin{aligned} |H\ \xi(t)|\ &\leq\ (c\ + M_1\ M_2)\ T + |\xi(0)| \\ &=\ \big(L - \big(k_1 + k_2 L^2\big) M_1 T\big)\ T + |\xi(0)| \\ &\leq\ L\ T + |\xi(0)| \leq T. \end{aligned}$$
This proves that $\{H\ \xi\}$ is uniformly bounded on S_L.
Next, we show that $H : S_L \longrightarrow S_L$ and $\{H\ \xi\}$ is equi-continuous on the set S_L.
Let $x \in S_L$ and $t_1,\ t_2 \in [0,T]$ with $|t_2, -t_1| < \delta$, then
$$\begin{aligned} & |H\ \xi(t_2) - H\ \xi(t_1)| = \big|\ \lambda(t_2) - \lambda(t_1) \\ +\ & \eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) \int_0^{t_2} \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds \\ -\ & \eta\big(t_1, \xi(\xi(\vartheta(t_1)))\big) \int_0^{t_1} \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds| \\ =\ & \big|\ \lambda(t_2) - \lambda(t_1) + \eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) \\ & \Big(\int_0^{t_2} \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds - \int_0^{t_1} \psi\big(s, \xi(\xi(\vartheta(s))\big)\ ds\Big) \\ +\ & \big(\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) - \eta\big(t_1, \xi(\xi(\vartheta(t_1)))\big)\big) \\ & \int_0^{t_1} \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds| \\ \leq\ & |\lambda(t_2) - \lambda(t_1)| + |\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big)| \\ & |\int_{t_1}^{t_2} \psi\ \big(s, \xi(\xi(\vartheta(s)))\big)\ ds| \\ +\ & |\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) - \eta\big(t_1, \xi(\xi(\vartheta(t_1)))\big)| \\ & |\int_0^{t_1} \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds| \\ \leq\ & |\lambda(t_2) - \lambda(t_1)| + |\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big)| \\ & \int_{t_1}^{t_2} |\psi\big(s, \xi(\xi(\vartheta(s)))\big)|ds \end{aligned}$$

$$+ \quad |\eta\big(t_2, \xi(\xi(\vartheta(t_2)))\big) - \eta\big(t_1, \xi(\xi(\vartheta(t_1)))\big)| \int_0^{t_1} |\psi\big(s, \xi(\xi(\vartheta(s)))\big)| ds.$$

From (4)-(8) we can get

$$\begin{aligned}
& |H\ \xi(t_2) - H\ \xi(t_1)| \\
\leq\ & c|t_2 - t_1| + \big(k_2|\xi(\xi(\vartheta(t_2)))| + A_2\big) \\
& \int_{t_1}^{t_2} \big(k_3|\xi(\xi(\vartheta(s)))| + A_1\big) ds \\
+\ & \big(k_1 + k_2\ L^2\big)\ |t_2 - t_1| \int_0^{t_1} \big(k_3|\xi(\xi(\vartheta(s)))| + A_1\big) ds \\
\leq\ & c|t_2 - t_1| + \big(k_2\ T + A_2\big)\big(k_3\ T + A_1\big)\ |t_2 - t_1| \\
+\ & \big(k_1 + k_2\ L^2\big)\ \big(k_3\ T + A_1\big)\ |t_2 - t_1|\ t_1 \\
\leq\ & c|t_2 - t_1| + M_2 M_1 |t_2 - t_1| \\
+\ & \big(k_1 + k_2 L^2\big) M_1 T\ |t_2 - t_1| \\
=\ & \big(c + M_2 M_1 + \big(k_1 + k_2\ L^2\big) M_1 T\big)|t_2 - t_1| \\
=\ & L\ |t_2 - t_1|.
\end{aligned}$$

Then we proved that $H : S_L \longrightarrow S_L$ and $\{H\ \xi\}$ is equi-continuous on S_L. Applying Arzela-Ascoli Theorem ([24] page (54)), we deduce that H is compact.

Next, we show that H is continuous. Let $\{\xi_n\} \subset S_L$ be such that $x_n \longrightarrow x$, $(i.e, |\xi_n(t) - \xi(t)| \leq \epsilon\ )$ this implies that $|\xi_n(\xi(\vartheta(t))) - \xi(\xi(\vartheta(t)))| \leq \epsilon^*$, then for arbitrary $\epsilon,\ \epsilon^* \geq 0$, we have

$$\begin{aligned}
& |\xi_n(\xi_n(\vartheta(t))) - \xi(\xi(\vartheta(t)))| \\
=\ & |\xi_n(\xi_n(\vartheta(t))) - \xi_n(\xi(\vartheta(t))) \\
+\ & \xi_n(\xi(\vartheta(t))) - \xi(\xi(\vartheta(t)))| \\
\leq\ & |\xi_n(\xi_n(\vartheta(t))) - \xi_n(\xi(\vartheta(t)))| \\
+\ & |\xi_n(\xi(\vartheta(t))) - \xi(\xi(\vartheta(t)))| \\
\leq\ & L\ |\xi_n(\vartheta(t)) - \xi(\vartheta(t))| \\
+\ & |\xi_n(\xi(\vartheta(t))) - \xi(\xi(\vartheta(t)))| \\
\leq\ & L\ \epsilon + \epsilon^*
\end{aligned}$$

which implies that

$$\xi_n(\xi_n(\vartheta(t)))) \longrightarrow (\xi(\xi(\vartheta(t))))\ in\ S_L.$$

Now from assumptions (2-4) and Lebesgues dominated convergence Theorem ([10] page(151)), we get

$$\lim_{n\longrightarrow\infty} \eta\big(t, \xi_n(\xi_n(\vartheta(t)))\big) = \eta\big(t, \xi(\xi(\vartheta(t)))\big)$$

and

$$\lim_{n\longrightarrow\infty} \int_0^t \psi\big(s, \xi_n(\xi_n(\vartheta(s)))\big) ds = \int_0^t \psi\big(s, \xi(\xi(\vartheta(s)))\big) ds,$$

hence

$$\begin{aligned}
& \lim_{n\longrightarrow\infty} \big(H\xi_n\big)(t) =\ \lambda\ (t) + \lim_{n\longrightarrow\infty} \eta\big(t, \xi_n(\xi_n(\vartheta(t)))\big) \\
& \lim_{n\longrightarrow\infty} \int_0^t \psi\big(s, \xi_n(\xi_n(\vartheta(s)))\big) ds \\
=\ & \lambda\ (t) + \eta\big(t, \xi(\xi(\vartheta(t)))\big) \int_0^t \psi\big(s, \xi(\xi(\vartheta(s)))\big) ds \\
=\ & \big(H\ \xi\big)(t).
\end{aligned}$$

This proves that H is continuous. Now according to ([23] page (482)), there exist solutions $\xi \in C[0,T]$ of (3).

Also, from our assumptions, we can get

$$\begin{aligned}
\xi(t) \ &=\ \lambda(t) + \eta(t, \xi(\xi(\vartheta(t))) \int_0^t \psi\ (s, \xi(\xi(\vartheta(s)))ds\ \geq 0, \\
& t \in [0,T].
\end{aligned}$$

Theorem 2.2. Let the assumptions $(1) - (7)$ be satisfied, then (1)-(2) has positive solutions $\xi \in C[0,T]$.

Proof. Let ξ be a solution of (1)-(2). Integrate (1), then substitute by (2), we get (3). Let ξ be a solution of (3). Differentiate (3) we obtain (1) and the initial value (2).

This proves the equivalence between (1)-(2) and (3). Then (1)-(2) has positive solutions $\xi \in C[0,T]$.

The next corollary relax the assumptions and generalize the results in [8] and [19].

Corollary 2.1. Using to Theorem 2.1 with $\eta(t,\xi) = 1$, then the state-dependent equation

$$\xi(t) = \lambda(t) + \int_0^t \psi\big(s, \xi(\xi(\vartheta(s)))\big)\ ds, \quad t \in [0,T]$$

has solution $\xi \in C[0,T]$. Consequently if $\lambda(t) = \xi_0$, then

$$\begin{aligned}
\xi(t)' \ &=\ \psi\ (t, \big(\xi(\xi(\vartheta(t)))\big)\ \ a.e. \quad t \in (0,T] \qquad (9) \\
\xi(0) \ &=\ \xi_0 \qquad (10)
\end{aligned}$$

has positive solutions $\xi \in C[0,T]$.

III. Uniqueness of the solution

Here we introduce the uniqueness theorem of (1)-(2). For this aim, we assume that:

$(1')$ $|\psi\ (t,\xi) - \psi\ (t,y)| \leq k_3\ |\xi - y|$
$(2')$ $\sup |\psi\ (t,0)| \leq A_1, \quad t \in [0,T]$.

Theorem 3.3. Let the assumptions (1)-(3), (5)-(7), $(1')$ *and* $(2')$ be satisfied, if

$$(M_2 k_3 + M_1 k_2)(L+1)T < 1,$$

then The solution of (1) and (2) is unique.

Proof. From assumptions $(1')$ *and* $(2')$ we can deduce assumption (4)

$$\begin{aligned}
|\psi\ (t,\xi)| \ &\leq\ k_3\ |\xi| + |\psi\ (t,0)| \\
&\leq\ k_3\ |\xi| + A_1,
\end{aligned}$$

thus using Theorem 2.1, (3) has at least a solution. Now, if (3) has two solutions $\xi\ and\ y$, then

$$\begin{aligned}
& |\xi(t)-y(t)| \\
= & \Big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)\int_0^t \psi\big(s,\xi(\xi(\vartheta(s)))\big)ds \\
- & \eta\big(t,y(y(\vartheta(t)))\big)\int_0^t \psi\big(s,y(y(\vartheta(s)))\big)ds\Big| \\
= & \Big|\eta\big(t,\xi(\xi(\vartheta(t)))\big) \\
& \int_0^t \big[\psi\big(s,\xi(\xi(\vartheta(s)))\big)-\psi\big(s,y(y(\vartheta(s)))\big)\big]ds \\
+ & \big[\eta\big(t,\xi(\xi(\vartheta(t)))\big)-\eta\big(t,y(y(\vartheta(t)))\big)\big] \\
& \int_0^t \psi\big(s,y(y(\vartheta(s)))\big)ds\Big| \\
\leq & \big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)\big| \\
& \int_0^t \big|\psi\big(s,\xi(\xi(\vartheta(s)))\big)-\psi\big(s,y(y(\vartheta(s)))\big)\big|ds \\
+ & \big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)-\eta\big(t,y(y(\vartheta(t)))\big)\big| \\
& \int_0^t \big|\psi\big(s,y(y(\vartheta(s)))\big)\big|ds \\
\leq & (k_2\ T+A_2)\ k_3\int_0^t |\xi(\xi(\vartheta(s)))-y(y(\vartheta(s)))|ds \\
+ & k_2\ |\xi(\xi(\vartheta(t)))-y(y(\vartheta(t)))|\int_0^t \big(k_3\ T+A_1\big)ds \\
\leq & M_2\ k_3\int_0^t |\xi(\xi(\vartheta(s)))-y(y(\vartheta(s)))|ds \\
+ & M_1\ k_2\ T\ |\xi(\xi(\vartheta(t)))-y(y(\vartheta(t)))|
\end{aligned}$$

and

$$\begin{aligned}
& |\xi(\xi(\vartheta(t)))-y(y(\vartheta(t)))| \\
= & |\xi(\xi(\vartheta(t)))-\xi(y(\vartheta(t))) \\
+ & \xi(y(\vartheta(t)))-y(y(\vartheta(t)))| \\
\leq & |\xi(\xi(\vartheta(t)))-\xi(y(\vartheta(t)))| \\
+ & |\xi(y(\vartheta(t)))-y(y(\vartheta(t)))| \\
\leq & L|\xi(\vartheta(t))-y(\vartheta(t))| \\
+ & |\xi(y(\vartheta(t)))-y(y(\vartheta(t)))| \\
\leq & L\|\xi-y\|+\|\xi-y\| \\
= & (L+1)\|\xi-y\|,
\end{aligned}$$

thus we have

$$\|\xi-y\| \ \leq\ \big(M_2\ k_3+M_1\ k_2\big)\ (L+1)\ T\|\xi-y\|$$

and

$$\big(1-(M_2k_3+M_1k_2)(L+1)T\big)\ ||\xi-y||\leq 0,$$

since $(M_2k_3+M_1k_2)(L+1)T<1$, then we get $\xi=y$ and the solution of (3) is unique. Consequently the solution of (1)-(2) is unique.

Corollary 3.2. According to Theorem 3.3, let $\eta(t,\xi)=1$ and $\lambda(t)=\xi_0$, then (9) and (10) has a unique positive solution $\xi\in C(0,T]$.

Now, as in [17] and [18], we introduce the next theorem.

Theorem 3.4. According to Theorem 3.3, we can approximate the numerical solution of (3) by

$$\xi(t)\ =\ \lim_{n\longrightarrow\infty}\xi_n(t)$$

where $\xi_n(t)$ is constructed by

$$\begin{aligned}
\xi_n(t) &= \lambda(t)+\eta\big(t,\xi_{n-1}(\xi_{n-1}(\vartheta(t)))\big) \\
& \int_0^t \psi\big(s,\xi_{n-1}(\xi_{n-1}(\vartheta(s)))\big)ds, \\
\xi_0(t) &= \lambda(t).
\end{aligned}$$

IV. Continuous dependence

A. Continuous dependence on the function λ

Definition 4.1. The solution of (1) and (2) continuously depends on λ if $\forall\epsilon_1>0\ \exists\ \delta_1(\epsilon_1)>0$ such that:

$$\|\lambda-\lambda^*\|\leq\delta_1\Longrightarrow\|\xi-\xi^*\|\leq\epsilon_1$$

where

$$\frac{d}{dt}\left\{\frac{\xi^*(t)-\lambda^*(t)}{\eta\big(t,\xi^*(\xi^*(t))\big)}\right\}=\psi\big(s,\xi^*(\xi^*(s))\big),\quad a.e.\quad t\in[0,T] \tag{11}$$

with the initial data

$$\xi^*(0)=\lambda^*(0). \tag{12}$$

Theorem 4.5. According to Theorem 3.3, the solution of (1) and (2) continuously depends on λ.

Proof. Assume that the functions ξ and ξ^* are the two unique solutions of (1)-(2) and (11)-(12) respectively, thus we have

$$\begin{aligned}
& |\xi(t)-\xi^*(t)| \\
= & \Big|\lambda(t)+\eta\big(t,\xi(\xi(\vartheta(t)))\big)\int_0^t \psi\big(s,\xi(\xi(\vartheta(s)))\big)ds \\
- & \lambda^*(t)-\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\int_0^t \psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds\Big| \\
= & \Big|\lambda(t)-\lambda^*(t)+\eta\big(t,\xi(\xi(\vartheta(t)))\big) \\
& \Big(\int_0^t \psi\big(s,\xi(\xi(\vartheta(s)))\big)ds-\int_0^t \psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds\Big) \\
+ & \Big(\eta\big(t,\xi(\xi(\vartheta(t)))\big)-\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\Big) \\
& \int_0^t \psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds \\
\leq & |\lambda(t)-\lambda^*(t)|+\big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)\big| \\
& \int_0^t \big|\psi\big(s,\xi(\xi(\vartheta(s)))\big)-\psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)\big|ds \\
+ & \Big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)-\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\Big| \\
& \int_0^t \big|\psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)\big|ds \\
\leq & \delta_1+(k_2|\xi(\xi(\vartheta(t)))|+A_2) \\
& \Big(\int_0^t k_3|\xi(\xi(\vartheta(s)))-\xi^*(\xi^*(\vartheta(s)))|ds\Big) \\
+ & k_2|\xi(\xi(\vartheta(t)))-\xi^*(\xi^*(\vartheta(t)))| \\
& \Big(\int_0^t (k_3|\xi^*(\xi^*(\vartheta(s)))|+A_1)ds\Big) \\
\leq & \delta_1+(k_2\ T+A_2)\ k_3\ \int_0^t |\xi(\xi(\vartheta(s)))-\xi^*(\xi^*(\vartheta(s)))|ds \\
+ & k_2|\xi(\xi(\vartheta(s)))-\xi^*(\xi^*(\vartheta(s)))|\ (k_3\ T+A_1)\ \int_0^t ds \\
\leq & \delta_1+M_2\ k_3(L+1)\|\xi-\xi^*\|\ t \\
+ & k_2\ M_1(L+1)\|\xi-\xi^*\|\ t
\end{aligned}$$

$$\begin{aligned} &\leq \delta_1 + M_2\ k_3(L+1)\|\xi-\xi^*\|\ T \\ &+ k_2\ M_1(L+1)\|\xi-\xi^*\|\ T \\ &= \delta_1 + (k_2\ M_1 + k_3\ M_2)\ (L+1)\ T\ \|\xi-\xi^*\| \end{aligned}$$

thus

$$\|\xi-\xi^*\|\big(1-(k_2\ M_1+M_2\ k_3)\ (L+1)\ T\big) \leq \delta_1$$

and

$$\|\xi-\xi^*\| \leq \frac{\delta_1}{\big(1-(k_2\ M_1+M_2\ k_3)\ (L+1)\ T\big)} = \epsilon_1$$

since $(k_2\ M_1+M_2\ k_3)\ (L+1)\ T<1$, then the solution of (3) continuously depends on the functions λ. Consequently the solution of (1) and (2) continuously depends on λ.

B. Continuous dependence on the function ψ

Definition 4.2. The solution of (1) and (2) continuously depends on the function $\psi,\ if\ \forall\ \epsilon_2>0\ \exists\ \delta_2(\epsilon_2)>0\ such\ that:$

$$\|\psi-\psi^*\| \leq \delta_2 \Longrightarrow \|x-\xi^*\| \leq \epsilon_2$$

where

$$\frac{d}{dt}\left\{\frac{\xi^*(t)-\lambda(t)}{\eta\big(t,\xi^*(\xi^*(t))\big)}\right\} = \psi^*\big(s,\xi^*(\xi^*(s))\big), \quad a.e.\quad t\in[0,T] \tag{13}$$

with the initial data

$$\xi^*(0)=\lambda(0). \tag{14}$$

Theorem 4.6. According to Theorem 3.3, the solution of (1) and (2) continuously depends on the function ψ.

Proof. Assume that ξ and ξ^* are the two unique solutions of (1)-(2) and (13)-(14) respectively, then we get

$$\begin{aligned} & |\xi(t)-\xi^*(t)| \\ =\ & \Big|\lambda(t)+\eta\big(t,\xi(\xi(\vartheta(t)))\big)\int_0^t \psi\big(s,\xi(\xi(\vartheta(s)))\big)ds \\ -\ & \lambda(t)-\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\int_0^t \psi^*\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds\Big| \\ =\ & \Big|\eta\big(t,\xi(\xi(\vartheta(t)))\big) \\ & \bigg(\int_0^t \psi\big(s,\xi(\xi(\vartheta(s)))\big)ds \\ -\ & \int_0^t \psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds\bigg) \\ +\ & \eta\big(t,\xi(\xi(\vartheta(t)))\big)\int_0^t \psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds \\ -\ & \eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\int_0^t \psi^*\big(s,\xi^*(\xi^*(\vartheta(s)))\big)ds\Big| \\ \leq\ & \big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)\big| \\ & \int_0^t \big|\psi\big(s,\xi(\xi(\vartheta(s)))\big)-\psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)\big|ds \\ +\ & \Big|\eta\big(t,\xi(\xi(\vartheta(t)))\big)-\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\Big| \\ & \int_0^t \big|\psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)\big|ds + \big|\eta\big(t,\xi^*(\xi^*(\vartheta(t)))\big)\big| \\ & \Big(\int_0^t \big|\psi\big(s,\xi^*(\xi^*(\vartheta(s)))\big)-\psi^*\big(s,\xi^*(\xi^*(\vartheta(s)))\big)\big|ds\Big) \\ \leq\ & (k_2|\xi(\xi(\vartheta(t)))|+A_2) \\ & \Big(\int_0^t k_3|\xi(\xi(\vartheta(s)))-\xi^*(\xi^*(\vartheta(s)))|ds\Big) \\ +\ & k_2|\xi(\xi(\vartheta(t)))-\xi^*(\xi^*(\vartheta(t)))| \\ & \Big(\int_0^t (k_3|\xi^*(\xi^*(\vartheta(s)))|+A_1)ds\Big) \\ +\ & (k_2|\xi^*(\xi^*(\vartheta(t)))|+A_2)\ \int_0^t \delta_2 ds \\ \leq\ & (k_2\ T+A_2)\int_0^t k_3(L+1)\|\xi-\xi^*\|ds \\ +\ & k_2(L+1)\|\xi-\xi^*\|\int_0^t (k_3\ T+A_1)ds \\ +\ & (k_2\ T+A_2)\ t\delta_2 \\ \leq\ & M_2\ k_3(L+1)\|\xi-\xi^*\|\ t \\ +\ & k_2\ M_1(L+1)\|\xi-\xi^*\|\ t+M_2\ T\delta_2 \\ \leq\ & M_2\ k_3(L+1)\|\xi-\xi^*\|\ T \\ +\ & k_2\ M_1(L+1)\|\xi-\xi^*\|\ T+M_2\ T\delta_2 \\ =\ & (k_2\ M_1+k_3\ M_2)\ (L+1)\ T\ \|\xi-\xi^*\|+M_2\ T\ \delta_2, \end{aligned}$$

thus

$$\|\xi-\xi^*\|\big(1-(k_2\ M_1+M_2\ k_3)\ (L+1)\ T\big) \leq M_2\ T\ \delta_2$$

and

$$\|\xi-\xi^*\| \leq \frac{M_2\ T\ \delta_2}{\big(1-(k_2\ M_1+M_2\ k_3)\ (L+1)\ T\big)} = \epsilon_2$$

since $(k_2\ M_1+M_2\ k_3)\ (L+1)\ T<1$, then the solution of (1) and (2) continuously depends on ψ.

V. Examples

Example 5.1. Consider the state-dependent problem

$$\begin{aligned} & \frac{d}{dt}\left\{\frac{\xi(t)-\frac{1+2t}{32}}{\big(\frac{1}{12}t+\frac{5}{32}|\xi\big(\xi(\beta\ t)\big)|\big)}\right\} \\ =\ & \frac{1}{5-t}\ \big|\sin\big(5(t+1)\big)\big| + \frac{e^{-t}}{15}\ \frac{|\xi\big(\xi(\beta\ t)\big)|}{1+|\xi\big(\xi(\beta\ t)\big)|} \end{aligned} \tag{15}$$

with the initial data

$$\xi(0)=\frac{1}{32}. \tag{16}$$

where $t\in(0,2]$ and $\beta\in(0,1]$. Here we have: $\vartheta(t)=\beta\ t$,

$$\lambda(t)=\frac{1+2t}{32},$$

$$\eta\big(t,\xi(\xi(\vartheta(t)))\big)=\big(\frac{1}{12}t+\frac{5}{32}|\xi\big(\xi(\beta\ t)\big)|\big),$$

$$\begin{aligned} & \psi\big(t,\xi(\xi(\vartheta(t)))\big) \\ =\ & \big(\frac{1}{5-t}\ \big|\sin\big(5(t+1)\big)\big| + \frac{e^{-t}}{15}\ \frac{|\xi\big(\xi(\beta\ t)\big)|}{1+|\xi\big(\xi(\beta\ t)\big)|}\big), \end{aligned}$$

thus we have: $c=\frac{1}{16},\ A_1=\frac{1}{3},\ A_2=\frac{1}{6},\ k_1=\frac{1}{12},\ k_2=\frac{5}{32},\ k_3=\frac{1}{15},\ M_1=\frac{7}{15},\ M_2=\frac{23}{48},\ L\simeq 0.386<1\ and\ L\ T+|\xi(0)|\approx 0.80\leq T=2.$

Using Theorem 2.2, then (15)-(16) has positive solutions $\xi\in C[0,T]$.

Example 5.2. Consider the state-dependent problem

$$\frac{d}{dt}\left\{\frac{\xi(t)-\frac{1}{15}(1-t^2)}{\left(\frac{1}{4}t+\frac{1}{9}\left|\xi\left(\xi(\alpha t)\right)\right|\right)}\right\}=\frac{1}{8}\left(t+\left|\sin\left(\xi(\xi(\alpha t))\right)\right|\right) \tag{17}$$

with the initial data

$$\xi(0)=\frac{1}{15}. \tag{18}$$

where $t \in (0, \frac{1}{2}]$ and $\alpha \in (0,1]$. Here: $\vartheta(t)=\alpha\ t$,

$$\lambda(t)=\frac{1}{15}(1-t^2),$$

$$\eta\big(t,\xi(\xi(\vartheta(t)))\big)=\big(\frac{1}{4}t+\frac{1}{9}|\xi\big(\xi(\alpha\ t)\big)|\big),$$

$$\psi\big(t,\xi(\xi(\vartheta(t)))\big) \;=\; \frac{t+|\sin\big(\xi(\xi(\alpha\ t))\big)|}{8},$$

thus we have: $c=\frac{1}{15},\ A_1=\frac{1}{16},\ A_2=\frac{1}{8},\ k_1=\frac{1}{4}$, $k_2=\frac{1}{9},\ k_3=\frac{1}{8},\ M_1=\frac{1}{8},\ M_2=\frac{13}{72}$, $L \simeq 0.1079 < 1\ and\ L\ T+|\xi(0)| \approx 0.121 \leq T=\frac{1}{2}$. Also we have

$$(k_2\ M_1+k_3\ M_2)\ T\ (L+1) \simeq 0.020 < 1.$$

Now form Theorem 3.3, the solution of (17)-(18) is unique.

VI. Application

Finally, we introduce an application of Theorem 3.4 in the next numerical example.

Example 6.3. Consider the state-dependent problem

$$\frac{d}{dt}\left\{\frac{\xi(t)-\left(\frac{1}{2}t-\frac{5}{512}t^3\right)}{\frac{1}{24}\left(t+|\xi\left(\xi(t)|\right)\right)}\right\} \;=\; \frac{1}{2}\left(t-|\xi\left(\xi(t)\right)|\right) \tag{19}$$

where $t \in (0,1]$ with the initial data

$$\xi(0)=0. \tag{20}$$

with

$$\begin{aligned}\xi(t) \;&=\; \big(\frac{1}{2}t-\frac{5}{512}t^3\big)\\ &+\; \frac{1}{24}\big(t+|\xi\big(\xi(t)\big)|\big)\ \int_0^t \frac{1}{2}\big(s-|\xi\big(\xi(s)\big)|\big)ds,\\ &\quad t\in[0,1].\end{aligned}$$

The exact solution of (19)-(20) is $\xi(t)=\frac{1}{2}t$. Here: $\vartheta(t)=t$,

$$\lambda(t)=\big(\frac{1}{2}t-\frac{5}{512}t^3\big),$$

$$\eta\big(t,\big(\xi\big(\xi(\vartheta(t))\big)\big)\big)=\frac{1}{24}\big(t+|\xi\big(\xi(t)\big)|\big),$$

$$\psi\big(t,\big(\xi\big(\xi(\vartheta(t))\big)\big)\big)=\frac{1}{2}\big(t-|\xi\big(\xi(t)\big)|\big),$$

then we have $c=\frac{271}{512},\ A_1=\frac{1}{2},\ A_2=\frac{1}{24},\ k_1=k_2=\frac{1}{24},\ k_3=\frac{1}{2},\ M_1=1,\ M_2=\frac{1}{12},\ L\simeq 0.673<1\ and\ L\ T+|\xi(0)| \approx 0.673 \leq T=1$. Also we have

$$(k_2\ M_1+k_3\ M_2)\ T\ (L+1) \simeq 0.139 < 1.$$

Now using Theorem 3.3, the solution of (19)-(20) is unique.

TABLE I
COMPARISON BETWEEN THE EXACT SOLUTION AND APPROXIMATE SOLUTION

t	First approximation	Second approximation	Error 1	Error 2
0.1	0.0499999999205326	0.049999999999999	7.945E-11	1.006E-15
0.2	0.0999999974575626	0.099999999999812	2.54E-09	1.880 E-13
0.3	0.149999980699926	0.149999999996796	1.93E-08	3.204E-12
0.4	0.199999918708124	0.199999999976014	8.13E-08	2.398E-11
0.5	0.249999752066693	0.249999999885713	2.48E-07	1.143E-10
0.6	0.299999383513753	0.299999999590865	6.16E-07	4.0914E-10
0.7	0.349998668669012	0.349999998797650	1.33E-06	1.202E-09
0.8	0.399997406877573	0.399999996941931	2.59E-06	3.058E-09
0.9	0.449995332181913	0.449999993034889	4.67E-06	6.965E-09
1	0.499992104428274	0.499999985459222	7.90E-06	1.454E-08

Now, we use the method of successive approximation (Picard Method) to estimate the solution of (19)-(20).

$$\begin{aligned}\xi_n(t) \;&=\; \big(\frac{1}{2}t-\frac{5}{512}t^3\big)+\frac{1}{24}\Big(t+\xi_{n-1}\big(\xi_{n-1}(t)\big)\Big)\\ &\quad \int_0^t \frac{1}{2}\big(s-\xi_{n-1}\big(\xi_{n-1}(s)\big)\big)ds \qquad (21)\\ \xi_0(t) \;&=\; \big(\frac{1}{2}t-\frac{5}{512}t^3\big). \qquad (22)\end{aligned}$$

and

$$\xi(t)=\lim_{n\longrightarrow\infty}\xi_n(t).$$

(19)-(20) was programmed using Matlap software and the results are shown in Table 1.
Table 1 shows the values of the solution obtained in the first and the second iterations for different values of t. We also calculated the error ($|Exact\ solution - Approximated\ solution|$) at every step. Columns 4 and 5 in Table 1 give the errors corresponding to the first and the second iterations respectively.
We deduce that the error will be very small when the number of iterations increase. So, when $n \longrightarrow \infty$ the approximated solution and the exact solution will coincide. Here we calculated the error for the first and the second iterations only since the error is very small in the next iterations.

VII. Conclusion

In this paper, we have proved the existence, the uniqueness, and the continuous dependence of the unique solution of a state-dependent hybrid differential equation under suitable assumptions. Here we have generalized the results in [7], [8], [17], and [18]. Some examples, to illustrate the obtained results, have been given. Also, the method of successive approximation has been used to estimate the solution of a given example.

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References

[1] P. K. Anh, N. T. T. Lan and N. M. Tuan, "Solutions to systems of partial differential equations with weighted self-reference and heredity," *Electronic Journal of Differential Equations*, vol 2012, no 117, pp. 1-14, 2012.
[2] J. Bana's and A. Martinon, "Monotonic solutions of a quadratic integral equation of Volterra type," *Comput. Math. Appl.*, vol 47, no 2-3, pp. 271 - 279, 2004.

[3] J. Bana's, J. Caballero and J. Rocha, K. Sadarangani, "Monotonic solutions of a class of quadratic integral equations of Volterra type," *Computers and Mathematics with Applications*, vol 49, no 5-6, pp. 943-952, 2005.
[4] J. Bana's, J. Rocha Martin and K. Sadarangani, "On the solution of a quadratic integral equation of Hammerstein type," *Mathematical and Computer Modelling*, vol 43, no 1-2, pp. 97-104, 2006.
[5] J. Bana's and B. Rzepka, " Nondecreasing solutions of a quadratic singular Volterra integral equation," *Math. Comput. Modelling*, vol. 49, no. 3-4, pp. 488-496, 2009.
[6] J. Bana's and B. Rzepka, "Monotonic solutions of a quadratic integral equations of fractional order, " *J. Math. Anal. Appl.*,vol. 332, no.2, pp. 1371-1379, 2007.
[7] V. Berinde, "Existence and approximation of solutions of some first order iterative differential equations," *Miskolc Mathematical Notes*, vol. 11, no. 1, pp. 13–26, 2010.
[8] A. Buicá, "Existence and continuous dependence of solutions of some functional differential equations," *Seminar on Fixed Point Theory*, vol. 3, no. 1, pp. 1–14, 1995.
[9] B.C. Dhage, V. Lakshmikantham, "Basic results on hybrid differential equations", *Nonlinear Anal. Hybrid*, vol. 4, no.3, pp. 414-424, 2010.
[10] N. Dunford, J. T. Schwartz, Linear Operators, (Part 1), General Theory, NewYork Interscience, 1957.
[11] E. Eder, "The functional differential equation $x'(t) = x(x(t))$," *J. Differential Equations*, pp. 390–400, 1984.
[12] A. M. A. El-Sayed, R. G. Aahmed, "Solvability of the functional integro-differential equation with self-reference and state-dependence," *J. Nonlinear Sci. Appl.*, pp. 13, 1-8, 2020.
[13] A. M. A. El-Sayed, R. G. Aahmed, "Solvability of a boundary value problem of self-reference functional differential equation with infinite point and integral conditions," *J. Math. Computer Sci.*, pp. 296-308, 2020.
[14] A. M. A. El-Sayed and , H. R. Ebead, "On an initial value problem of delay-refereed differential Equation," *International Journal of Mathematics Trends and Technology*, vol. 66, no.5, pp. 32-37, 2020.
[15] A. M. A. El-Sayed and , H. R. Ebead, "Positive solutions of an initial value problem of a delay-self-reference nonlinear differential equation," *Malaya J. of matematik*, vol 8, no. 3, pp. 1001-1006, 2020.
[16] A. M. A. El-Sayed, H. R. Ebead, "On the solvability of a self-reference functional and quadratic functional integral equations," *FILOMAT*, vol 34, no 1, pp. 129-141, 2020.
[17] A. M. A. El-Sayed, H. H. G. Hashem and E. A. A. Ziada, "Picard and Adomian methods for quadratic integral equation," *Computational & Applied Mathematics*, vol. 29, no. 3, pp. 447–463, 2010.
[18] A. M. A. El-Sayed, H. H. G. Hashem and E. A. A. Ziada, "Picard and Adomian decomposition methods for a quadratic integral equation of fractional order," *Computational & Applied Mathematics*, vol 33, no 1, pp. 95–109, 2014.
[19] M. Fečkan, "On a certain type of functional differential equations," *Mathematica Slovaca*, vol 43, no 1, pp. 39–43, 2014.
[20] C. G. Gal, "Nonlinear abstract differential equations with deviated argument," *Journal of mathematical analysis and applications*, vol. 333, no. 1 pp. 971–983, 2007.
[21] R. Haloi, P. Kumar and D. N. Pandey, "Sufficient conditions for the existence and uniqueness of solutions to impulsive fractional integro-differential equations with deviating arguments," *Journal of Fractional Calculus and Applications*, vol. 5, no. 1, pp. 73–84, 2014.
[22] K. Hilal and A.Kajouni, "Boundary value problems for hybrid differential equations," *Mathematical Theory and Modeling*, vol. 5, no. 1, pp. 2224-5804, 2015.
[23] L. V. Kantorovich, G. P. Akilov, Functional analysis, (translated by howard L. silcock), Pergamon Press, New York, 1982.
[24] A. N. Kolmogorov, S. V. Fomin, Elements of the theory of functions and functional analysis, (Vol. 1), Metric and normed spaces, 1957.
[25] N. T. Lan and E. Pascali, "A two-point boundary value problem for a differential equation with self-refrence," *Electronic Journal of Mathematical Analysis and Applications*, vol. 6, no. 1, pp. 25–30, 2018.
[26] J. Letelier, T. Kuboyama, H. Yasuda, M. Cárdenas and A. Cornish-Bowden, "A self-referential equation, $f(f) = f$, obtained by using the theory of $(m; r)$ systems: Overview and applications," *Algebraic Biology*, pp. 115–126, 2005.
[27] M. Miranda, E. Pascali, "On a type of evolution of self-referred and hereditary phenomena," *Aequationes mathematicae*, vol. 71, no. 3, pp. 253–268, 2006.
[28] N. M. Tuan and L. T. Nguyen, "On solutions of a system of hereditary and self-referred partial-differential equations," *Numerical Algorithms*, vol. 55, no. 1, pp. 101–113, 2010.
[29] D. Yang, W. Zhang, "Solutions of equivariance for iterative differential equations,"*Applied mathematics letters*, vol.17, no. 7, pp. 759–765, 2004.
[30] Y. Zhao, S. Sun, Z. Han, Q. Li, "Theory of fractional hybrid differential equations," *Computers & Mathematics with Applications*, vol.62, no.3, pp. 1312–1324, 2011.