Compactness of Hausdorff Fuzzy Metric Spaces

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***Abstract*—Hausdorff fuzzy metric space is often considered in fuzzy analysis, since for a given fuzzy metric space (X^*,d^*), we can define a certain new fuzzy metric space of all nonempty compact subsets of *X*, which we call it the Hausdorff fuzzy metric space (X^*,D^*). The introduction of the distance function defined over a fuzzy subset $\tilde{A}$ of a universal set *X* is is given first and then prove some related properties to this distance function. In addition, the generalization of this definition for a distance function between two fuzzy subsets of *X* is obtained based on the Hausdorff distance. Finally, the completeness of the Hausdorff fuzzy metric space and then its compactness are studied and proved.**

***Index Terms*—"Fuzzy metric space, Fuzzy distance function, Complete fuzzy metric space, Compact fuzzy metric space, Hausdorff distance".**

I. INTRODUCTION

Given any metric space (X,d), the Hausdorff distance defined on the space of all nonempty compact fuzzy subsets of (X,d) would express a fuzzy metric space, which is ascribed by (X^*,D^*), whereby the Hausdorff distance gives the largest length of the fuzzy set of all distances between each fuzzy point of a set.

The introduction of the concepts of fuzzy sets can be traced back in 1965 by Zadeh in 1965 [1]; whereby it became an important source topic of mathematicians, such as Kramosil and Michale who introduced the notions of fuzzy metric spaces which is modified then by George and Vermani in 1994 using M-metric spaces, [2-9]. Thereafter, Rodrignezlapez and Romagnera introduced the Hausdorff metric defined over the collection of all compact nonempty sets and studied the completeness, pre-compactness and compactness of such Hausdorff fuzzy metric [10]. Whilst, the identification theorem for the completeness of the Hausdorff fuzzy metric space was initially presented by Garcia et al. in 2013 [11-12]. However, several equivalent conditions for the Hausdorff fuzzy metric spaces defined over the family of compact nonempty sets are given recently in [13]. After that, it is promising to persue for other properties of the Hausdorff fuzzy metric space. Furtheremore, fundamental concepts of fuzzy analysis has many applications, such as in the statement and the proof of the existence and uniqueness of different types of fuzzy differential or difference equations (see for example [14]). Also, Prasertpong R. and Siripitukdet M. presents some results concerning advanced sets in approximation spaces based on overlaping classes of successor restricted to levels in closed unit intervals under serial fuzzy relations between two universal sets [15], while Julath P. and Siripitukdet M. in 2017 characterize fuzzy subsemigroups, fuzzy bi-ideals, fuzzy generalized bi-ideals, and fuzzy quasi-ideals of semigroups, [16].

In this artical, the distance function between fuzzy sets is defined along with several related properties in real analysis which are stated and proved. Additionally, this function is utulized to define the Hausdorff distance function between fuzzy sets. This in turn is used to construct the Hausdorff fuzzy metric space proving the completness and compactness when the nonfuzzy metric space is complete and copact.

It must be noted that the completeness of the Hausdorff induced fuzzy metric space is obtained when the original metric space is complete. While, the compactness of the function space would induce the Hausdorff-induced fuzzy metric space to be compact.

II. PRELIMINARIES

In this section, some basic and fundamental concepts related to this paper are given, while more elementary concepts are not introduced, by which the basic definition of fuzzy sets is given.

Definition (1), [1], [17]: A subset $\tilde{A}$ of the universal set *X* is said to be fuzzy subset of *X* if it is characterized by a membership function $\mu_{\tilde{A}} : X \longrightarrow [0,1]$. In other words, $\tilde{A}$ can be defined mathematically as:

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right) : x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1 \right\}$$

On the other hand, the most important relationships regarding fuzzy sets are given for completeness purpose of this paper in the next remark, which appears in most classical literatures, such as [1], [18-22].

Remark (2): Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of the universal set *X*, with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, respectively. Then for all $x \in X$:

1. $\tilde{A} \subseteq \tilde{B}$ if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$.
2. $\tilde{A} = \tilde{B}$ if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$.
3. The support of $\tilde{A}$ is a crisp set consisting of all $x \in X$, such that $\mu_{\tilde{A}}(x) > 0$.
4. The union of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{C}$ with membership function:
$$\mu_{\tilde{C}}(x) = \max\left\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \right\}$$
5. The intersection of $\tilde{A}$ and $\tilde{B}$ is also a fuzzy set $\tilde{D}$ with membership function:
$$\mu_{\tilde{D}}(x) = \min\left\{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \right\}$$
6. The complement of $\tilde{A}$ is also a fuzzy set (denoted by $\tilde{A}^c$) with membership function $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$

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Definition (3), [23]: A fuzzy point $\tilde{q}_x^\lambda$ in a fuzzy set $\tilde{A}$ is a fuzzy subset of *X*, where *x* belongs to the the support of the fuzzy point, and $\lambda \in (0,1]$ is the grade of this fuzzy point, with membership function:

$$\mu_{\tilde{q}_x^\lambda}(y) = \begin{cases} \lambda & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$

It is notable that, a fuzzy point $\tilde{q}_x^\lambda$ belong to a fuzzy set $\tilde{A}$ (denoted by $\tilde{q}_x^\lambda \tilde{\in} \tilde{A}$) if and only if $\mu_{\tilde{A}}(x) \geq \lambda$, where the symbol $\tilde{\in}$ is used in order to distinguish the belong in fuzzy set with the ordinary belong in nonfuzzy sets, [24-25].

As an abbreviation, the set of all fuzzy subsets of *X* or fuzzy points in *X* will be denoted in general by *X**.

Next, the definition of the function in terms of fuzzy points is given, as follows:

Definition (4), [26], [23]: If X^* be the set of all fuzzy points in *X*, i.e.,

$$X^* = \left\{ \tilde{q}_x^\lambda : x \in X, \lambda \in (0,1] \right\}$$

and let $\tilde{q}_x^\lambda$, $\tilde{q}_{x'}^{\lambda'}$, $\tilde{q}_{x_i}^{\lambda_i}$, $i = 1,2,3$; be fuzzy points in X^*, where λ, λ', $\lambda_i \in (0,1]$, x, $x_i \in X$, for all $i = 1,2,3$. The a function $d^*: X^* \times X^* \longrightarrow [0,\infty)$ is called fuzzy distance function if it satisfies the following conditions:

1. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = 0$ if and only if $\lambda_1 \leq \lambda_2$ and $x_1 = x_2$.
2. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = d^*({}^c\tilde{q}_{x_2}^{\lambda_2}, {}^c\tilde{q}_{x_1}^{\lambda_1})$.
3. $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \leq d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) + d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3})$.
4. If $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) < r$, where $r > 0$, then there exist $\lambda' > \lambda_1 > \lambda_2$, such that $d^*(\tilde{q}_{x'}^{\lambda'}, \tilde{q}_{x_2}^{\lambda_2}) < r$.

Also, (X^*, d^*) is called fuzzy metric space.

Furthermore, among the well-known examples of fuzzy distance function are those given below [26], [23]:

1. If $X = R$ and $d(x_1,x_2) = |x_1 - x_2|$, then we get the fuzzy distance function:
$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} - |x_1 - x_2|$$
2. If $X = R^2$ and $d(x_1, x_2) = \sqrt{(x_1^1 - x_2^1)^2 + (x_1^2 - x_2^2)^2}$, then the fuzzy distance function takes the form:
$$d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + \sqrt{(x_1^1 - x_2^1)^2 + (x_1^2 - x_2^2)^2}$$
where $x_1 = (x_1^1, x_1^2)$, $x_2 = (x_2^1, x_2^2)$.
3. If $X = C[a,b]$ and $d(f,g) = \left(\int_a^b | f - g |^2 \, dx \right)^{1/2}$, for all $f, g \in C[a,b]$, and hence the fuzzy distance function takes the form:
$$d^*(\tilde{q}_f^{\lambda_1}, \tilde{q}_g^{\lambda_2}) = \max\{\lambda_1 - \lambda_2, 0\} + \left(\int_a^b | f - g |^2 \, dx \right)^{1/2}$$

While the basic concepts in metric spaces are those given in the next definitions:

Definition (5), [26], [27]: If (X^*, d^*) is a fuzzy metric space, then the fuzzy neighborhood $U_\varepsilon(\tilde{q}_x^\lambda)$ of a fuzzy point $\tilde{q}_x^\lambda$ is the fuzzy set which consist of all $\tilde{q}_{x'}^{\lambda'} \in X^*$, such that:

$$d^*(\tilde{q}_x^\lambda, \tilde{q}_{x'}^{\lambda'}) < \varepsilon$$

where the number $\varepsilon > 0$ is called the radius of $U_\varepsilon(\tilde{q}_x^\lambda)$, and $\tilde{q}_x^\lambda$ is the center of the neighborhood, i.e.,

$$U_\varepsilon(\tilde{q}_x^\lambda) = \left\{ \tilde{q}_{x'}^{\lambda'} \tilde{\in} X^* : d^*(\tilde{q}_x^\lambda, \tilde{q}_{x'}^{\lambda'}) < \varepsilon \right\}$$

Definition (6), [26], [27]: A fuzzy set $\tilde{A}$ is called open fuzzy set if for all $\tilde{q}_x^\lambda \tilde{\in} \tilde{A}$, there exists $\varepsilon > 0$, such that $U_\varepsilon(\tilde{q}_x^\lambda) \subset \tilde{A}$.

Definition (7), [26-27]: A fuzzy set $\tilde{A} \tilde{\in} X^*$ is said to be fuzzy closed set if $\tilde{A}^c$ is a fuzzy open set in *X**, or every fuzzy point of $\tilde{A}$ is a "fuzzy limit point" of $\tilde{A}$.

Definition (8), [26]: A fuzzy set $\tilde{A}$ is bounded if there exists a real number $h > 0$, such that $d^*(\tilde{q}_{x_0}^{\lambda_0}, \tilde{q}_x^\lambda) < h$, for all $\tilde{q}_x^\lambda \tilde{\in} \tilde{A}$, where $\tilde{q}_{x_0}^{\lambda_0} \tilde{\in} X^*$ and $x_0, x \in X$, $\lambda_0, \lambda \in (0,1]$.

Definition (9), [23]: A fuzzy subset $\tilde{A}$ of a fuzzy metric space (X^*, d^*) is closed and bounded if for all $\alpha \in (0,1]$, A_α is "closed and bounded" nonfuzzy subset of *X*.

Definition (10), [26]: If $\tilde{A}$ is a fuzzy subset of a "fuzzy metric space" (X^*, d^*) and let $\varepsilon > 0$. A finite fuzzy set $\tilde{W}$ of fuzzy points:

$\tilde{W} = \left\{ \tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n} \right\}$, where $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0, 1]$

is called an "ε-fuzzy net" for $\tilde{A}$ if for every fuzzy point $\tilde{p}_x^\lambda \tilde{\in} \tilde{A}$, there exists $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{W}$, for some $i \in \{1, 2, \ldots, n\}$; such that $d^*(\tilde{p}_x^\lambda, \tilde{q}_{x_i}^{\lambda_i}) < \varepsilon$.

III. Distance Between Fuzzy Sets

In this section, the distance function between fuzzy sets (based on the usual definitions between two nonfuzzy sets) and the definition of the distance function in terms of fuzzy points given in definition (4) is introduced. Therefore, the distance between fuzzy point $\tilde{q}_x^\lambda$ in *X** and a fuzzy set $\tilde{A}$ in *X** is initially defined by:

$$d^*(\tilde{q}_x^\lambda, \tilde{A}) = \inf_{\tilde{q}_a^\mu \in \tilde{A}} d^*(\tilde{q}_x^\lambda, \tilde{q}_a^\mu), \ \forall \ \tilde{q}_a^\mu \tilde{\in} \tilde{A}$$

while the distance between any two fuzzy subsets $\tilde{A}$ and $\tilde{B}$ of X^* is defined as:

$$d^*(\tilde{A},\tilde{B}) = \sup_{\tilde{q}_a^{\mu}\in\tilde{A}} d^*(\tilde{q}_a^{\mu},\tilde{B})$$

$$= \sup_{\tilde{q}_a^{\mu}\in\tilde{A}} \inf_{\tilde{q}_b^{\lambda}\in\tilde{B}} d^*(\tilde{q}_a^{\mu},\tilde{q}_b^{\lambda}) \qquad \ldots(1)$$

The next theorems presents several basic properties related to the distance between fuzzy points and/or fuzzy subsets of X.

Theorem (11): If $\tilde{A}$ is a fuzzy subset of the universal metric space (X,d), then $d^*(\tilde{q}_x^{\lambda},\tilde{A})=0$ if and only if $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$.

Proof: If $d^*(\tilde{q}_x^{\lambda},\tilde{A})=0$. To prove that $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$

From the definition $d^*(\tilde{q}_x^{\lambda},\tilde{A}) = 0$ implies that:

$$\inf_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{A}} \{d^*(\tilde{q}_x^{\lambda},\tilde{q}_{x_1}^{\lambda_1}) : \tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{A}\} = 0$$

Hence, for each $x_1\in X$, $\lambda_1\in R$ yields $d^*(\tilde{q}_x^{\lambda},\tilde{q}_{x_1}^{\lambda_1}) = 0$ (by definition (2.4), [26])

Now, $\mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(x_1) = \lambda_1 \geq \lambda$

Therefore $\mu_{\tilde{A}}(x) \geq \lambda$, i.e., $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$

Conversely, if $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$, i.e., $\mu_{\tilde{A}}(x) \geq \lambda$. To prove that $d^*(\tilde{q}_x^{\lambda},\tilde{A}) = 0$, i.e., to prove that there exists $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{A}$ in which the infimum distance occur, i.e., $d^*(\tilde{q}_x^{\lambda},\tilde{q}_{x_1}^{\lambda_1}) = 0$ which is satisfied if $x = x_1$, $\lambda = \lambda_1$ and hence $d^*(\tilde{q}_x^{\lambda},\tilde{A}) = 0$. ■

Theorem (12): If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of the universal metric space (X,d), then $d^*(\tilde{A},\tilde{B})=0$ if and only if $\tilde{A}\subseteq\tilde{B}$.

Proof: If $d^*(\tilde{A},\tilde{B})=0$, then by the distance function (1), which implies:

$$\sup_{\tilde{q}_a^{\lambda}\tilde{\in}\tilde{A}} \inf_{\tilde{q}_b^{\lambda'}\tilde{\in}\tilde{B}} d^*(\tilde{q}_a^{\lambda},\tilde{q}_b^{\lambda'}) = 0$$

and hence by the uniqueness of the infimum and the supremum, there exists $\tilde{q}_a^{\lambda_1} \,\tilde{\in}\, \tilde{A}$ and $\tilde{q}_b^{\lambda_2} \,\tilde{\in}\, \tilde{B}$, such that $d^*(\tilde{q}_a^{\lambda_1},\tilde{q}_b^{\lambda_2}) = 0$

From definition (4)(1), $\lambda_1 \leq \lambda_2$ and $a = b$, and hence:

$$\begin{aligned}\mu_{\tilde{A}}(a) &= \lambda_1\\ &\leq \lambda_2\\ &= \mu_{\tilde{B}}(b),\ \mathrm{a = b}\\ &= \mu_{\tilde{B}}(a)\end{aligned}$$

Therefore, from Remark (2)(1) it implies that $\tilde{A}\subseteq\tilde{B}$. ■

Theorem (13): If $\tilde{A}$ and $\tilde{B}$ are any two fuzzy subsets of the universal metric space (X,d), such that $\tilde{A}\subseteq\tilde{B}$, then:

$$d^*(\tilde{q}_x^{\lambda},\tilde{B}) \leq d^*(\tilde{q}_x^{\lambda},\tilde{A}),\ \forall\, \tilde{q}_x^{\lambda} \,\tilde{\in}\, X^*.$$

Proof: Let $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{A}$ and since $\tilde{A}\subseteq\tilde{B}$, then $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{B}$

It follows that:

$$d^*(\tilde{q}_x^{\lambda},\tilde{q}_{x_1}^{\lambda_1}) \geq \inf\{d^*(\tilde{q}_x^{\lambda},\tilde{q}_y^{\mu}) : \tilde{q}_y^{\mu} \,\tilde{\in}\, \tilde{B}\}$$

$$= d^*(\tilde{q}_x^{\lambda},\tilde{B}) \qquad \ldots(2)$$

which is also true for all $\tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{A}$

We defined that

$$d^*(\tilde{q}_{x_1}^{\lambda_1},\tilde{A}) = \inf\{d^*(\tilde{q}_x^{\lambda},\tilde{q}_{x_1}^{\lambda_1}) : \tilde{q}_{x_1}^{\lambda_1} \,\tilde{\in}\, \tilde{A}\} \qquad \ldots(3)$$

substitute (3) in (2), we get:

$$d^*(\tilde{q}_x^{\lambda},\tilde{B}) \leq d^*(\tilde{q}_x^{\lambda},\tilde{A})$$

which completes the proof of the theorem. ■

Theorem (14): If $\tilde{A},\tilde{B}$ and $\tilde{C}$ be fuzzy subsets of the universal metric space (X,d), such that $\tilde{B}\subseteq\tilde{C}$, then:

$$d^*(\tilde{A},\tilde{C}) \leq d^*(\tilde{A},\tilde{B})$$

Proof: Let $\tilde{A} \,\tilde{\in}\, X^*$ be any fuzzy subset of X and take a fuzzy point $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$

By theorem (13), $d^*(\tilde{q}_x^{\lambda},\tilde{C}) \leq d^*(\tilde{q}_x^{\lambda},\tilde{B})$

Now, take the supremum over all $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}$, then:

$$\sup_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{A}} d^*(\tilde{q}_x^{\lambda},\tilde{C}) \leq \sup_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{A}} d^*(\tilde{q}_x^{\lambda},\tilde{B})$$

Hence using the definition of d^*, this implies to $d^*(\tilde{A},\tilde{C}) \leq d^*(\tilde{A},\tilde{B})$. ■

Theorem (15): If $\tilde{A},\tilde{B}$ and $\tilde{C}$ be fuzzy subsets of the universal metric space (X,d), then:

$$d^*(\tilde{A}\cup\tilde{B},\tilde{C}) = \max\{d^*(\tilde{A},\tilde{C}), d^*(\tilde{B},\tilde{C})\}$$

Proof: Let $\tilde{D} = \tilde{A}\cup\tilde{B}$, which has the membership function:

$$\mu_{\tilde{D}}(x) = \max\{\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)\} \qquad \ldots(4)$$

Now:

$$d^*(\tilde{A}\cup\tilde{B},\tilde{C}) = d^*(\tilde{D},\tilde{C})$$

$$= \sup_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{D}} d^*(\tilde{q}_x^{\lambda},\tilde{C}),\ \text{for all } \tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}\cup\tilde{B}$$

By (3), implies to:

$$d^*(\tilde{A}\cup\tilde{B},\tilde{C}) = \max\left\{\sup_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{A}} d^*(\tilde{q}_x^{\lambda},\tilde{C}),\ \sup_{\tilde{q}_x^{\lambda}\tilde{\in}\tilde{B}} d^*(\tilde{q}_x^{\lambda},\tilde{C})\right\}$$

$$= \max\{d^*(\tilde{A},\tilde{C}), d^*(\tilde{B},\tilde{C})\}.$$ ■

Theorem (16): Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be fuzzy subsets of the universal metric space (X,d), then:

$$d^*(\tilde{A},\tilde{B}) \leq d^*(\tilde{A},\tilde{C}) + d^*(\tilde{C},\tilde{B})$$

Proof: Since:

$$d^*(\tilde{A},\tilde{B}) = \sup_{\tilde{q}_{x_1}^{\lambda_1}\tilde{\in}\tilde{A}} d^*(\tilde{q}_{x_1}^{\lambda_1},\tilde{B})$$

$$= \sup_{\tilde{q}_{x_1}^{\lambda_1}\tilde{\in}\tilde{A}} \inf_{\tilde{q}_{x_2}^{\lambda_2}\tilde{\in}\tilde{B}} d^*(\tilde{q}_{x_1}^{\lambda_1},\tilde{q}_{x_2}^{\lambda_2})$$

$$\leq \sup_{\tilde{q}_{x_1}^{\lambda_1} \tilde{\in} \tilde{A}} \inf_{\tilde{q}_{x_2}^{\lambda_2} \tilde{\in} \tilde{B}} \left\{ d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3})+d^*(\tilde{q}_{x_3}^{\lambda_3}, \tilde{q}_{x_2}^{\lambda_2}) \right\}$$

$$\leq \sup_{\tilde{q}_{x_1}^{\lambda_1} \tilde{\in} \tilde{A}} \inf_{\tilde{q}_{x_3}^{\lambda_3} \tilde{\in} \tilde{C}} d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) +$$

$$\sup_{\tilde{q}_{x_3}^{\lambda_3} \tilde{\in} \tilde{C}} \inf_{\tilde{q}_{x_2}^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_{x_3}^{\lambda_3}, \tilde{q}_{x_2}^{\lambda_2})$$

$$= \sup_{\tilde{q}_{x_1}^{\lambda_1} \tilde{\in} \tilde{A}} d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{C}) + \sup_{\tilde{q}_{x_3}^{\lambda_3} \tilde{\in} \tilde{C}} d^*(\tilde{q}_{x_3}^{\lambda_3}, \tilde{B})$$

$$= d^*(\tilde{A}, \tilde{C}) + d^*(\tilde{C}, \tilde{B})$$

which completes the proof of the theorem. ■

IV. Hausdorff Fuzzy Metric Space

In the last section, the distance fuction between fuzzy sets is given which does not define a "fuzzy metric space", since the property of symmetry becomes, in general, not true. Therefore, using the definition of Hausdorff distance between two closed and bounded fuzzy subsets of X, would induce [28-29]:

$$D^*(\tilde{A},\tilde{B}) = \max\{d^*(\tilde{A}, \tilde{B}),d^*(\tilde{B}, \tilde{A})\} \qquad \ldots(5)$$

In the next theorem, it is proved that X^* with the Hausdorff distance (5) form a metric space, which will be called "fuzzy metric space" defined over X is proven

Theorem (17): Let (X,d) be a nonfuzzy or the universal metric space, then (X^*,D^*) is a fuzzy metric space.

Proof: To prove that (X^*,D^*) is a fuzzy metric space, we need to verify the following properties for all $\tilde{A}, \tilde{B}, \tilde{C} \tilde{\in} X^*$:

1. $D^*(\tilde{A},\tilde{B}) \geq 0$.
2. $D^*(\tilde{A},\tilde{B}) = 0$ if and only if $\tilde{A} = \tilde{B}$.
3. $D^*(\tilde{A},\tilde{B}) = D^*(\tilde{B},\tilde{A})$.
4. $D^*(\tilde{A},\tilde{B}) \leq D^*(\tilde{A},\tilde{C}) + D^*(\tilde{C},\tilde{B})$.

The proof proceeds as follows:

1. By definition (4), we have $d^*(\tilde{A}, \tilde{B}) \geq 0$ and $d^*(\tilde{B}, \tilde{A}) \geq 0$ and hence by the definition of D^*, implies that $D^*(\tilde{A}, \tilde{B}) \geq 0$, for all $\tilde{A}, \tilde{B} \tilde{\in} X^*$.
2. If $D^*(\tilde{A}, \tilde{B}) = 0$, then from (4),

   $$\max\left\{d^*(\tilde{A},\tilde{B}), d^*(\tilde{B},\tilde{A})\right\} = 0$$

   Hence $d^*(\tilde{A},\tilde{B}) = 0$ and $d^*(\tilde{B},\tilde{A}) = 0$

   By theorem (12), $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$, i.e., $\tilde{A} = \tilde{B}$

   Conversely, if $\tilde{A} = \tilde{B}$, then $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$

   Hence, from theorem (12) it implies that $d^*(\tilde{A},\tilde{B}) = 0$ and $d^*(\tilde{B},\tilde{A}) = 0$

   Thus from (5), it follows that $D^*(\tilde{A}, \tilde{B}) = 0$
3. $D^*(\tilde{A}, \tilde{B}) = \max\{d^*(\tilde{A}, \tilde{B}),d^*(\tilde{B}, \tilde{A})\}$

   $= \max\{d^*(\tilde{B}, \tilde{A}),d^*(\tilde{A}, \tilde{B})\}$

   $= D^*(\tilde{B}, \tilde{A})$, for all $\tilde{A}$, $\tilde{B} \in X^*$
4. Since $D^*(\tilde{A}, \tilde{B}) = \max\{d^*(\tilde{A}, \tilde{B}),d^*(\tilde{B}, \tilde{A})\}$, therefore from theorem (16) it follows that:

$$\begin{aligned} d^*(\tilde{A},\tilde{B}) &\leq d^*(\tilde{A},\tilde{C}) + d^*(\tilde{C},\tilde{B}) \\ &\leq \max\left\{d^*(\tilde{A},\tilde{C}), d^*(\tilde{C},\tilde{A})\right\} + \\ &\quad \max\left\{d^*(\tilde{C},\tilde{B}), d^*(\tilde{B},\tilde{C})\right\} \\ &= D^*(\tilde{A},\tilde{C}) + D^*(\tilde{C},\tilde{B}) \qquad \ldots(6) \end{aligned}$$

Similarly:

$$\begin{aligned} d^*(\tilde{B},\tilde{A}) &\leq d^*(\tilde{B},\tilde{C}) + d^*(\tilde{C},\tilde{A}) \\ &\leq \max\left\{d^*(\tilde{B},\tilde{C}), d^*(\tilde{C},\tilde{B})\right\} + \\ &\quad \max\left\{d^*(\tilde{C},\tilde{A}), d^*(\tilde{A},\tilde{C})\right\} \\ &= D^*(\tilde{B},\tilde{C}) + D^*(\tilde{C},\tilde{A}) \\ &= D^*(\tilde{A},\tilde{C}) + D^*(\tilde{C},\tilde{B}) \qquad \ldots(7) \end{aligned}$$

From inequalities (6) and (7), it follows that:

$$\begin{aligned} D^*(\tilde{A},\tilde{B}) &= \max\left\{d^*(\tilde{A},\tilde{B}), d^*(\tilde{B},\tilde{A})\right\} \\ &\leq \max\left\{D^*(\tilde{A},\tilde{C}) + D^*(\tilde{C},\tilde{B}), D^*(\tilde{A},\tilde{C}) + \right. \\ &\qquad \left. D^*(\tilde{C},\tilde{B})\right\} \\ &= D^*(\tilde{A},\tilde{C}) + D^*(\tilde{C},\tilde{B}) \end{aligned}$$

Therefore, (X^*,D^*) is a fuzzy metric space. ■

For the rest of this paper, we further need to introduce several definitions in fuzzy metric spaces, which are considered as a generalization to definitions those given in section II above.

Definition (18): The "Hausdorff fuzzy metric space" (X^*,D^*) is said to be compact if every open cover of fuzzy open sets $\tilde{U}_i$, $i = 1,2,...,n$; which is covering X^* has a finite subcover.

Theorem (19): Let (X^*,D^*) be a Hausdorff compact fuzzy metric space, then every infinite fuzzy subset of X^* has at least one fuzzy limit point in X^*.

Proof: Let $\tilde{A}$ be an infinite fuzzy subset of X^*

To prove that there exist a fuzzy point $\tilde{p}_x^{\lambda} \tilde{\in} X^*$, $x \in X$, $\lambda \in (0, 1]$; such that $\tilde{p}_x^{\lambda}$ is a fuzzy limit point of $\tilde{A}$.

For contrary, suppose that $\tilde{A}$ has no fuzzy limit point in X^*, i.e., every fuzzy point of X^* is not a fuzzy limit point of $\tilde{A}$.

Let $\tilde{V}(\tilde{p}_x^{\lambda})$ be a fuzzy neighborhood of $\tilde{p}_x^{\lambda}$, then:

$$(\tilde{V}(\tilde{p}_x^{\lambda}) \,\tilde{-}\, \{ \tilde{p}_x^{\lambda} \}) \cap \tilde{A} = \tilde{\emptyset}$$

where $\tilde{-}$ is the subtraction between fuzzy sets using the extension principle

Hence, $\{ {}^c\tilde{p}_x^{\lambda} \} \cap \tilde{A} = \tilde{\emptyset}$, where ${}^c\tilde{p}_x^{\lambda} = \tilde{p}_x^{1-\lambda}$.

Then $\mu_{({}^c\tilde{p}_x^{\lambda} \cap \tilde{A})}(y) = 0$, $\forall\, y \in X$, which means that either:

$$\tilde{V}(\tilde{p}_x^{\lambda}) \cap \tilde{A} = \tilde{\emptyset}, \text{ if } \tilde{p}_x^{\lambda} \tilde{\notin} \tilde{A}$$

or:

$$\tilde{V}(\tilde{p}_x^{\lambda}) \cap \tilde{A} = \tilde{p}_x^{\lambda}, \text{ if } \tilde{p}_x^{\lambda} \tilde{\in} \tilde{A}$$

Since, $\tilde{V}(\tilde{p}_x^{\lambda})$ is a fuzzy neighborhood, then $\tilde{V}(\tilde{p}_x^{\lambda})$ is a fuzzy open set, i.e., $\{\tilde{V}(\tilde{p}_x^{\lambda})\}$ form a fuzzy open cover of X^*, $\forall\ \tilde{V}(\tilde{p}_x^{\lambda}) \tilde{\in} X^*$

Since every $\tilde{V}(\tilde{p}_x^{\lambda})$ contains at most one fuzzy point of $\tilde{A}$ and since $\tilde{A}$ is an infinite fuzzy set

Hence, $\{\tilde{V}(\tilde{p}_x^{\lambda})\}$ has an infinite fuzzy open cover which is converging $\tilde{A}$

Therefore, $\{\tilde{V}(\tilde{p}_x^{\lambda})\}$ has an infinite fuzzy subcover, which is covering X^*

Thus, X^* is not compact fuzzy metric space, which contradicts that X^* is a compact Hausdorff fuzzy metric space and hence $\tilde{A}$ has fuzzy limit point in X^*. ■

Definition (20): The Hausdorff fuzzy metric space (X^*, D^*) is said to be "totally bounded (or pre-compact)" if every fuzzy subset of X^* has an ε-fuzzy net for every $\varepsilon > 0$.

Definitio (21): A sequence $\{\tilde{A}_n\}_{n=1}^{\infty}$ of fuzzy subsets of a Hausdorff fuzzy metric space (X^*, D^*) is said to be converge to a fuzzy subset $\tilde{A}_0$ of X^* if there exists a natural number N, such that:

$D^*(\tilde{A}_n, \tilde{A}_0) < \varepsilon$, for all $n \geq N$

where $\tilde{A}_0$ is given by.

$\tilde{A}_0 = \{\tilde{q}_x^{\lambda} \,\tilde{\in}\, X^* \mid$ there exists a sequence $\{\tilde{q}_{x_n}^{\lambda_n}\} \,\tilde{\in}\, \tilde{A}_n$, such that $\tilde{q}_{x_n}^{\lambda_n} \longrightarrow \tilde{q}_x^{\lambda}$ if and only if $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda\}$

Definition (22): A sequence $\{\tilde{A}_n\}_{n=1}^{\infty}$ of fuzzy subsets of a Hausdorff fuzzy metric space (X^*, D^*) is said to be Cauchy sequence if there exists two narural numbers n and m, such that:

$D^*(\tilde{A}_n, \tilde{A}_m) < \varepsilon$, for all $m \geq n$

Definition (23): The Hausdorff fuzzy metric space (X^*, D^*) is complete if every Cauchy sequence is converge.

Theorem (24): The fuzzy subset $\tilde{A}_0$ of the Hausdorff fuzzy metric space (X^*, D^*) is bounded.

Proof: Let $\{\tilde{A}_n\}_{n=1}^{\infty}$ be a sequence of fuzzy subsets of (X^*, D^*), then for each $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}_n$, such that $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$, $x_n, x \in X$, $\lambda_n, \lambda \in (0,1]$

Hence $\{x_n\}$ and $\{\lambda_n\}$ are bounded sequences, which implies to that there exists two natural numbers M_1 and M_2, such that:

$d(x_n,x) \leq M_1$ and $\max\{\lambda_n - \lambda, 0\} \leq M_2$

Thus:

$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) = \max\{\lambda_n - \lambda, 0\} + d(x_n,x)$

$$D^*(\tilde{A}_n, \tilde{A}_0) = \max\{d^*(\tilde{A}_n, \tilde{A}_0), d^*(\tilde{A}_0, \tilde{A}_n)\} \qquad \ldots(8)$$

In equation (8), suppose that:

$\max\{d^*(\tilde{A}_n, \tilde{A}_0), d^*(\tilde{A}_0, \tilde{A}_n)\} = d^*(\tilde{A}_n, \tilde{A}_0)$

then:

$$\begin{aligned} D^*(\tilde{A}_n, \tilde{A}_0) &= d^*(\tilde{A}_n, \tilde{A}_0) \\ &= \sup_{\tilde{q}_{x_n}^{\lambda_n} \tilde{\in} \tilde{A}_0} \ \inf_{\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}_0} d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) \\ &= \sup \inf\{\max\{\lambda_n - \lambda, 0\} + d(x_n,x)\}, \text{ for all } x_n, x \in X, \lambda_n, \lambda \in (0,1] \\ &\leq M_1 + M_2 = M \in N \end{aligned}$$

Similarly, when $d^*(\tilde{A}_0, \tilde{A}_n)$ is the max in equation (8), hence the sequence $\{\tilde{A}_n\}_{n=1}^{\infty}$ is a bounded sequence of fuzzy point in $\tilde{A}_0$, which implies that $\tilde{A}_0$ is bounded. ■

Theorem (25): Let (X, d) be a complete metric space, then (X^*, D^*) is complete Hausdorff fuzzy metric space.

Proof: Let $\{\tilde{A}_n\}_{n=1}^{\infty}$ be a fuzzy Cauchy sequence in (X^*, D^*)

To prove that $\lim_{n \to \infty} \tilde{A}_n = \tilde{A}_0$, i.e., to prove that for each $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}_0$ there exists a sequence of fuzzy points $\tilde{q}_{x_n}^{\lambda_n} \,\tilde{\in}\, \tilde{A}_n$, such that $\tilde{q}_{x_n}^{\lambda_n} \longrightarrow \tilde{q}_x^{\lambda}$, which is happend when $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$

Since for all $\varepsilon > 0$, there exists N, which is a natural number, such that:

$$d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_{x_m}^{\lambda_m}) < \varepsilon, \tilde{q}_{x_m}^{\lambda_m}, \text{ for all } n, m > N \qquad \ldots(9)$$

From Definition (19), we have (9) is equivalent to:

$d^*(x_n,x_m) < \varepsilon$ and $\max\{\lambda_n - \lambda_m, 0\} < \varepsilon$, for all $n, m > N$

Now, to prove that $\tilde{q}_{x_n}^{\lambda_n} \longrightarrow \tilde{q}_x^{\lambda}$, where $\tilde{q}_{x_n}^{\lambda_n} \,\tilde{\in}\, \tilde{A}_n$, $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}_0$, i.e., to prove that:

$d^*(x_n,x) < \varepsilon$ and $\max\{\lambda_n - \lambda, 0\} < \varepsilon$

Since $\{x_n\}$, where n is a natural number, is Cauchy sequence in (X, d) which is a complete metric space, hence it is converge to $x \in X$, i.e., $d(x_n,x) < \varepsilon/2$

Also, $\{\lambda_n\}$, $n \in N$ is a Cauchy sequence of real numbers in (0,1], hence it is converge to $\lambda \in (0,1]$, i.e., $\max\{\lambda_n - \lambda, 0\} < \varepsilon/2$

Then $d^*(\tilde{q}_{x_n}^{\lambda_n}, \tilde{q}_x^{\lambda}) < \varepsilon$, which means that it is a convergent sequence of fuzzy points in $\{\tilde{A}_n\}_{n=1}^{\infty}$ to a fuzzy point $\tilde{q}_x^{\lambda} \,\tilde{\in}\, \tilde{A}_0$ and since:

$D^*(\tilde{A}_n, \tilde{A}_0) = \max\{d^*(\tilde{A}_n, \tilde{A}_0), d^*(\tilde{A}_0, \tilde{A}_n)\}$

Thus $\{\tilde{A}_n\}_{n=1}^{\infty}$ is a convergent sequence of fuzzy sets in (X^*, D^*)

Therefore, (X^*, D^*) is a complete Hausdorff fuzzy metric space. ■

Definition (26): Let (X^*, D^*) be a Hausdorff fuzzy metric space and $\tilde{A}$ is a fuzzy subset of X^* and let $\varepsilon > 0$ any given number, the ε-neighborhood $\tilde{A} + \varepsilon$ of $\tilde{A}$ in X^* is defined by:

$\tilde{A} + \varepsilon = \{y \in X : \exists\, x \in \tilde{A}$, such that $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \leq \varepsilon\}$

Theorem (27): Let $\tilde{A}, \tilde{B} \tilde{\in} X^*$ and $\varepsilon > 0$, then $D^*(\tilde{A}, \tilde{B}) < \varepsilon$, if and only if $\tilde{A} \subseteq \tilde{B} + \varepsilon$ and $\tilde{B} \subseteq \tilde{A} + \varepsilon$.

Proof: Suppose that $D^*(\tilde{A}, \tilde{B}) < \varepsilon$, for all $\tilde{A}, \tilde{B} \tilde{\in} X^*$. Hence:

$$\sup_{\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}} \inf_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_a^{\lambda_1}, \tilde{q}_b^{\lambda_2}) \le \varepsilon$$

and

$$\sup_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} \inf_{\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}} d^*(\tilde{q}_b^{\lambda_2}, \tilde{q}_a^{\lambda_1}) \le \varepsilon$$

Now, $\sup_{\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}} \inf_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_a^{\lambda_1}, \tilde{q}_b^{\lambda_2}) \le \varepsilon$ implies for all $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}$

that $\inf_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_a^{\lambda_1}, \tilde{q}_b^{\lambda_2}) \le \varepsilon$

i.e., for all $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}$, then $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{B} + \varepsilon$, which yields to $\tilde{A} \subseteq \tilde{B} + \varepsilon$

Similarly, $\sup_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} \inf_{\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}} d^*(\tilde{q}_b^{\lambda_2}, \tilde{q}_a^{\lambda_1}) \le \varepsilon$ implies that

$\tilde{B} \subseteq \tilde{A} + \varepsilon$

Conversely, if $\tilde{A} \subseteq \tilde{B} + \varepsilon$ and $\tilde{B} \subseteq \tilde{A} + \varepsilon$

Since $\tilde{A} \subseteq \tilde{B} + \varepsilon$, then taking $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}$ implies to $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{B} + \varepsilon$

Hence there exists $\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}$, such that $d^*(\tilde{q}_b^{\lambda_2}, \tilde{q}_a^{\lambda_1}) \le \varepsilon$, for all $\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}$

Thus $\inf_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_a^{\lambda_1}, \tilde{q}_b^{\lambda_2}) \le \varepsilon$ and hence:

$$\sup_{\tilde{q}_a^{\lambda_1} \in \tilde{A}} \inf_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} d^*(\tilde{q}_a^{\lambda_1}, \tilde{q}_b^{\lambda_2}) \le \varepsilon$$

Similarly, $\tilde{B} \subseteq \tilde{A} + \varepsilon$ will implies to:

$$\sup_{\tilde{q}_b^{\lambda_2} \tilde{\in} \tilde{B}} \inf_{\tilde{q}_a^{\lambda_1} \tilde{\in} \tilde{A}} d^*(\tilde{q}_b^{\lambda_2}, \tilde{q}_a^{\lambda_1}) \le \varepsilon$$

Therefore, $D^*(\tilde{A}, \tilde{B}) < \varepsilon$. ■

Theorem (28): The Hausdorff fuzzy metric space (X^*, D^*) is compact whenever (X, d) is a compact metric space.

Proof: To prove that (X^*, D^*) is compact, i.e., to prove that (X^*, D^*) is complete and also totally bounded fuzzy metric space

From Theorem (25), (X^*, D^*) is complete Hausdorff fuzzy metric space

Since (X, d) is compact metric space, then it is also totally bounded metric space and hence it remains that to prove (X^*, D^*) is a "totally bounded" Hausdorff fuzzy metric space

Let $\tilde{A}_i$, $i = 1, 2, \ldots, n$; be a finite set of fuzzy open sets, such that for each fuzzy subset $\tilde{A}$ of X^*, if $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}$ and $\tilde{q}_x^{\lambda} \tilde{\in} \bigcup_{i=1}^{n} \tilde{A}_i$

Let $\varepsilon > 0$ and $\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{A}_i$, for all $i = 1, 2, \ldots, n$; with radius ε

Hence, the following finite set of fuzzy point may be constructed:

$$\tilde{W} = \left\{ \tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n} \right\}$$

Since $\bigcup_{i=1}^{n} \tilde{A}_i$ is also a fuzzy set with maximum membership function, then using Theorem (4.1) [27] implies to $d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_x^{\lambda}) = 0$, i.e., $d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_x^{\lambda}) < \varepsilon$, for each *i*.

Similarly, it may be proved that $d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_i}^{\lambda_i}) < \varepsilon$, for each *i*. Hence:

$$\max\left\{ \sup_{\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{A}_i} \inf_{\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}} d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_x^{\lambda}), \sup_{\tilde{q}_x^{\lambda} \tilde{\in} \tilde{A}} \inf_{\tilde{q}_{x_i}^{\lambda_i} \tilde{\in} \tilde{A}_i} d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_i}^{\lambda_i}) \right\} < \varepsilon$$

i.e., $D^*(\tilde{A}_i, \tilde{A}) < \varepsilon$, which means that $\tilde{A}$ possess an ε-fuzzy net.

Thus (X^*, D^*) is totally bounded Hausdorff fuzzy metric space

Therefore, (X^*, D^*) is a compact Hausdorff fuzzy metric space. ■

Definition (29): The Hausdorff fuzzy metric space (X^*, D^*) is said to be countably compact if every infinite fuzzy subset $\tilde{A}$ of X^* has a fuzzy limit point $\tilde{q}_x^{\lambda} \tilde{\in} X^*$.

Theorem (30): Let (X^*, D^*) be a compact Hausdorff fuzzy metric space, then (X^*, D^*) is countably compact Hausdorff fuzzy metric space.

Proof: Suppose that (X^*, D^*) is not countably compact Hausdorff fuzzy metric space

Let $\tilde{A}$ be an infinite subset of X^*, which has no fuzzy limit point in X^*, i.e., there is no $\tilde{q}_x^{\lambda} \tilde{\in} X^*$, such that there exists $\tilde{q}_{x_n}^{\lambda_n} \tilde{\in} \tilde{A}$, and $d^*(\tilde{q}_x^{\lambda}, \tilde{q}_{x_n}^{\lambda_n}) < \varepsilon$

Since X^* is a compact Hausdorff fuzzy metric space, hence there exist a fuzzy neighborhood system $\mathcal{U}_{\tilde{q}_x^{\lambda}}$ and for all $\tilde{q}_x^{\lambda} \tilde{\in} X^*$, there exists finite subcover $\tilde{U}_i$, $i = 1, 2, \ldots, n$; such that:

$$\tilde{q}_x^{\lambda} \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_i, \ \tilde{U}_i \tilde{\in} \mathcal{U}_{\tilde{q}_x^{\lambda}}, \ \forall i$$

Let each fuzzy point $\tilde{q}_x^{\lambda} \tilde{\in} X^*$ belong to an open fuzzy set $\tilde{U}_{i_{\tilde{q}_{x_i}^{\lambda_i}}}$, i.e., $\tilde{q}_x^{\lambda} \tilde{\in} \tilde{U}_{i_{\tilde{q}_{x_i}^{\lambda_i}}}$, which contains at most one fuzzy point of X^* and since $\mathcal{U}_{\tilde{q}_x^{\lambda}}$ is the fuzzy neighborhood system of X^* with open cover $\tilde{U}_{i_{\tilde{q}_{x_i}^{\lambda_i}}}$ of the compact Hausdorff fuzzy metric space X^*. Hence:

$$\tilde{q}_x^\lambda \tilde{\in} \bigcup_{i=1}^{n} \tilde{U}_{i_{\tilde{q}_{x_i}^{\lambda_i}}} = \tilde{U}_{1_{\tilde{q}_{x_1}^{\lambda_1}}} \cup \tilde{U}_{2_{\tilde{q}_{x_2}^{\lambda_2}}} \cup \ldots \cup \tilde{U}_{m_{\tilde{q}_{x_m}^{\lambda_m}}}$$

Since $\tilde{A} \tilde{\in} X^*$, then:

$$\tilde{A} \tilde{\in} \tilde{U}_{1_{\tilde{q}_{x_1}^{\lambda_1}}} \cup \tilde{U}_{2_{\tilde{q}_{x_2}^{\lambda_2}}} \cup \ldots \cup \tilde{U}_{m_{\tilde{q}_{x_m}^{\lambda_m}}}$$

and since each $\tilde{U}_{i_{\tilde{q}_{x_i}^{\lambda_i}}}$ contains at most one fuzzy point $\tilde{q}_{x_i}^{\lambda_i}$, for all $i = 1, 2, \ldots, m$

Therefore, $\tilde{A}$ contains at most m fuzzy points $\tilde{q}_{x_i}^{\lambda_i}$, $i = 1, 2, \ldots, m$; and hence $\tilde{A}$ is finite which has no accumulation fuzzy point, which is a contradiction.

Thus, (X^*, D^*) is "countably compact" Hausdorff fuzzy metric space. ■

Theorem (31): Let (X^*, D^*) be a countably compact Hausdorff "fuzzy metric space", then (X^*, D^*) is a totally bounded Hausdorff "fuzzy metric space".

Proof: Suppose that the Hausdorff fuzzy metric space (X^*, D^*) is countably fuzzy compact, but not fuzzy totally bounded.

Hence, there exists $\varepsilon > 0$, such that there is no ε-fuzzy net of X^*

Let $\tilde{q}_{x_1}^{\lambda_1}$ be a fuzzy point in X^*

Then the finite fuzzy set $\{ \tilde{q}_{x_1}^{\lambda_1} \}$ is not an ε-fuzzy net of X^*

Therefore, there exist a fuzzy point $\tilde{q}_{x_2}^{\lambda_2} \tilde{\notin} \tilde{U}(\tilde{q}_{x_1}^{\lambda_1}, \varepsilon)$, i.e., $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}) \geq \varepsilon$. Also the finite fuzzy set $\{ \tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2} \}$ is not an ε-fuzzy net of X^*

Also, there exist another fuzzy point $\tilde{q}_{x_3}^{\lambda_3}$ such that $\tilde{q}_{x_3}^{\lambda_3} \tilde{\notin} \bigcup_{i=1}^{2} \tilde{U}(\tilde{q}_{x_i}^{\lambda_i}, \varepsilon)$, i.e., $d^*(\tilde{q}_{x_2}^{\lambda_2}, \tilde{q}_{x_3}^{\lambda_3}) \geq \varepsilon$ and $d^*(\tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_3}^{\lambda_3}) \geq \varepsilon$

Continue the proof by using the mathematical induction. Suppose that there exists finite fuzzy set $\{ \tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n} \}$ such that $d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_{x_j}^{\lambda_j}) \geq \varepsilon$, when $i \neq j$

Therefore, the fuzzy set $\{ \tilde{q}_{x_1}^{\lambda_1}, \tilde{q}_{x_2}^{\lambda_2}, \ldots, \tilde{q}_{x_n}^{\lambda_n} \}$ is not an ε-fuzzy net of X^* which implies that there exists a fuzzy point $\tilde{q}_{x_{n+1}}^{\lambda_{n+1}}$, in which $\tilde{q}_{x_{n+1}}^{\lambda_{n+1}} \tilde{\notin} \bigcup_{i=1}^{n} \tilde{U}(\tilde{q}_{x_i}^{\lambda_i}, \varepsilon)$, i.e.,

$$d^*(\tilde{q}_{x_{n+1}}^{\lambda_{n+1}}, \tilde{q}_{x_i}^{\lambda_i}) \geq \varepsilon, \text{ when } i \neq n+1$$

By induction, we can define the sequence $\{ \tilde{q}_{x_n}^{\lambda_n} \}$, $n \in N$ from different fuzzy points in countable compact space X^*, such that $d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_{x_j}^{\lambda_j}) \geq \varepsilon$, when $i \neq j$

Let $\tilde{E} = \bigcup_{n \in N} \{ \tilde{q}_{x_n}^{\lambda_n} \}$

Now, by countably compact Hausdorff fuzzy metric space, we means that there exists $\tilde{q}_x^\lambda \tilde{\in} X^*$ which is a fuzzy limit point of $\tilde{U}$, hence $\tilde{U}(\tilde{q}_x^\lambda, \varepsilon)$ contains infinitely many fuzzy point of $\tilde{E}$ and this contradicts the fact that $d^*(\tilde{q}_{x_i}^{\lambda_i}, \tilde{q}_{x_j}^{\lambda_j}) \geq \varepsilon$, if $i \neq j$. ■

ACKNOWLEDGMENT

The author would like to extensively thanks and appreciate "Asst. Prof. Dr. Fadhel S. Fadhel the head of Mathematics and Computer Applications Department, College of Science, Al-Nahrain University, Baghdad-Iraq" for his general efforts in reviewing and verifying the contents of this paper.

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