Some New Results in Theory and Application on Positive Definiteness of Portfolio Covariance Matrix

Xue Deng, Cuirong Huang

***Abstract*—Covariance matrix carries a big weight in the domain of portfolio. Most researchers spend more time on model construction, risk measurement and algorithm analysis, but paid little attention to the deeper theory research and rigorous proof on the positive definiteness of the covariance matrix. This paper aims to obtain the equivalent condition about the positive definite of covariance matrix. In theory, some innovative and significant results are identified by matrix theory on risk value range of equal weight portfolio and optimal portfolio. These results obtained by using rigorous mathematical proof in this paper can explain some actual portfolio problems about risk. In practice, four numerical examples are shown to verify the validity of the proposed lemmas and theorems. Finally, the results of this paper show that the positive definiteness of the covariance matrix will no longer be an assumption, but a condition of the rigorous theoretical basis, which provides theoretical support for the research of portfolio theory.**

***Index Terms*—covariance matrix, positive definiteness, value range of risk, eigenvalue, non-singular matrix**

I. Introduction

In the gradual evolution of modern portfolio theory, it is precisely because of the M-V model proposed by Markowitz [1] that it has laid the foundation stone for the majority of scholars. The kernel idea of this theory is put forward two quantitative indicators. The conventional M-V model is double criteria portfolio that explains a series of trade-offs between benefits and risks. That is to say, the trade-off scheme is to minimize risk under a certain return, and maximize the expected return under a certain risk. As a practical reference tool for investors, M-V model is queried and challenged in complex reality, which is the motivation for scholars to study. Recently, many scholars have developed modified models and hybrid algorithms based on M-V model to simulate the complex environment in the real financial world. For instance, Zheng and Yao [2] optimized the portfolio in view of risk measurement and disintegration of ensemble empirical pattern. Yu et al. [3] researched fuzzy multi-objective portfolio through hybrid genetic algorithm. Nazir [4] presented an efficient financial portfolio selection and optimization implementation of Anticor's algorithm.

Covariance matrix is one of the critical points in portfolio study. Some researchers have made some achievements in the portfolio field by utilizing the covariance matrix as well as its expanded information. On this point of view, the representatives of the literature are as follows. From the perspective of optimizing the investment portfolio, Alali and Cagri [5] studied the portfolio selection of return and risk factors based on the covariance matrix; Hitoshi et al. [6] focused on variance–covariance matrices with multiple objective model under fuzzy random framework; Richard et al. [7] proposed a mixed multivariate exponentially weighted moving average estimation in regard to variance–covariance matrix; Thomas [8] discussed the hypothesis test problem that covariance matrix being identity matrix when the dimension of covariance matrix is equal to or greater than the sample size; Vincent et al. [9] studied a global and explicit sensitivity analysis of portfolio models with respect to the semi-definite covariance; Sun et al. [10] calculated portfolio proportions under the Stein-type shrinkage framework through the properties of Cholesky decomposition. Ismail and Pham [11] studied a Markowitz portfolio with robust continuous-time, among them, and the uncertainty is determined by covariance matrix.

On the other hand, some researchers have obtained many research results about covariance matrix by estimating the parameters or elements. As for related literatures and researches in this field, the latest major achievements are mainly as following: Bouriga et al. [12] focused on covariance matrix estimation by utilizing Bayesian shrinkage methods; Dimitrios et al. [13] examined the covariance matrixes via the estimation of intraday nonparametric; Gillen et al. [14] developed the conjugate Bayesian regression model to address the issues, for the goal of obtaining the covariance matrix of multiple securities; Kourtis et al. [15] conducted empirical research on multiple decisions from the perspective of inverse estimation of covariance matrix; Deng [16]-[18] studied the portfolio with the information of covariance matrix.

It is worth mentioning that despite the importance of the covariance matrix is really recognized, most researchers regard the positive definiteness of the covariance matrix as an assumption, and there are few literatures on its theoretical

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study. So, it is necessary to do theoretical research about the positive definiteness of the covariance matrix. Thus, to handle this problem, this paper aims to elaborate the positive definite property about covariance matrix and present rigorous mathematical proofs and effective numerical examples.

In Section II, the definitions of notations are given to outline the mean-variance model. In Section III, new lemmas and theorems about the positive definiteness of covariance matrix and value range of risk are put forward simultaneously. In Section IV, the consistency between theory and practice is emphasized by four numerical examples. Finally, the main work and its significance are summarized in Section V.

II. Problem Descriptions

In this section, we will retrospect descriptions of portfolio concisely. For the better comprehension of this paper, we need to explain the meaning of notations in detail (see TABLE I), which will be utilized in the paper subsequently.

TABLE I
NOTATIONS AND STATEMENTS

Notations	Statements
n	number of securities $(n \geq 2)$
i	the $i-\text{th}$ security
F	n dimensional column vector with elements being 1
x_i	investment proportion of the $i-\text{th}$ security
X	investment proportional vector, $X=(x_1,x_2,\cdots,x_n)^T$
r_i	return of the $i-\text{th}$ security
R_i	mean of r_i, quantitative indicator of security return
σ_{ii}	variance of r_i, quantitative indicator of security risk
r	return of portfolio
R	mean of r
σ_{ij}	the covariance of r_i and r_j
Σ	covariance matrix of portfolio, where $\Sigma=(\sigma_{ij})_{n\times n}$
σ^2	variance of portfolio, where $\sigma^2=X^T\Sigma X$

It is worth mentioning that the security returns $r_i,\ i=1,\cdots,n$ are affected by the liquidity and uncertainty of financial market. In order to describe the uncertainty of security return in the real world, we regard r_i as random variable. In addition, rational investors pursue the optimal portfolio, that is, maximize R_i or minimize σ_{ii}. Consider the portfolio return r, where $r=\sum_{i=1}^n x_i r_i$. While due to investment proportion x_i is a constant and r_i is a random variable, it is clearly that r is also a random variable. In other words, we have $R=\sum_{i=1}^n x_i R_i$.

Depending on the statements mentioned above, the philosophy of portfolio is to pursue minimize portfolio risk σ^2 among the restriction condition $F^T X=1$ whose portfolio without short selling. To express it in mathematical way, the corresponding model to find investment proportional vector X will be as follows:

$$\text{Model (I)}: \begin{cases} \min \quad \sigma^2 = X^T \Sigma X \\ \text{s.t.} \quad F^T X = 1. \end{cases}$$

III. Some New Results

In this section, some new results about portfolio covariance matrix and value range of risk are discussed.

Let Σ be positive definite matrix. Denote $\lambda_1,\lambda_2,\cdots,\lambda_n$ as the eigenvalues of Σ. Without loss of generality, we let $\lambda_1>\lambda_2>\cdots>\lambda_n>0$, $\lambda_{\max}=\lambda_1,\lambda_{\min}=\lambda_n$, $\sigma_{\max}=\max\limits_i\{\sigma_{ii}\}$, $\sigma_{\min}=\min\limits_i\{\sigma_{ii}\}$. With regard to the positive definite properties of covariance matrix and value range of risk, we have the following results.

A. New Lemmas about Positive Definite Properties of the Covariance Matrix

Lemma 1 : *The covariance matrix Σ must be positive semi-definite matrix.*

Proof:

On the basic definition about covariance matrix $\Sigma=(\sigma_{ij})_{n\times n},\ i,j=1,\cdots,n$. Obviously, $\Sigma=\Sigma^T$. We have

$$\sigma_{ij}=\text{cov}(r_i,r_j)=E[(r_i-E(r_i))(r_j-E(r_j))] \tag{1}$$

and

$$\begin{aligned} X^T\Sigma X &= \sum_{i=1}^n\sum_{j=1}^n x_i\sigma_{ij}x_j \\ &= \sum_{i=1}^n\sum_{j=1}^n x_i\left(E[(r_i-E(r_i))(r_j-E(r_j))]\right)x_j \\ &= E\sum_{i=1}^n\sum_{j=1}^n\left([x_i(r_i-E(r_i))x_j(r_j-E(r_j))]\right) \\ &= E\left(\sum_{i=1}^n x_i(r_i-E(r_i))\right)^2 \geq 0. \end{aligned} \tag{2}$$

So $X^T\Sigma X\geq 0$, which means Σ must be positive semi-definite matrix. Lemma 1 has been proved.

Lemma 2 : *The following sentences (a), (b), (c) and (d) are equivalent:*

(a) *The covariance matrix Σ is positive definite matrix;*

(b) *The covariance matrix Σ is non-singular matrix;*

(c) *The covariance matrix Σ is full-rank;*

(d) *Denote σ_i as the row vectors of matrix Σ, where $\sigma_i=(\sigma_{i1},\sigma_{i2},\cdots,\sigma_{in}),\ i=1,2,\cdots,n.$ Then, σ_i are linearly independent.*

Proof:

Step 1: (a) $\Rightarrow$ (b)

By the relation between matrix determinant and matrix eigenvalues, we always have

$$|\Sigma|=\prod_{i=1}^n\lambda_i. \tag{3}$$

Supposing the matrix Σ is positive definite matrix, we have

$$\lambda_i>0,\ (i=1,2,\cdots,n). \tag{4}$$

By utilizing (3), we obtain

$|\Sigma|>0 \Rightarrow |\Sigma|\neq 0 \Rightarrow \Sigma$ is non-singular;

Step 2: (b) $\Rightarrow$ (c)

In accordance with the definitions of non-singular matrix and full-rank matrix, we have

$$\Sigma \text{ is non-singular} \Leftrightarrow |\Sigma|\neq 0 \Leftrightarrow \Sigma \text{ is full-rank}; \tag{5}$$

Step 3: (c) $\Rightarrow$ (d)

Consider the definition of linear dependence of the vectors, it is simple to prove that:

$$\sum \text{ is full}-\text{rank} \Leftrightarrow \text{row vectors } \sigma_i \text{ of matrix } \sum \text{ are linearly independent;} \tag{6}$$

Step 4: (d) $\Rightarrow$ (a)

According to (5) and (6), we have

$$\begin{aligned}&\text{row vectors } \sigma_i \text{ of matrix } \sum \text{ are linearly independent}\\ &\Leftrightarrow |\Sigma| \neq 0 \Leftrightarrow \prod_{i=1}^{n} \lambda_i \neq 0.\end{aligned} \tag{7}$$

By Lemma 1, we obtain

$$\lambda_i \geq 0,\ (i=1,2,\cdots,n). \tag{8}$$

Combined with (7), we can get $\lambda_i > 0,\ (i=1,2,\cdots,n)$. Thus, it is comprehensible that the matrix Σ is positive definite matrix.

To sum up, (a), (b), (c) and (d) are equivalent conditions. Lemma 2 has been proved by Steps 1-4.

B. New Theorems about Value Range of Risk

Lemma 3[19]: *As to Model (1):* $\begin{cases}\min & \sigma^2 = X^T \sum X \\ \text{s.t.} & F^T X = 1\end{cases}$*, let* X_* *be the coefficient vector of optimal investment proportional and* σ_*^2 *be the variance of optimal portfolio about Model (1). Concretely,* $X_* = \dfrac{\sum^{-1} F}{F^T \sum^{-1} F}$ *and* $\sigma_*^2 = \dfrac{1}{F^T \sum^{-1} F}$*, where* $F = (1,1,\cdots,1)^T$.

Theorem 1: *The value range of risk in the optimal portfolio model* $\sigma_*^2 < \sigma_{\max}$.

Proof:

On the one hand, the given covariance matrix $\sum$ is positive definite matrix, as to arbitrary 2-order principal minor determinant of $\sum$ is positive definite, so we have

$$\begin{vmatrix}\sigma_{ii} & \sigma_{ij}\\ \sigma_{ji} & \sigma_{ii}\end{vmatrix} = \sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji} = \sigma_{ii}\sigma_{jj} - (\sigma_{ij})^2 > 0, \tag{9}$$

where $\forall i \neq j, \sigma_{ij} = \sigma_{ji}$. Then

$$(\sigma_{ij})^2 < \sigma_{ii}\sigma_{jj} \leq (\sigma_{\max})^2 \tag{10}$$

and

$$\sigma_{ij} < \sigma_{\max}. \tag{11}$$

On the other hand, the portfolio risk is

$$\sigma^2 = X^T \sum X = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j, \tag{12}$$

by (11) and (12), we obtain

$$\begin{aligned}\sigma_*^2 &< \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{\max} x_i x_j = \sigma_{\max} \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\\ &= \sigma_{\max} \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} x_j\right)\\ &= \sigma_{\max} \cdot 1 \cdot 1 = \sigma_{\max}.\end{aligned} \tag{13}$$

Namely, we have $\sigma_*^2 < \sigma_{\max}$. (14)

Theorem 1 has been proved. Hence, it is reasonable to contend that the optimal risk is less than the maximal risk, which has reduced the portfolio risk.

Theorem 2: *The value range of the minimizing risk portfolio* $\sigma_*^2 \leq \sigma_{\min}$.

Proof:

As we know,

$$\sigma_*^2 = \min\{\sigma^2\} = \min\{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j\}. \tag{15}$$

If $X = (1,0,0,\cdots,0)^T$, $\sigma^2 = X^T \sum X = \sigma_{11}$;

If $X = (0,1,0,\cdots,0)^T$, $\sigma^2 = X^T \sum X = \sigma_{22}$;

If $X = (0,0,\cdots,0,1)^T$, $\sigma^2 = X^T \sum X = \sigma_{nn}$.

Then, we obtain

$$\sigma_*^2 = \min\{\sigma^2\} \leq \min\{\sigma_{11}, \sigma_{22}, \cdots, \sigma_{nn}\} = \sigma_{\min}, \tag{16}$$

accordingly, we have

$$\sigma_*^2 \leq \sigma_{\min}. \tag{17}$$

Theorem 2 has been proved. The above result shows that the optimal risk of minimizing risk portfolio is less than or equal to the minimal risk of risky assets.

Theorem 3: *If* $X_A = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$*, then we will have*

$$\sigma_A^2 = X_A^T \sum X_A < \left(\frac{\sum_{i=1}^{n}\sqrt{\sigma_{ii}}}{n}\right)^2.$$

Proof:

On the basic definition about equal weight portfolio, we have

$$\sigma_A^2 = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}) \sum \begin{pmatrix}\frac{1}{n}\\ \frac{1}{n}\\ \vdots\\ \frac{1}{n}\end{pmatrix} = \frac{1}{n^2}\sum_{i,j=1}^{n} \sigma_{ij} \leq \frac{1}{n^2}\left|\sum_{i,j=1}^{n} \sigma_{ij}\right|. \tag{18}$$

Since $\sum$ is a positive definite matrix, then

$$(\sigma_{ij})^2 < \sigma_{ii}\sigma_{jj} \Rightarrow |\sigma_{ij}| < \sqrt{\sigma_{ii}} \cdot \sqrt{\sigma_{jj}}. \tag{19}$$

By utilizing the property of absolute value inequality and (19), now it is easily computed that

$$\begin{aligned}\left|\sum_{i,j=1}^{n} \sigma_{ij}\right| &\leq \sum_{i,j=1}^{n} |\sigma_{ij}| < \sum_{i,j=1}^{n} (\sqrt{\sigma_{ii}} \cdot \sqrt{\sigma_{jj}})\\ &= (\sum_{i=1}^{n}\sqrt{\sigma_{ii}}) \cdot (\sum_{j=1}^{n}\sqrt{\sigma_{jj}}) = (\sum_{i=1}^{n}\sqrt{\sigma_{ii}})^2.\end{aligned} \tag{20}$$

From (18) and (20), we obtain

$$\sigma_A^2 < \frac{1}{n^2}(\sum_{i=1}^{n}\sqrt{\sigma_{ii}})^2 = \left(\frac{\sum_{i=1}^{n}\sqrt{\sigma_{ii}}}{n}\right)^2. \tag{21}$$

Theorem 3 has been proved.

Theorem 4: $\dfrac{1}{n\left(\dfrac{1}{\lambda_1} + \cdots + \dfrac{1}{\lambda_n}\right)} < \sigma_A^2 < \dfrac{\lambda_1 + \cdots + \lambda_n}{n}$.

Proof:

By Theorem 3, $\sigma_A^2 = X_A^T \sum X_A < \left(\frac{\sum_{i=1}^{n} \sqrt{\sigma_{ii}}}{n} \right)^2$. By the result of Lemma 3, the equal weight risk must be larger than the optimal risk, so we have $\sigma_A^2 \geq \sigma_*^2 = \frac{1}{F^T \Sigma^{-1} F}$.

1) On the one hand, by Theorem 3, we have

$$\sigma_A^2 = X_A^T \sum X_A < \left(\frac{\sum_{i=1}^{n} \sqrt{\sigma_{ii}}}{n} \right)^2 \leq \left(\sqrt{\frac{\sum_{i=1}^{n} \sigma_{ii}}{n}} \right)^2 = \frac{\sum_{i=1}^{n} \sigma_{ii}}{n}. \tag{22}$$

Furthermore, we know

$$\sum_{i=1}^{n} \sigma_{ii} = \mathrm{tr}(\Sigma) = \sum_{i=1}^{n} \lambda_i, \tag{23}$$

then

$$\sigma_A^2 < \frac{\sum_{i=1}^{n} \lambda_i}{n}. \tag{24}$$

2) On the other hand, by Lemma 3, we have

$$\sigma_*^2 = \frac{1}{F^T \Sigma^{-1} F} \leq \sigma_A{}^2, \tag{25}$$

since Σ^{-1} is positive definite matrix, which means $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ are corresponding eigenvalues. By (22), we get

$$X_A^T \Sigma^{-1} X_A < \frac{\sum_{i=1}^{n} \frac{1}{\lambda_i}}{n}. \tag{26}$$

Note that

$$\begin{aligned} F^T \Sigma^{-1} F &= n^2 \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right) \Sigma^{-1} \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T \\ &< n^2 \frac{\sum_{i=1}^{n} \frac{1}{\lambda_i}}{n} = n \sum_{i=1}^{n} \frac{1}{\lambda_i}. \end{aligned} \tag{27}$$

Equivalently, we get

$$\frac{1}{n \sum_{i=1}^{n} \frac{1}{\lambda_i}} < \frac{1}{F^T \Sigma^{-1} F}, \tag{28}$$

by (25), we have

$$\frac{1}{n \sum_{i=1}^{n} \frac{1}{\lambda_i}} < \frac{1}{F^T \Sigma^{-1} F} \leq \sigma_A{}^2. \tag{29}$$

Consequently, by 1) and 2), we have proved $\frac{1}{n \sum_{i=1}^{n} \frac{1}{\lambda_i}} < \sigma_A^2 < \frac{\sum_{i=1}^{n} \lambda_i}{n}$. Theorem 4 has been proved.

C. *The Equivalent Condition about $X_A = X_*$*

Theorem 5: *The equivalent condition about $X_A = X_*$ is that the row sum of matrix $\sum$ is equal to constant, that is*

$$d_i = \sum_{j=1}^{n} \sigma_{ij} \equiv \frac{n}{F^T \Sigma^{-1} F} \ (i = 1, 2, \cdots, n).$$

Proof:

On the basic definition about equal weight portfolio and minimizing risk portfolio, we have

$$X_A = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T = \frac{1}{n}(1, 1, \cdots, 1)^T = \frac{1}{n} F \tag{30}$$

and

$$X_* = \frac{\Sigma^{-1} F}{F^T \Sigma^{-1} F}. \tag{31}$$

Then,

$$X_* = X_A \Leftrightarrow \frac{\Sigma^{-1} F}{F^T \Sigma^{-1} F} = \frac{F}{n} \Leftrightarrow \frac{nF}{F^T \Sigma^{-1} F} = \sum F. \tag{32}$$

Namely,

$$\begin{aligned} \frac{n}{F^T \Sigma^{-1} F} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} &= \sum F = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{21} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= \left[\sum_{j=1}^{n} \sigma_{1j} \quad \sum_{j=1}^{n} \sigma_{2j} \quad \cdots \quad \sum_{j=1}^{n} \sigma_{nj} \right]^T. \end{aligned} \tag{33}$$

Note that the expression $F^T \Sigma^{-1} F$ is a real number, so the expression $\frac{n}{F^T \Sigma^{-1} F}$ is a real number too. We can obtain

$$d_i = \sum_{j=1}^{n} \sigma_{ij} \equiv \frac{n}{F^T \Sigma^{-1} F} \ (i = 1, 2, \cdots, n). \tag{34}$$

Theorem 5 has been proved.

IV. Numerical Examples

In this section, let us elaborate that lemmas and theorems are consistent with numerical examples.

A. *Example Verification of Lemmas 1-2*

Example 1: Suppose there are two portfolios including three stocks, and the covariance matrices Σ_1 and Σ_2 of each portfolio are given as below:

$$\Sigma_1 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{8}{3} \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 2.50 & -0.75 & 0.75 \\ -0.75 & 1.30 & -1.85 \\ 0.75 & -1.85 & 2.70 \end{bmatrix},$$

please try to verify the result of Lemma 1.

Solution:

As to Σ_1, the eigenvalues of Σ_1 are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 4$, thus Σ_1 is positive semi-definite matrix;

As to Σ_2, the eigenvalues of Σ_2 are $\lambda_1 = 0.0056$, $\lambda_3 = 4.5193$, $\lambda_2 = 1.9751$, thus Σ_2 is a positive definite matrix.

Lemma 1 is verified.

Example 2: Suppose there is a portfolio including three stocks, and the covariance matrices $\sum$ of portfolio are given as below:

$$\Sigma = \begin{bmatrix} 2.50 & -0.75 & 0.75 \\ -0.75 & 1.30 & -1.85 \\ 0.75 & -1.85 & 2.70 \end{bmatrix},$$

please try to verify the results of Lemma 2.

Solution:

It's remarkable that by the properties of matrix determinant and matrix eigenvalues, we can get $|\Sigma| = \prod_{i=1}^{n} \lambda_i$.

Step 1: $(a) \Rightarrow (b)$

The eigenvalues of Σ are $\lambda_1 = 0.0056$, $\lambda_2 = 1.9751$, $\lambda_3 = 4.5193$. Obviously, $\lambda_i > 0, (i = 1, 2, 3)$. Thus, Σ is positive definite $\Rightarrow |\Sigma| = 0.05 \neq 0 \Rightarrow$ Σ is non-singular;

Step 2: $(b) \Rightarrow (c)$

It is obvious to prove that Σ is non-singular $\Leftrightarrow$ $|\Sigma| = 0.05 \neq 0 \Leftrightarrow \text{rank}(\Sigma) = 3 \Leftrightarrow \Sigma$ is full-rank;

Step 3: $(c) \Rightarrow (d)$

Consider the row vectors $\sigma_1, \sigma_2, \sigma_3$ of Σ , we have

$\sigma_1 = (2.50, -0.75, 0.75),$

$\sigma_2 = (-0.75, 1.30, -1.85),$

$\sigma_3 = (0.75, -1.85, 2.70).$

It is not difficult to prove that they are linearly independent, thus we obtain that Σ is full-rank $\Leftrightarrow$ row vectors $\sigma_1, \sigma_2, \sigma_3$ of Σ are linearly independent;

Step 4: $(d) \Rightarrow (a)$

Since the row vectors $\sigma_1, \sigma_2, \sigma_3$ of matrix Σ are linearly independent, and $\lambda_1 = 0.0056$, $\lambda_2 = 1.9751$, $\lambda_3 = 4.5193$, then, row vectors $\sigma_1, \sigma_2, \sigma_3$ of matrix Σ are linearly independent $\Leftrightarrow$ $\lambda_i > 0, (i = 1, 2, 3)$. By the properties of positive definite matrix, we can obtain that row vectors $\sigma_1, \sigma_2, \sigma_3$ of matrix Σ are linearly independent $\Leftrightarrow$ Σ is positive definite.

Thus, Lemma 2 is verified by Steps 1-4.

B. Example Verification of Theorems 1-5

Example 3: Some investor possesses four stocks, the covariance matrix Σ of four stocks is shown:

$$\Sigma = \begin{bmatrix} 140 & -120 & 110 & -150 \\ -120 & 240 & -80 & 100 \\ 110 & -80 & 100 & -100 \\ -150 & 100 & -100 & 300 \end{bmatrix},$$

please formulate risk minimization portfolio and verify Theorems 1-4.

Solution:

The risk minimization portfolio is:

$$\text{Model (I)}: \begin{cases} \min \quad \sigma^2 = X^T \Sigma X \\ \text{s.t.} \quad F^T X = 1 \end{cases}$$

As to Model (I), the optimal weight vector X_* is calculated as

$X_* = (0.5385, 0.2240, 0, 0.2375)^T$,

the optimal risk σ_*^2 is calculated as

$\sigma_*^2 = 12.8837.$

The corresponding equal weight vector X_A and portfolio risk σ_A^2 are as follows:

$X_A = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, $\sigma_A^2 = 33.3333.$

As to Σ, the maximal risk $\sigma_{\max} = 300$ and the minimal risk $\sigma_{\min} = 100$. The eigenvalues of Σ are $\lambda_1 = 544.1257$, $\lambda_2 = 165.9447$, $\lambda_3 = 66.1456$, $\lambda_4 = 3.7839$. From the above results, we can easily obtain

(a) $\sigma_*^2 = 12.8837 < \sigma_{\max} = 300$, this verifies Theorem 1: $\sigma_*^2 < \sigma_{\max}$.

(b) $\sigma_*^2 = 12.8837 < \sigma_{\min} = 100$, this verifies Theorem 2: $\sigma_*^2 < \sigma_{\min}$.

(c) We can see that

$$\left(\frac{\sum_{i=1}^{4} \sqrt{\sigma_{ii}}}{4} \right)^2 = \left(\frac{\sqrt{140} + \sqrt{240} + \sqrt{100} + \sqrt{300}}{4} \right)^2 = 186.63.$$

Obviously,

$$\sigma_A^2 = 33.3333 < \left(\frac{\sum_{i=1}^{4} \sqrt{\sigma_{ii}}}{4} \right)^2 = 186.63.$$

This verifies Theorem 3.

(a) We can easily compute that

$$\frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{4} = \frac{779.9999}{4} = 195,$$

$$\frac{1}{4\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \right)} = \frac{1}{4(0.2872)} = 0.8705,$$

then $0.8705 < \sigma_A^2 = 33.3333 < 195,$

that is,

$$\frac{1}{4\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \right)} < \sigma_A^2 < \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{4}.$$

This verifies Theorem 4.

Example 4: Suppose that investor invests in three stocks. The covariance matrix Σ of portfolio is as follows:

$$\Sigma = \begin{bmatrix} 500 & 250 & 150 \\ 250 & 600 & 50 \\ 150 & 50 & 700 \end{bmatrix},$$

please attempt to verify Theorem 5.

Solution:

The optimal weight vectors X_* can be computed via Model (I), i.e.

$X_* = (0.3333, 0.3333, 0.3333)^T$.

Similarly, the equal weight vectors X_A is calculated as

$X_A = (0.3333, 0.3333, 0.3333)^T$.

It is patently obvious that $X_A = X_*$. And we have

$d_1 = \sigma_{11} + \sigma_{12} + \sigma_{13} = 500 + 250 + 150 = 900,$

$d_2 = \sigma_{21} + \sigma_{22} + \sigma_{23} = 250 + 600 + 50 = 900,$

$d_3 = \sigma_{31} + \sigma_{32} + \sigma_{33} = 150 + 50 + 700 = 900,$

$d_1 = d_2 = d_3 = 900.$

At the same time, we can obtain the inverse covariance matrix

$$\Sigma^{-1} = \begin{bmatrix} 0.2689 & -0.1079 & -0.0499 \\ -0.1079 & 0.2110 & 0.0081 \\ -0.0499 & 0.0081 & 0.1530 \end{bmatrix} \times 10^{-2},$$

then we can compute the result

$$\frac{n}{F^T \Sigma^{-1} F} = \frac{3}{0.3333 \times 10^{-2}} = 900,$$

as previously mentioned, we can see

$$d_i = \sum_{j=1}^{n} \sigma_{ij} \equiv \frac{n}{F^T \Sigma^{-1} F}, \ i = 1, 2, 3.$$

Theorem 5 is verified.

V. CONCLUSIONS

On the research of portfolio, most researchers spend more time on model construction, risk measurement and algorithm analysis, and pay a little attention to the theory research on covariance matrix positive definiteness. It is essential to study the positive definiteness of covariance matrix. Some sufficient and necessary conditions are given for the covariance matrix being positive definite. Furthermore, some new results about risk value range in optimal portfolio model are presented. In addition, several accurate results of risk value range of equal weight portfolio model are obtained, too. These results are indicated by using rigorous mathematical proof of matrix theory. These results proposed in this paper are effective by some numerical examples, which are significant to theory research of portfolio.

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