Application of Chebyshev Polynomials of the Second Kind to the Numerical Solution of Weakly Singular Fredholm Integral Equations of the First Kind

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Abstract— **We present a numerical method for solving a certain potential-type weakly singular Fredholm integral equation of the first kind. The considered equation symbolizes the density functions of the Dirichlet problem for the Laplace equation for an open electrostatically charged thin contour in the plane. This equation is singular due to the singularity of the unknown function near and at the endpoints of the integration domain and the singularity of the logarithmic kernel. The method is based on Chebyshev polynomials of the second kind in matrix form and facilitates the procedures, saving the CPU time, and reducing the roundoff error. We separate the Chebyshev polynomials coefficients from the monomial basis functions and perform a square known coefficients matrix. By isolating the singular behavior of the unknown function near the endpoints and putting it in a separate badly-behaved function, the unknown function is represented by the product of two functions; the first is the known badly-behaved function and the second is a regular unknown function to be determined. The unknown function and the given data function are expanded into Chebyshev polynomials of degree at most n; each function is expressed via three matrices; one of which is the monomial basis matrix. Furthermore, we merge the kernel with the badly-behaved function with the approximant regular function in one entity integrand function that is defined on two half intervals, then we change the variables and integrate numerically. In this way, we effectively remove the singularities of both the kernel and the unknown function. Moreover, we reformulate the Gauss- Legendre quadrature formula in a simple matrix form and apply it to integrate numerically the one entity integrand function. To avoid the solution's singularity near and at the endpoints of the integration domain, we develop a collocation points distribution formula by using two different sets of points to compare for the best solution. The convergence in the mean of the numerical solutions is proved. To show the efficiency of the presented method, computational results with tables and graphs are demonstrated and discussed.**

Index Terms— **electromagnetism, electro-optics, singular Fredholm integral equations, logarithmic kernels, numerical methods.**

I. Introduction

The method of integral equations plays an important and pivotal role in solving many different mathematical applications in electromagnetics, heat transfer, antenna, radar, radioactivity, scattering theory, potential theory and electron-optics, which described by singular Fredholm integral equations of the first kind [1-6].

The goal of this paper is to present a new numerical method for solving Fredholm integral equations of the first kind with singular logarithmic kernel and singular unknown function. This equation is equivalent to the solution of Dirichlet boundary value problem for Laplace equation for an open straight electrostatically charged thin contour in the plane.

There are many published methods for solving similar equations by different techniques and methods [7,13]. V. I. Dmitriev, *et al.* in [7] provided an interesting iteration method with analytical analysis and solutions but without isolating the solution's singularity and the obtained solution was varied very slowly on the middle of the integration domain. The authors in [8-13] presented methods for solving this type of equation with analytical treatment of the solution's singularity near the endpoint of the integration domain as well as the logarithmic singularity of the kernel. Although the resulting solutions existing are somewhat accurate, the authors have employed in all these methods the trial and error method to determine an appropriate distribution of the collection points to obtain the best possible results.

Moreover, all these studies were not subjected to the study of the convergence or the maximum norm error. This was a strong motivation for presenting not only a new method but also creating a formula for the collocation points distribution to form an explicit formula for the perfect distribution so that the collocation points distributed far away from the endpoints of the integration domain. In addition, we developed the Gauss-Legendre quadrature formula in a matrix form for integrating numerically the obtained integrand function.

The presented method is based on applying the Chebyshev polynomials of the second kind [14] of degree n at most, via matrices in order to simplify the algebraic operations and integration. First, we extract the Chebyshev polynomials coefficients and separate them from the monomial basis functions and preform a known square Chebyshev coefficients matrix. Furthermore, we isolate the singular behavior of the unknown function near and at the endpoints in a separate function named a badly-behaved function. Thus, the unknown function may be represented by the product of

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two functions; the first is the known badly-behaved function which has a specific form and the second is a regular unknown function to be determined.

By expanding the unknown regular function and the given data function into Chebyshev polynomials of degree at most n, we can express each function via three matrices; one of which is the monomial basis matrix.

This mathematical procedure significantly reduces the round off errors since the integration of the monomials basis functions is easier and simpler than integrating Chebyshev polynomials of different degrees, particularly for higher degree polynomials. After substituting the approximate unknown function into the integral equation, we obtain an integral equation with one entity integrand equal to the product of the three quantities: the integral equation's logarithmic kernel, the badly-behaved function, and the monomial basis polynomials. Since the collocation points distribution is very sensitive due to the Chebyshev equioscillation theorem [14], we use some $\Delta x \geq 0$ to preform a formula for choosing the perfect collocation points distribution that avoids the singularities occurring when substituting the collocation points near the endpoints of the integration domain.

Moreover, we divided the integration domain into two intervals; in the first integral we change the main argument by a specific function and in the other interval we use another different specific function. Then we modify the Gauss-Legendre formula via matrices and numerically integrate the integrand function. Thus, the singularities of both the unknown function and the kernel are completely removed and the solution of the integral equation is transformed to the solution of a linear system of algebraic equations. By solving this system using the direct methods, we get the unknown coefficients matrix of the unknown function and thereby can find the unknown function itself.

II. Chebyshev polynomials of the second kind for solving singular Fredholm integral equations of the first kind.

Consider the Fredholm integral equation of the first kind

$$\int_{-1}^{1} k(x,t)\tau(t)dt = f(x); \ -1 \leq x \leq 1 \tag{1}$$

Here the kernel $k(x,t)=ln\frac{1}{|t-x|}$ is weakly singular when $t \to x$, where $\int_{-1}^{1}\int_{-1}^{1} |k(x,t)|^2 dxdt \leq M$ for a real number M and the unknown function $\tau(t)$ is also singular at the endpoints $t=\pm 1$ where $\|\tau(t)\|_2 \leq N$ for a real number N. The problem is to find the unknown function $\tau(t)$ as well as isolating the singularities of both functions $\tau(t), k(x,t)$. Assume $\tau(t)$ in the form

$$\tau(t)=\gamma(t)\theta(t); \ \gamma(t)=\frac{1}{\sqrt{1-t^2}} \tag{2}$$

Here, $\gamma(t)$ demonstrates the singular behavior of $\tau(t)$ whereas $\theta(t)$ is a regular (well-behaved) unknown function. Now, we retain $\gamma(t)$ and the kernel $k(x,t)$ without any approximations and only expanding $\theta(t)$ into Chebyshev polynomials of the second kind $\{U_i(t)\}_{i=0}^{n}$. Thus, we get the approximant $\tilde{\theta}_n(t)$ of degree n at most in the form

$$\tilde{\theta}_n(t)=\sum_{i=0}^{n} a_i U_i(t) \tag{3}$$

Here, the coefficients a_i are unknowns. Based on some matrix operations we can simplify (3). We extract the coefficients of the Chebyshev polynomials of the second kind $U_i(t)$ and place these coefficients in a square known coefficients Chebyshev matrix C in ascending powers. Thus, any set of Chebyshev polynomials may be expressed through these coefficients Chebyshev matrix C multiplied by the monomial basis matrix $\mathrm{T}(t)=[t^i]_{i=0}^{n}$. Furthermore $\tilde{\theta}_n(t)$ can be represented in the matrix form

$$\tilde{\theta}_n(t)=\mathrm{ACT}(t) \tag{4}$$

where $\mathrm{A}=\left[a_i\right]_{i=0}^{n}$ is the $1\times(n+1)$ row matrix whose entries a_i are unknowns to be determined, $\mathrm{C}=\left[c_{ij}\right]_{i,j=0}^{n}$ is the $(n+1)\times(n+1)$ square coefficients Chebyshev polynomials matrix which is a known matrix and $\mathrm{T}(t)=[t^i]_{i=0}^{n}$ is the monomial basis column matrix of order $(n+1)\times 1$. This simple mathematical treatment greatly contributes to reducing the roundoff errors. This is due to the fact that integrating the monomial basis functions is significantly easier than integrating Chebyshev polynomials, especially for higher degree n. Substituting $\tilde{\theta}_n(t)$ from eq. (4) into (1) we get the approximate unknown function $\tilde{\tau}_n(t)$ in the matrix form

$$\tilde{\tau}_n(t)=\mathrm{AC}\tilde{\mathrm{T}}(t) \tag{5}$$

where the $(n+1)\times 1$ column matrix $\tilde{\mathrm{T}}(t)$ is defined by $\tilde{\mathrm{T}}(t)=\left[\frac{t^i}{\sqrt{1-t^2}}\right]_{i=0}^{n}$. Similar to $\tilde{\theta}_n(t)$, we find the approximant $\tilde{f}_n(x)$ of the given data function in the form

$$\tilde{f}_n(x)=\mathrm{FCT}(x) \tag{6}$$

where, $U_i(x)$ are the Chebyshev polynomials of the second kind [11] and $\mathrm{F}=\left[f_i\right]_{i=0}^{n}$ is a column matrix of order $(n+1)\times n$ whose entries f_i can be determined by

$$f_i=\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\, f(x)\; U_i(x)dx; \ i \geq 0 \tag{7}$$

Substituting (5) and (6) into (1) yields

$$\mathrm{AC}\Phi(x)=\mathrm{FCT}(x) \tag{8}$$

Here $\Phi(x)=[\varphi_i(x)]_{i=0}^{n}$ is a column matrix of order $(n+1)\times1$. whose entries $\varphi_i(x)$ are defined by

$$\varphi_i(x)=\int_{-1}^{0}\frac{t^i}{\sqrt{1-t^2}}\times\ln\frac{1}{|t-x|}dt+\int_{0}^{1}\frac{t^i}{\sqrt{1-t^2}}\times\ln\frac{1}{|t-x|}dt = \varphi_i^1(x)+\varphi_i^2(x) \tag{9}$$

In the first integrals $\varphi_i^1(x)$ we put $1+t=\beta^2$, in the second integrals $\varphi_i^2(x)$ we put $1-t=\gamma^2$. Then, we return to the normalized interval $[-1,1]$ by using the linear transformations $\beta=\frac{1}{2}(\alpha+1),\ \gamma=\frac{1}{2}(\mu+1)$. Since the integration variable is a dummy variable, we get

$$\varphi_i(x)=2\int_{-1}^{1}\frac{\left(\frac{(t+1)^2}{4}-1\right)^i}{\sqrt{8-(t+1)^2}}\times\ln\frac{1}{\left|\frac{(t+1)^2}{4}-1-x\right|}dt \tag{10}$$

$$+2\int_{-1}^{1}\frac{\left(1-\frac{(t+1)^2}{4}\right)^i}{\sqrt{8-(t+1)^2}}\times\ln\frac{1}{\left|1-\frac{(t+1)^2}{4}-x\right|}dt$$

With this mathematical procedure, the singularities of both the unknown function and the kernel are permanently removed. Now, we design a formula for the collocation points $\{\tilde{x}_j\}_{j=0}^{n}$ distribution for small number $\Delta x\geq0$, the first when $\Delta x=0$ and the second for $\Delta x=0.1$.

$$\tilde{x}_j=(a+\Delta x)+jh;\ h=\frac{(b-\Delta x)-(a+\Delta x)}{n}, \tag{11}$$
$$j=\overline{0,n};\ \Delta x=0.0,0.1$$

We will study the accuracy of obtained solutions by using this formula for both quantities $\Delta x=0$ and $\Delta x=0.1$. We defined the collocation points distribution $\Delta x=0.1$ in order to avoid the possible singularities that may occur when substituting by the collocation points near the endpoints of the integration domain. Applying the collocation points $\{\tilde{x}_j\}_{j=0}^{n}$ to (8) we get the following the linear system of algebraic equations

$$\mathrm{AC}\tilde{\Phi}(\tilde{x}_j)=\mathrm{FCT}(\tilde{x}_j) \tag{12}$$

where $\tilde{\Phi}=\left[\tilde{\varphi}_i(\tilde{x}_j)\right]_{i=0}^{n}\ \forall j=\overline{0,n}$ is a column matrix of order $(n+1)\times1$ whose entries $\tilde{\varphi}_i(\tilde{x}_j)$ are defined by

$$\varphi_i(x)=2\int_{-1}^{1}\frac{\left(\frac{(t+1)^2}{4}-1\right)^i}{\sqrt{8-(t+1)^2}}\times\ln\frac{1}{\left|\frac{(t+1)^2}{4}-1-\tilde{x}_j\right|}dt \tag{13}$$

$$+2\int_{-1}^{1}\frac{\left(1-\frac{(t+1)^2}{4}\right)^i}{\sqrt{8-(t+1)^2}}\times\ln\frac{1}{\left|1-\frac{(t+1)^2}{4}-\tilde{x}_j\right|}dt$$

Now, we reformulate the $N-$ nodes Gauss-Legendre formula in the following matrix form

$$\int_a^b f(x)dx=\sum_{s=1}^{N}\delta_s f(\omega_s)=\Delta(\delta_s)\mathrm{F}(\omega_s) \tag{14}$$

Here $\Delta(\delta_s)=[\delta_s]_{s=1}^{N}$ is an $1\times N$ row matrix and $\mathrm{F}(\omega_s)=[\omega_s]_{s=1}^{N}$ is a $m\times1$ column matrix. The entries ω_s,δ_s are computed by

$$\omega_s=\frac{b-a}{2}\alpha_s+\frac{b+a}{2},$$
$$\delta_s=\frac{b-a}{(1-\alpha_s^2)(P_N'(\alpha_s))^2}\quad\forall s=\overline{1,N} \tag{15}$$

where α_s are the roots of Legendre polynomials defined on $[-1,1]$. By solving the integrals $\tilde{\varphi}_i(\tilde{x}_j)$ of system (12) numerically by using formula (14), we get the unknown coefficients matrix A, and thereby, by substituting into (5) we can obtain the approximate unknown function $\tilde{\tau}_n(t)$.

III. The Convergence in the Mean

Here, we give many approaches for the convergence in the mean of the approximate unknown function $\tilde{\tau}_n(t)$. Putting the exact solution which we denoted by $\tau(t)$ in the form of the infinite Chebyshev polynomials of the second kind U_i whereas expanding $\tilde{\tau}_n(t)$ into U_i with degree $\leq n$ and multiply it by the badly-behaved function $\gamma(t)$ such that

$$\tau(t)=\sum_{i=0}^{\infty}b_iU_i(t)=\mathrm{BU},\ \tilde{\tau}_n(t)=\gamma(t)\sum_{i=0}^{n}a_iU_i(t)=\mathrm{A}\tilde{\mathrm{U}} \tag{16}$$

where, $\mathrm{B}=[b_i]_{i=0}^{\infty}$ is infinite row matrix of order whose entries b_i are the exact coefficients of the unknown function $\tau(t)$, $\mathrm{A}=[a_i]_{i=0}^{n}$ is a row matrix of order $1\times(n+1)$ whose entries a_i are unknowns, $\mathrm{U}=[U_i]_{i=0}^{\infty}$ is infinite column matrix whose entries are the second kind Chebyshev polynomials and $\tilde{\mathrm{U}}=[u_i]_{i=0}^{n}$ is a column matrix of order

$(n+1)\times 1$ whose entries u_i; $i=\overline{0,n}$ are defined by $u_i=\frac{U_i(t)}{\sqrt{1-t^2}}$.

Hence, we have $|\tau(t)|^2$ in the form

$$|\tau(t)|^2=\mathrm{BUU}^T\mathrm{B}^T=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\mathrm{U}_{ij};\quad b_{ij}=b_ib_j,\mathrm{U}_{ij}(t)=\mathrm{U}_i(t)\mathrm{U}_j \tag{17}$$

Furthermore,

$$\int_{-1}^{1}|\tau(t)|^2dt=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\delta_{ij};\quad \delta_{ij}=\int_{-1}^{1}\mathrm{U}_{ij}(t)dt;\ b_{ij}=b_ib_j,\quad \mathrm{U}_{ij}(t)=\mathrm{U}_i(t)\mathrm{U}_j(t) \tag{18}$$

Now, to simplify the integration, we extract the coefficients of each product of Chebyshev polynomials of the second kind $\mathrm{U}_{ij}(t)$ and place these coefficients in a square known coefficients Chebyshev matrix in ascending powers and denote it by $\tilde{\mathrm{C}}$ so we can isolate the integration variable t. But it is easier if we represent $|\tau(t)|^2$ through the matrices $\mathrm{C}_\infty,\mathrm{T}_\infty(t)$ like (4) to get

$$|\tau(t)|^2=\mathrm{BC}_\infty\mathrm{T}_\infty(t)\times\mathrm{T}_\infty^T(t)\mathrm{C}^T\mathrm{B}^T=\mathrm{BC}_\infty\Psi(t)\mathrm{C}_\infty^T\mathrm{B}^T,\quad \Psi(t)=\left[\psi_{ij}(t)\right]_{i,j=0}^{\infty};\ \psi_{ij}(t)=t^{i+j} \tag{19}$$

Hence,

$$\int_{-1}^{1}|\tau(t)|^2dt=\mathrm{BC}_\infty\tilde{\Psi}(t)\mathrm{C}_\infty^T\mathrm{B}^T;\tilde{\Psi}(t)=\left[\tilde{\psi}_{ij}(t)\right]_{i,j=0}^{\infty};\quad \tilde{\psi}_{ij}(t)=\int_{-1}^{1}t^{i+j}dt=\frac{1}{i+j+1}t^{i+j+1}\Big|_{t=-1}^{t=1} \tag{20}$$

Here, $\mathrm{B}=\left[b_i\right]_{i=0}^{\infty}$ is an infinite row matrix and $\mathrm{C}_\infty=\left[c_{ij}\right]_{i,j=0}^{\infty}$ is the infinite square coefficients Chebyshev polynomials matrix which can be found by extracting the coefficient of Chebyshev polynomials from the monomial basis infinite column matrix $\mathrm{T}_\infty(t)=\left[1;t;t^2;t^3,\ldots\right]$. Similar to $\int_{-1}^{1}|\tau(t)|^2dt$ we can find $|\tilde{\tau}_n(t)|^2$ in the form

$$|\tilde{\tau}_n(t)|^2=\mathrm{AC}\tilde{\mathrm{T}}(t)\tilde{\mathrm{T}}^T(t)\mathrm{C}^T\mathrm{A}^T=\mathrm{AC}\Phi(t)\mathrm{C}^T\mathrm{A}^T;\Phi(t)=\tilde{\mathrm{T}}(t)\tilde{\mathrm{T}}^T(t) \tag{21}$$

Here, $\tilde{\mathrm{T}}(t)=\left[\frac{t^i}{\sqrt{1-t^2}}\right]_{i=0}^{n}$ is $(n+1)\times 1$ column matrix and $\Phi(t)=\left[\phi_{ij}(t)\right]_{i,j=0}^{n};\phi_{ij}(t)=\frac{t^{i+j}}{1-t^2}$. By expanding each entry $\phi_{ij}(t)$ into Maclaurin polynomials $m_{ij}(x)$ and by extracting the Maclaurin coefficients, we get $m_{ij}(x)=\mathrm{MT}(t)$. Integrating (21), we get

$$\int_{-1}^{1}|\tilde{\tau}_n(t)|^2dt=\mathrm{AC}\tilde{\Phi}(t)\mathrm{C}^T\mathrm{A}^T;\quad \tilde{\Phi}(t)=\left[m_{ij}(t)\right]_{i,j=0}^{n},\int_{-1}^{1}m_{ij}(t)dt=\mathrm{M}\int_{-1}^{1}t^idt=\mathrm{MZ};\ Z=\left[z_{ij}\right]_{i,j=0}^{n}\ \&\ zij=\frac{1}{i+j+1} \tag{22}$$

By applying the Schwarz inequality, we have

$$\int_{-1}^{1}|\tau(t)\tilde{\tau}_n(t)|dt\le\left[\int_{-1}^{1}|\tau(t)|^2dx\right]^{\frac{1}{2}}\times\left[\int_{-1}^{1}|\tilde{\tau}_n(t)|^2dt\right]^{\frac{1}{2}}\le\left[\mathrm{BC}_\infty\tilde{\Psi}(t)\mathrm{C}_\infty^T\mathrm{B}^T\right]^{\frac{1}{2}}\cdot\left[\mathrm{AC}\tilde{\Phi}(t)\mathrm{C}^T\mathrm{A}^T\right]^{\frac{1}{2}} \tag{23}$$

Consequently,

$$\|\tau(t)-\tilde{\tau}_n(t)\|_2=\int_{-1}^{1}|\tau(t)-\tilde{\tau}_n(t)|^2dt=\int_{-1}^{1}|\tau(t)|^2dt+\int_{-1}^{1}|\tilde{\tau}_n(t)|^2dt-2\int_{-1}^{1}|\tau(t)\tilde{\tau}_n(t)|dt \tag{24}$$

Thus, we get in the limit $n\to\infty$

$$\lim_{n\to\infty}\|\tau(t)-\tilde{\tau}_n(t)\|_2=0 \tag{25}$$

We also have

$$\|\tau(t)-\tilde{\tau}_n(t)\|_2\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\delta_{ij}+\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\Delta_{ij}-2\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\delta_{ij}\right]^{\frac{1}{2}}\cdot\left[\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\Delta_{ij}\right]^{\frac{1}{2}} \tag{26}$$

Let $\delta_1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\delta_{ij},\delta_2=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\Delta_{ij}$, then we get

$$\lim_{n\to\infty}\|\tau(t)-\tilde{\tau}_n(t)\|_2\le\lim_{n\to\infty}\left(\delta_1+\delta_2-2\sqrt{\delta_1}\cdot\sqrt{\delta_2}\right)=\lim_{n\to\infty}\left(\sqrt{\delta_1}+\sqrt{\delta_2}\right)^2=0 \tag{27}$$

IV. Computational results

Consider the weakly singular Fredholm integral equation of the first kind

$$\int_{-1}^{1}k(x,t)\tau(t)dt=1;\ -1\le x\le 1 \tag{28}$$

whose exact solution [8] is given by

$$\tau(t)=\frac{1}{(\pi ln(2))\sqrt{1-t^2}};\ |t|<1 \tag{29}$$

We designed the codes for the numerical solutions of (28) by using MATLAB 2014a and plotted the graphs by using Excel 2016. We take $n=2,4,6,8$, where n denotes the degree of Chebyshev polynomials of the second kind. The number of $N-$ Gauss-Legendre nodes is chosen equal to $N=15{:}5{:}40$. Tables 1-8 show a comparison of the exact solutions $\tau(t_i)$ at $t_i=-0.9{:}0.1{:}0.9$ with the numerical solution $\tilde{\tau}_n(t_i)$ (denoted by $App(n,N,\Delta x)$) for $n=8,6,4,2$ respectively.

The tables 1-4 are constructed for $\Delta x=0$ whereas the tables 5-8 are for $\Delta x=0.1$. Tables 9-16 show the corresponding

absolute errors $\left|\tau(t_i)-\tilde{\tau}_n(t_i)\right|$ (denoted by $E(n,N,\Delta x)$) for $n=8,6,4,2$ respectively. The tables 9-12 are presented for $\Delta x=0$ whereas the tables 13-16 are presented for $\Delta x=0.1$. The graphs of the exact solutions $\tau(t_i)$ at $t_i=-0.9{:}0.1{:}0.9$ and the obtained numerical solutions $\tilde{\tau}_n(t_i)$ for $n=8,6,4$ and $N=30$ for $\Delta x=0$ and $\Delta x=0.1$ are plotted in figures 1-3.
Figure 4 shows the graphs of the exact solutions $\tau(t_i)$ at $t_i=-0.9{:}0.1{:}0.9$ and the numerical solutions $App(8,30,0.1)$ and $App(6,30,0.1)$. Figure 5 shows the graphs of the exact solutions $\tau(t_i)$ at $t_i=-0.9{:}0.1{:}0.9$ and the numerical solutions $App(8,30,0)$ and $App(6,30,0)$. All the numerical solutions $App(n,N,\Delta x)$ are symmetric. The best result was found for $n=8, N=30, \Delta x=0.1$, where the solutions are exact in $[-0.9.0.9]$. Each obtained approximate polynomials of degree at most n is characterized and agrees with the alternating or eqioscillation property [11]. We find that the numerical solutions depend on the distribution of the collocation points. The formula for the collocation points for $\Delta x=0.1$ is the better than for $\Delta x=0$. For all the obtained numerical solutions $App(n,N,\Delta x)$ the solutions at the endpoints are found as shown in the presented tables when the exact solution failed to find.

V. Conclusion

The singular Fredholm integral equation of the first kind that is equivalent to the Dirichlet boundary value problem for Laplace equation of an open contour in the plane is numerically solved. The factorization of the unknown function into two functions, the first expressing the bad behavior of the solution near the endpoints of the integration domain and the second being an unknown regular function facilitates to treat the singularity. The extracting of the coefficients of the Chebyshev polynomials and isolating them from the monomial basis functions in a matrix form significantly reduces the round-off errors and saves CPU time. The expanding of the unknown regular function into Chebyshev polynomials of degree n at most in matrix form via monomial basis matrix enables merging the badly behaved-function with the weakly singular logarithmic kernel and the approximate regular function in one entity integrand. Thus, by changing variables and by using the developed $N-$ Gauss- quadrature integration formula, the entity one integrand function is integrated on two intervals so that the singularities of the integral equation are completely isolated. The perfect choice of the collocation points in a specific formula gives rise to an equivalent linear system of algebraic equations for the solution. The obtained numerical solutions for the illustrated example were exact inside the integration domain and strongly converge to the exact near and at the endpoints. The presented method, apart from being a straightforward and easy method that provided accurate solutions and dealt with the singularities of the equation, it also succeeded in obtaining the solutions at the endpoints of the integration domain while the exact solution formula failed in this. This proves the efficiency, versatility of the presented method and its technical superiority in treating singular integral equations.

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Table I
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_8(t_i)$
for $N=15{:}5{:}40, \Delta x=0$, denoted by $App(8,N,0)$

t_i	$\tau(t_i)$	$App(8,15,0)$	$App(8,20,0)$	$App(8,25,0)$	$App(8,30,0)$	$App(8,35,0)$	$App(8,40,0)$
-1.0	None	3.6187	1.7272	-1.1233	1.2365	3.2072	1.378
-0.9	1.0535	1.8325	1.0878	0.0189	0.9285	1.6966	0.9730
-0.8	0.7654	0.9044	0.7578	0.5730	0.7425	0.8897	0.7458
-0.7	0.6430	0.5410	0.6170	0.7272	0.6334	0.5544	0.6285
-0.6	0.5740	0.4761	0.5704	0.6737	0.5695	0.4762	0.5705
-0.5	0.5303	0.5162	0.5550	0.5600	0.5303	0.4970	0.5385
-0.4	0.5011	0.5508	0.5376	0.4712	0.5038	0.5237	0.5145
-0.3	0.4814	0.5412	0.5091	0.4361	0.4844	0.5209	0.4919
-0.2	0.4687	0.4971	0.4751	0.4439	0.4702	0.4923	0.4713
-0.1	0.4615	0.4505	0.4475	0.4664	0.4611	0.4603	0.4564
0	0.4592	0.4310	0.4370	0.4774	0.4580	0.4467	0.4510
0.1	0.4615	0.4505	0.4475	0.4664	0.4611	0.4603	0.4564
0.2	0.4687	0.4971	0.4751	0.4439	0.4702	0.4923	0.4713
0.3	0.4814	0.5412	0.5091	0.4361	0.4844	0.5209	0.4919
0.4	0.5011	0.5508	0.5376	0.4712	0.5038	0.5237	0.5145
0.5	0.5303	0.5162	0.5550	0.5600	0.5303	0.4970	0.5385
0.6	0.5740	0.4761	0.5704	0.6737	0.5695	0.4762	0.5705
0.7	0.6430	0.5410	0.6170	0.7272	0.6334	0.5544	0.6285
0.8	0.7654	0.9044	0.7578	0.5730	0.7425	0.8897	0.7458
0.9	1.0535	1.8325	1.0878	0.0189	0.9285	1.6966	0.9730
1.0	None	3.6187	1.7272	-1.1233	1.2365	3.2072	1.378

Table II
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_6(t_i)$
for $N=15{:}5{:}40, \Delta x=0$, denoted by $App(6,N,0)$

t_i	$\tau(t_i)$	$App(6,15,0)$	$App(6,20,0)$	$App(6,25,0)$	$App(6,30,0)$	$App(6,35,0)$	$App(6,40,0)$
- 1.0	None	0.73299	0.8077	0.98773	1.0774	1.4145	1.4426
-0.9	1.0535	0.6751	0.8077	0.8249	0.8736	1.0891	1.1047
-0.8	0.7654	0.6469	0.7124	0.7107	0.7271	0.8374	0.8432
-0.7	0.6430	0.6263	0.6432	0.6304	0.6255	0.6585	0.6576
-0.6	0.5740	0.6022	0.5903	0.5736	0.5578	0.5443	0.5400
-0.5	0.5303	0.5714	0.5481	0.5329	0.5153	0.4823	0.4773
-0.4	0.5011	0.5356	0.5139	0.5036	0.4903	0.4583	0.4543
-0.3	0.4814	0.4994	0.4867	0.4829	0.4769	0.4576	0.4557
-0.2	0.4687	0.4683	0.4666	0.4690	0.4705	0.4674	0.4677
-0.1	0.4615	0.4474	0.4542	0.4609	0.4680	0.4776	0.4796
0	0.4592	0.4400	0.4500	0.4582	0.4673	0.4817	0.4843
0.1	0.4615	0.4474	0.4542	0.4609	0.4680	0.4776	0.4796
0.2	0.4687	0.4683	0.4666	0.4690	0.4705	0.4674	0.4677
0.3	0.4814	0.4994	0.4867	0.4829	0.4769	0.4576	0.4557
0.4	0.5011	0.5356	0.5139	0.5036	0.4903	0.4583	0.4543
0.5	0.5303	0.5714	0.5481	0.5329	0.5153	0.4823	0.4773
0.6	0.5740	0.6022	0.5903	0.5736	0.5578	0.5443	0.5400
0.7	0.6430	0.6263	0.6432	0.6304	0.6255	0.6585	0.6576
0.8	0.7654	0.6469	0.7124	0.7107	0.7271	0.8374	0.8432
0.9	1.0535	0.6751	0.8077	0.8249	0.8736	1.0891	1.1047
1.0	None	0.73299	0.8077	0.98773	1.0774	1.4145	1.4426

Table III
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_4(t_i)$
for $N=15:5:40, \Delta x=0$, denoted by $App(4,N,0)$

t_i	$\tau(t_i)$	$App(4,15,0)$	$App(4,20,0)$	$App(4,25,0)$	$App(4,30,0)$	$App(4,35,0)$	$App(4,40,0)$
-1.0	None	0.84888	0.90157	0.89514	0.86165	0.8176	87518
-0.9	1.0535	0.7459	0.7987	0.7922	0.7588	0.7148	0.7723
-0.8	0.7654	0.6657	0.7134	0.7076	0.6773	0.6373	0.6895
-0.7	0.6430	0.6041	0.6436	0.6388	0.6136	0.5804	0.6237
-0.6	0.5740	0.5578	0.5872	0.5835	0.5646	0.5398	0.5722
-0.5	0.5303	0.5237	0.5425	0.5400	0.5278	0.5118	0.5326
-0.4	0.5011	0.4992	0.5079	0.5066	0.5007	0.4931	0.5029
-0.3	0.4814	0.4824	0.4822	0.4819	0.4816	0.4812	0.4816
-0.2	0.4687	0.4715	0.4645	0.4651	0.4690	0.4742	0.4672
-0.1	0.4615	0.4655	0.4542	0.4552	0.4618	0.4706	0.4589
0	0.4592	0.4635	0.4508	0.4520	0.4595	0.4694	0.4562
0.1	0.4615	0.4655	0.4542	0.4552	0.4618	0.4706	0.4589
0.2	0.4687	0.4715	0.4645	0.4651	0.4690	0.4742	0.4672
0.3	0.4814	0.4824	0.4822	0.4819	0.4816	0.4812	0.4816
0.4	0.5011	0.4992	0.5079	0.5066	0.5007	0.4931	0.5029
0.5	0.5303	0.5237	0.5425	0.5400	0.5278	0.5118	0.5326
0.6	0.5740	0.5578	0.5872	0.5835	0.5646	0.5398	0.5722
0.7	0.6430	0.6041	0.6436	0.6388	0.6136	0.5804	0.6237
0.8	0.7654	0.6657	0.7134	0.7076	0.6773	0.6373	0.6895
0.9	1.0535	0.7459	0.7987	0.7922	0.7588	0.7148	0.7723
1.0	None	0.84888	0.90157	0.89514	0.86165	0.8176	87518

Table IV
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_2(t_i)$
for $N=15:5:40, \Delta x=0$, denoted by $App(2,N,0)$

t_i	$\tau(t_i)$	$App(2,15,0)$	$App(2,20,0)$	$App(2,25,0)$	$App(2,30,0)$	$App(2,35,0)$	$App(2,40,0)$
-1.0	None	0.69015	0.68958	0.68932	0.68917	0.68908	0.68904
-0.9	1.0535	0.6465	0.6459	0.6457	0.6455	0.6455	0.6454
-0.8	0.7654	0.6074	0.6069	0.6066	0.6065	0.6064	0.6064
-0.7	0.6430	0.5730	0.5724	0.5722	0.5721	0.5720	0.5719
-0.6	0.5740	0.5431	0.5426	0.5423	0.5422	0.5421	0.5421
-0.5	0.5303	0.5178	0.5173	0.5171	0.5169	0.5169	0.5168
-0.4	0.5011	0.4972	0.4966	0.4964	0.4963	0.4962	0.4961
.1 -0.3	0.4814	0.4811	0.4806	0.4803	0.4802	0.4801	0.4801
-0.2	0.4687	0.4696	0.4691	0.4688	0.4687	0.4686	0.4686
-0.1	0.4615	0.4627	0.4622	0.4620	0.4618	0.4617	0.4617
0	0.4592	0.4604	0.4599	0.4597	0.4595	0.4594	0.4594
0.1	0.4615	0.4627	0.4622	0.4620	0.4618	0.4617	0.4617
0.2	0.4687	0.4696	0.4691	0.4688	0.4687	0.4686	0.4686
0.3	0.4814	0.4811	0.4806	0.4803	0.4802	0.4801	0.4801
0.4	0.5011	0.4972	0.4966	0.4964	0.4963	0.4962	0.4961
0.5	0.5303	0.5178	0.5173	0.5171	0.5169	0.5169	0.5168
0.6	0.5740	0.5431	0.5426	0.5423	0.5422	0.5421	0.5421
0.7	0.6430	0.5730	0.5724	0.5722	0.5721	0.5720	0.5719
0.8	0.7654	0.6074	0.6069	0.6066	0.6065	0.6064	0.6064
0.9	1.0535	0.6465	0.6459	0.6457	0.6455	0.6455	0.6454
1.0	None	0.69015	0.68958	0.68932	0.68917	0.68908	0.68904

Table V
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_8(t_i)$
for N=15:5:40,Δx=0.1, denoted by $App(8,N,0.1)$

t_i	$\tau(t_i)$	$App(8,15,0.1)$	$App(8,20,0.1)$	$App(8,25,0.1)$	$App(8,30,0.1)$	$App(8,35,0.1)$	$App(8,40,0.1)$
-1.0	None	4.1171	-4.9194	3.9607	1.4294	-0.073976	5.0744
-0.9	1.0535	2.1549	-1.9613	2.2222	1.0567	0.2768	2.7988
-0.8	0.7654	1.0705	-0.2325	1.1960	0.8118	0.5018	1.4185
-0.7	0.6430	0.6027	0.5268	0.7044	0.6620	0.5967	0.7408
-0.6	0.5740	0.4833	0.6961	0.5352	0.5755	0.6018	0.5025
-0.5	0.5303	0.5035	0.6107	0.5115	0.5268	0.5641	0.4728
-0.4	0.5011	0.5365	0.4908	0.5196	0.4983	0.5192	0.4974
-0.3	0.4814	0.5320	0.4347	0.5093	0.4800	0.4856	0.5034
-0.2	0.4687	0.4934	0.4476	0.4766	0.4674	0.4675	0.4793
-0.1	0.4615	0.4501	0.4871	0.4418	0.4596	0.4607	0.4468
0	0.4592	0.4317	0.5066	0.4271	0.4569	0.4593	0.4323
0.1	0.4615	0.4501	0.4871	0.4418	0.4596	0.4607	0.4468
0.2	0.4687	0.4934	0.4476	0.4766	0.4674	0.4675	0.4793
0.3	0.4814	0.5320	0.4347	0.5093	0.4800	0.4856	0.5034
0.4	0.5011	0.5365	0.4908	0.5196	0.4983	0.5192	0.4974
0.5	0.5303	0.5035	0.6107	0.5115	0.5268	0.5641	0.4728
0.6	0.5740	0.4833	0.6961	0.5352	0.5755	0.6018	0.5025
0.7	0.6430	0.6027	0.5268	0.7044	0.6620	0.5967	0.7408
0.8	0.7654	1.0705	-0.2325	1.1960	0.8118	0.5018	1.4185
0.9	1.0535	2.1549	-1.9613	2.2222	1.0567	0.2768	2.7988
1.0	None	4.1171	-4.9194	3.9607	1.4294	-0.073976	5.0744

Table VI
A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_6(t_i)$
for N=15:5:40,Δx=0.1, denoted by $App(6,N,0.1)$

t_i	$\tau(t_i)$	$App(6,15,0.1)$	$App(6,20,0.1)$	$App(6,25,0.1)$	$App(6,30,0.1)$	$App(6,35,0.1)$	$App(6,40,0.1)$
-1.0	None	0.88941	1.0732	0.80339	1.5751	3.2153	1.4093
-0.9	1.0535	0.7686	0.8905	0.6600	1.2391	2.3907	1.1242
-0.8	0.7654	0.6907	0.7542	0.5852	0.9512	1.6153	0.8851
-0.7	0.6430	0.6358	0.6547	0.5475	0.7308	1.0031	0.7033
-0.6	0.5740	0.5923	0.5836	0.5275	0.5811	0.5951	0.5789
-0.5	0.5303	0.5544	0.5341	0.5140	0.4943	0.3814	0.5046
-0.4	0.5011	0.5203	0.5007	0.5021	0.4561	0.3209	0.4689
-0.3	0.4814	0.4906	0.4791	0.4906	0.4499	0.3575	0.4589
-0.2	0.4687	0.4672	0.4659	0.4805	0.4592	0.4337	0.4620
-0.1	0.4615	0.4522	0.4589	0.4735	0.4706	0.5009	0.4682
0	0.4592	0.4470	0.4567	0.4710	0.4754	0.5271	0.4711
0.1	0.4615	0.4522	0.4589	0.4735	0.4706	0.5009	0.4682
0.2	0.4687	0.4672	0.4659	0.4805	0.4592	0.4337	0.4620
0.3	0.4814	0.4906	0.4791	0.4906	0.4499	0.3575	0.4589
0.4	0.5011	0.5203	0.5007	0.5021	0.4561	0.3209	0.4689
0.5	0.5303	0.5544	0.5341	0.5140	0.4943	0.3814	0.5046
0.6	0.5740	0.5923	0.5836	0.5275	0.5811	0.5951	0.5789
0.7	0.6430	0.6358	0.6547	0.5475	0.7308	1.0031	0.7033
0.8	0.7654	0.6907	0.7542	0.5852	0.9512	1.6153	0.8851
0.9	1.0535	0.7686	0.8905	0.6600	1.2391	2.3907	1.1242
1.0	None	0.88941	1.0732	0.80339	1.5751	3.2153	1.4093

Table VII

A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_4(t_i)$

for N=15:5:40,Δx=0.1 ,denoted by $App(4,N,0.1)$

t_i	$\tau(t_i)$	$App(4,15,0.1)$	$App(4,20,0.1)$	$App(4,25,0.1)$	$App(4,30,0.1)$	$App(4,35,0.1)$	$App(4,40,0.1)$
-1.0	None	0.86168	0.89297	0.88322	0.85916	0.87727	0.82435
-0.9	1.053	0.7525	0.7913	0.7831	0.7591	0.7858	0.7228
-0.8	0.765	0.6682	0.7075	0.7009	0.6791	0.7081	0.6454
-0.7	0.643	0.6042	0.6391	0.6341	0.6160	0.6428	0.5876
-0.6	0.574	0.5566	0.5841	0.5806	0.5670	0.5887	0.5454
-0.5	0.530	0.5221	0.5406	0.5385	0.5296	0.5447	0.5155
-0.4	0.501	0.4978	0.5071	0.5061	0.5019	0.5099	0.4950
. -0.3	0.481	0.4814	0.4824	0.4824	0.4821	0.4837	0.4815
-0.2	0.468	0.4709	0.4654	0.4661	0.4689	0.4653	0.4731
-0.1	0.461	0.4651	0.4555	0.4566	0.4613	0.4545	0.4687
0	0.459	0.4633	0.4522	0.4535	0.4588	0.4509	0.4673
0.1	0.461	0.4651	0.4555	0.4566	0.4613	0.4545	0.4687
0.2	0.468	0.4709	0.4654	0.4661	0.4689	0.4653	0.4731
0.3	0.481	0.4814	0.4824	0.4824	0.4821	0.4837	0.4815
0.4	0.501	0.4978	0.5071	0.5061	0.5019	0.5099	0.4950
0.5	0.530	0.5221	0.5406	0.5385	0.5296	0.5447	0.5155
0.6	0.574	0.5566	0.5841	0.5806	0.5670	0.5887	0.5454
0.7	0.643	0.6042	0.6391	0.6341	0.6160	0.6428	0.5876
0.8	0.765	0.6682	0.7075	0.7009	0.6791	0.7081	0.6454
0.9	1.053	0.7525	0.7913	0.7831	0.7591	0.7858	0.7228
1.0	None	0.86168	0.89297	0.88322	0.85916	0.87727	0.82435

Table VIII

A comparison of the exact solutions $\tau(t_i)$ with the numerical solutions $\tilde{\tau}_2(t_i)$

for N=15:5:40,Δx=0.1 , denoted by $App(2,N,0.1)$

t_i	$\tau(t_i)$	$App(2,15,0.1)$	$App(2,20,0.1)$	$App(2,25,0.1)$	$App(2,30,0.1)$	$App(2,35,0.1)$	$App(2,40,0.1)$
-1.0	None	0.86168	0.89297	0.88322	0.85916	0.87727	0.82435
-0.9	1.0535	0.7525	0.7913	0.7831	0.7591	0.7858	0.7228
-0.8	0.7654	0.6682	0.7075	0.7009	0.6791	0.7081	0.6454
-0.7	0.6430	0.6042	0.6391	0.6341	0.6160	0.6428	0.5876
-0.6	0.5740	0.5566	0.5841	0.5806	0.5670	0.5887	0.5454
-0.5	0.5303	0.5221	0.5406	0.5385	0.5296	0.5447	0.5155
-0.4	0.5011	0.4978	0.5071	0.5061	0.5019	0.5099	0.4950
. -0.3	0.4814	0.4814	0.4824	0.4824	0.4821	0.4837	0.4815
-0.2	0.4687	0.4709	0.4654	0.4661	0.4689	0.4653	0.4731
-0.1	0.4615	0.4651	0.4555	0.4566	0.4613	0.4545	0.4687
0	0.4592	0.4633	0.4522	0.4535	0.4588	0.4509	0.4673
0.1	0.4615	0.4651	0.4555	0.4566	0.4613	0.4545	0.4687
0.2	0.4687	0.4709	0.4654	0.4661	0.4689	0.4653	0.4731
0.3	0.4814	0.4814	0.4824	0.4824	0.4821	0.4837	0.4815
0.4	0.5011	0.4978	0.5071	0.5061	0.5019	0.5099	0.4950
0.5	0.5303	0.5221	0.5406	0.5385	0.5296	0.5447	0.5155
0.6	0.5740	0.5566	0.5841	0.5806	0.5670	0.5887	0.5454
0.7	0.6430	0.6042	0.6391	0.6341	0.6160	0.6428	0.5876
0.8	0.7654	0.6682	0.7075	0.7009	0.6791	0.7081	0.6454
0.9	1.0535	0.7525	0.7913	0.7831	0.7591	0.7858	0.7228
1.0	None	0.86168	0.89297	0.88322	0.85916	0.87727	0.82435

Table IX
The absolute errors $E(8,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_8(t_i)\right|$ for $N=15{:}5{:}40,\Delta x=0$

t_i	$E(8,15,0)$	$E(8,20,0)$	$E(8,25,0)$	$E(8,30,0)$	$E(8,35,0)$	$E(8,40,0)$
-0.9	0.779	0.0343	1.0346	0.125	0.6431	0.0805
-0.8	0.139	0.0076	0.1924	0.0229	0.1243	0.0196
-0.7	0.102	0.026	0.0842	0.0096	0.0886	0.0145
-0.6	0.0979	0.0036	0.0997	0.0045	0.0978	0.0035
-0.5	0.0141	0.0247	0.0297	0	0.0333	0.0082
-0.4	0.0497	0.0365	0.0299	0.0027	0.0226	0.0134
-0.3	0.0598	0.0277	0.0453	0.003	0.0395	0.0105
-0.2	0.0284	0.0064	0.0248	0.0015	0.0236	0.0026
-0.1	0.011	0.014	0.0049	0.0004	0.0012	0.0051
0	0.0282	0.0222	0.0182	0.0012	0.0125	0.0082
0.1	0.011	0.014	0.0049	0.0004	0.0012	0.0051
0.2	0.0284	0.0064	0.0248	0.0015	0.0236	0.0026
0.3	0.0598	0.0277	0.0453	0.003	0.0395	0.0105
0.4	0.0497	0.0365	0.0299	0.0027	0.0226	0.0134
0.5	0.0141	0.0247	0.0297	0	0.0333	0.0082
0.6	0.0979	0.0036	0.0997	0.0045	0.0978	0.0035
0.7	0.102	0.026	0.0842	0.0096	0.0886	0.0145
0.8	0.139	0.0076	0.1924	0.0229	0.1243	0.0196
0.9	0.779	0.0343	1.0346	0.125	0.6431	0.0805

Table X
The absolute errors $E(6,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_6(t_i)\right|$ for $N=15{:}5{:}40,\Delta x=0$

t_i	$E(6,15,0)$	$E(6,20,0)$	$E(6,25,0)$	$E(6,30,0)$	$E(6,35,0)$	$E(6,40,0)$
-0.9	0.3784	0.2458	0.2286	0.1799	0.0356	0.0512
-0.8	0.1185	0.053	0.0547	0.0383	0.072	0.0778
-0.7	0.0167	0.0002	0.0126	0.0175	0.0155	0.0146
-0.6	0.0282	0.0163	0.0004	0.0162	0.0297	0.034
-0.5	0.0411	0.0178	0.0026	0.015	0.048	0.053
-0.4	0.0345	0.0128	0.0025	0.0108	0.0428	0.0468
-0.3	0.018	0.0053	0.0015	0.0045	0.0238	0.0257
-0.2	0.0004	0.0021	0.0003	0.0018	0.0013	0.001
-0.1	0.0141	0.0073	0.0006	0.0065	0.0161	0.0181
0	0.0192	0.0092	0.001	0.0081	0.0225	0.0251
0.1	0.0141	0.0073	0.0006	0.0065	0.0161	0.0181
0.2	0.0004	0.0021	0.0003	0.0018	0.0013	0.001
0.3	0.018	0.0053	0.0015	0.0045	0.0238	0.0257
0.4	0.0345	0.0128	0.0025	0.0108	0.0428	0.0468
0.5	0.0411	0.0178	0.0026	0.015	0.048	0.053
0.6	0.0282	0.0163	0.0004	0.0162	0.0297	0.034
0.7	0.0167	0.0002	0.0126	0.0175	0.0155	0.0146
0.8	0.1185	0.053	0.0547	0.0383	0.072	0.0778
0.9	0.3784	0.2458	0.2286	0.1799	0.0356	0.0512

Table XI
The absolute errors $E(4,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_4(t_i)\right|$ for $N=15{:}5{:}40,\Delta x=0$

t_i	$E(4,15,0)$	$E(4,20,0)$	$E(4,25,0)$	$E(4,30,0)$	$E(4,35,0)$	$E(4,40,0)$
-0.9	0.3076	0.2548	0.2613	0.2947	0.3387	0.2812
-0.8	0.0997	0.052	0.0578	0.0881	0.1281	0.0759
-0.7	0.0389	0.0006	0.0042	0.0294	0.0626	0.0193
-0.6	0.0162	0.0132	0.0095	0.0094	0.0342	0.0018
-0.5	0.0066	0.0122	0.0097	0.0025	0.0185	0.0023
-0.4	0.0019	0.0068	0.0055	0.0004	0.008	0.0018
-0.3	0.001	0.0008	0.0005	0.0002	0.0002	0.0002
-0.2	0.0028	0.0042	0.0036	0.0003	0.0055	0.0015
-0.1	0.004	0.0073	0.0063	0.0003	0.0091	0.0026
0	0.0043	0.0084	0.0072	0.0003	0.0102	0.003
0.1	0.004	0.0073	0.0063	0.0003	0.0091	0.0026
0.2	0.0028	0.0042	0.0036	0.0003	0.0055	0.0015
0.3	0.001	0.0008	0.0005	0.0002	0.0002	0.0002
0.4	0.0019	0.0068	0.0055	0.0004	0.008	0.0018
0.5	0.0066	0.0122	0.0097	0.0025	0.0185	0.0023
0.6	0.0162	0.0132	0.0095	0.0094	0.0342	0.0018
0.7	0.0389	0.0006	0.0042	0.0294	0.0626	0.0193
0.8	0.0997	0.052	0.0578	0.0881	0.1281	0.0759
0.9	0.3076	0.2548	0.2613	0.2947	0.3387	0.2812

Table XII
The absolute errors $E(2,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_2(t_i)\right|$ for $N=15{:}5{:}40,\Delta x=0$

t_i	$E(2,15,0)$	$E(2,20,0)$	$E(2,25,0)$	$E(2,30,0)$	$E(2,35,0)$	$E(2,40,0)$
-0.9	0.0744	0.4076	0.4078	0.408	0.408	0.4081
-0.8	0.0652	0.1585	0.1588	0.1589	0.159	0.159
-0.7	0.0561	0.0706	0.0708	0.0709	0.071	0.0711
-0.6	0.0468	0.0314	0.0317	0.0318	0.0319	0.0319
-0.5	0.0376	0.013	0.0132	0.0134	0.0134	0.0135
-0.4	0.0285	0.0045	0.0047	0.0048	0.0049	0.005
-0.3	0.0193	0.0008	0.0011	0.0012	0.0013	0.0013
-0.2	0.0101	0.0004	0.0000	0.0000	0.0000	0.0000
-0.1	0.0009	0.0007	0.0005	0.0003	0.0002	0.0002
0	0.0083	0.0007	0.0005	0.0003	0.0002	0.0002
0.1	0.0175	0.0007	0.0005	0.0003	0.0002	0.0002
0.2	0.0267	0.0004	0.0000	0.0000	0.0000	0.0000
0.3	0.0358	0.0008	0.0011	0.0012	0.0013	0.0013
0.4	0.045	0.0045	0.0047	0.0048	0.0049	0.005
0.5	0.0543	0.013	0.0132	0.0134	0.0134	0.0135
0.6	0.0634	0.0314	0.0317	0.0318	0.0319	0.0319
0.7	0.0725	0.0706	0.0708	0.0709	0.071	0.0711
0.8	0.6074	0.1585	0.1588	0.1589	0.159	0.159
0.9	0.6465	0.4076	0.4078	0.408	0.408	0.4081

Table XIII

The absolute errors $E(8,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_8(t_i)\right|$ for $N=15{:}5{:}40, \Delta x=0.1$

t_i	$E(8,15,0.1)$	$E(8,20,0.1)$	$E(8,25,0.1)$	$E(8,30,0.1)$	$E(8,35,0.1)$	$E(8,40,0.1)$
-0.9	1.1014	3.0148	1.1687	0.0032	0.7767	1.7453
-0.8	0.3051	0.9979	0.4306	0.0464	0.2636	0.6531
-0.7	0.0403	0.1162	0.0614	0.019	0.0463	0.0978
-0.6	0.0907	0.1221	0.0388	0.0015	0.0278	0.0715
-0.5	0.0268	0.0804	0.0188	0.0035	0.0338	0.0575
-0.4	0.0354	0.0103	0.0185	0.0028	0.0181	0.0037
-0.3	0.0506	0.0467	0.0279	0.0014	0.0042	0.022
-0.2	0.0247	0.0211	0.0079	0.0013	0.0012	0.0106
-0.1	0.0114	0.0256	0.0197	0.0019	0.0008	0.0147
0	0.0275	0.0474	0.0321	0.0023	0.0000	0.0269
0.1	0.0114	0.0256	0.0197	0.0019	0.0008	0.0147
0.2	0.0247	0.0211	0.0079	0.0013	0.0012	0.0106
0.3	0.0506	0.0467	0.0279	0.0014	0.0042	0.022
0.4	0.0354	0.0103	0.0185	0.0028	0.0181	0.0037
0.5	0.0268	0.0804	0.0188	0.0035	0.0338	0.0575
0.6	0.0907	0.1221	0.0388	0.0015	0.0278	0.0715
0.7	0.0403	0.1162	0.0614	0.019	0.0463	0.0978
0.8	0.3051	0.9979	0.4306	0.0464	0.2636	0.6531
0.9	1.1014	3.0148	1.1687	0.0032	0.7767	1.7453

Table XIV

The absolute errors $E(6,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_6(t_i)\right|$ for $N=15{:}5{:}40, \Delta x=0.1$

t_i	$E(6,15,0.1)$	$E(6,20,0.1)$	$E(6,25,0.1)$	$E(6,30,0.1)$	$E(6,35,0.1)$	$E(6,40,0.1)$
-0.9	0.2849	0.163	0.3935	0.1856	1.3372	0.0707
-0.8	0.0747	0.0112	0.1802	0.1858	0.8499	0.1197
-0.7	0.0072	0.0117	0.0955	0.0878	0.3601	0.0603
-0.6	0.0183	0.0096	0.0465	0.0071	0.0211	0.0049
-0.5	0.0241	0.0038	0.0163	0.036	0.1489	0.0257
-0.4	0.0192	0.0004	0.001	0.045	0.1802	0.0322
-0.3	0.0092	0.0023	0.0092	0.0315	0.1239	0.0225
-0.2	0.0015	0.0028	0.0118	0.0095	0.035	0.0067
-0.1	0.0093	0.0026	0.012	0.0091	0.0394	0.0067
0	0.0122	0.0025	0.0118	0.0162	0.0679	0.0119
0.1	0.0093	0.0026	0.012	0.0091	0.0394	0.0067
0.2	0.0015	0.0028	0.0118	0.0095	0.035	0.0067
0.3	0.0092	0.0023	0.0092	0.0315	0.1239	0.0225
0.4	0.0192	0.0004	0.001	0.045	0.1802	0.0322
0.5	0.0241	0.0038	0.0163	0.036	0.1489	0.0257
0.6	0.0183	0.0096	0.0465	0.0071	0.0211	0.0049
0.7	0.0072	0.0117	0.0955	0.0878	0.3601	0.0603
0.8	0.0747	0.0112	0.1802	0.1858	0.8499	0.1197
0.9	0.2849	0.163	0.3935	0.1856	1.3372	0.0707

Table XV

The absolute errors $E(4,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_4(t_i)\right|$ for $N=15{:}5{:}40, \Delta x=0.1$

t_i	$E(4,15,0.1)$	$E(4,20,0.1)$	$E(4,25,0.1)$	$E(4,30,0.1)$	$E(4,35,0.1)$	$E(4,40,0.1)$
-0.9	0.301	0.2622	0.2704	0.7591	0.7858	0.3307
-0.8	0.0972	0.0579	0.0645	0.6791	0.7081	0.12
-0.7	0.0388	0.0039	0.0089	0.616	0.6428	0.0554
-0.6	0.0174	0.0101	0.0066	0.567	0.5887	0.0286
-0.5	0.0082	0.0103	0.0082	0.5296	0.5447	0.0148
-0.4	0.0033	0.006	0.005	0.5019	0.5099	0.0061
-0.3	0.0000	0.001	0.001	0.4821	0.4837	0.0000
-0.2	0.0022	0.0033	0.0026	0.4689	0.4653	0.0044
-0.1	0.0036	0.006	0.0049	0.4613	0.4545	0.0072
0	0.0041	0.007	0.0057	0.4588	0.4509	0.0081
0.1	0.0036	0.006	0.0049	0.4613	0.4545	0.0072
0.2	0.0022	0.0033	0.0026	0.4689	0.4653	0.0044
0.3	0.0000	0.001	0.001	0.4821	0.4837	0.0000
0.4	0.0033	0.006	0.005	0.5019	0.5099	0.0061
0.5	0.0082	0.0103	0.0082	0.5296	0.5447	0.0148
0.6	0.0174	0.0101	0.0066	0.567	0.5887	0.0286
0.7	0.0388	0.0039	0.0089	0.616	0.6428	0.0554
0.8	0.0972	0.0579	0.0645	0.6791	0.7081	0.12
0.9	0.301	0.2622	0.2704	0.7591	0.7858	0.3307

Table XVI

The absolute errors $E(2,N,\Delta x)=\left|\tau(t_i)-\tilde{\tau}_2(t_i)\right|$ for $N=15{:}5{:}40, \Delta x=0.1$

t_i	$E(2,15,0.1)$	$E(2,20,0.1)$	$E(2,25,0.1)$	$E(2,30,0.1)$	$E(2,35,0.1)$	$E(2,40,0.1)$
-0.9	0.3995	0.404	0.4077	0.4118	0.4194	0.4159
-0.8	0.1524	0.1558	0.1587	0.1617	0.1674	0.1648
-0.7	0.0661	0.0687	0.0707	0.0729	0.0769	0.0751
-0.6	0.0285	0.0303	0.0316	0.033	0.0355	0.0344
-0.5	0.0113	0.0124	0.0132	0.0139	0.0152	0.0147
-0.4	0.0038	0.0044	0.0047	0.0049	0.0051	0.0051
-0.3	0.001	0.0012	0.0011	0.0009	0.0003	0.0006
-0.2	0.0004	0.0002	1E-04	0.0006	0.0018	0.0012
-0.1	0.0004	0.0001	0.0004	0.0011	0.0026	0.0019
0	0.0005	1E-04	0.0004	0.0012	0.0028	0.002
0.1	0.0004	0.0001	0.0004	0.0011	0.0026	0.0019
0.2	0.0004	0.0002	1E-04	0.0006	0.0018	0.0012
0.3	0.001	0.0012	0.0011	0.0009	0.0003	0.0006
0.4	0.0038	0.0044	0.0047	0.0049	0.0051	0.0051
0.5	0.0113	0.0124	0.0132	0.0139	0.0152	0.0147
0.6	0.0285	0.0303	0.0316	0.033	0.0355	0.0344
0.7	0.0661	0.0687	0.0707	0.0729	0.0769	0.0751
0.8	0.1524	0.1558	0.1587	0.1617	0.1674	0.1648
0.9	0.3995	0.404	0.4077	0.4118	0.4194	0.4159

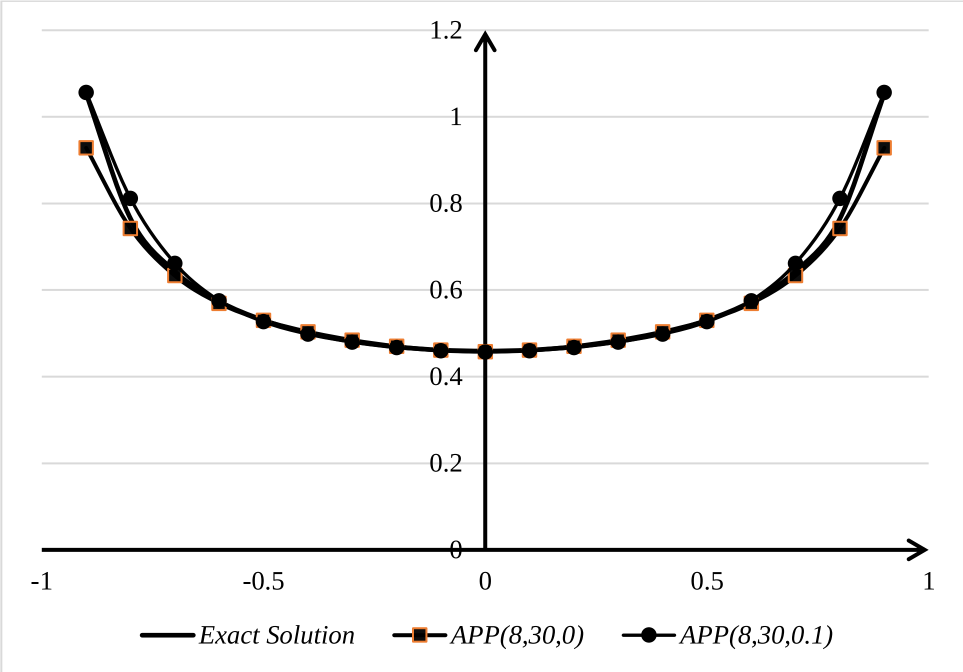

Fig. 1. A comparison of the exact solution $\tau(t_i)$ at the set op points $t_i = -0.9{:}0.1{:}0.9$ with the numerical solutions $APP(8,30,0)$ and $APP(8,30,0.1)$

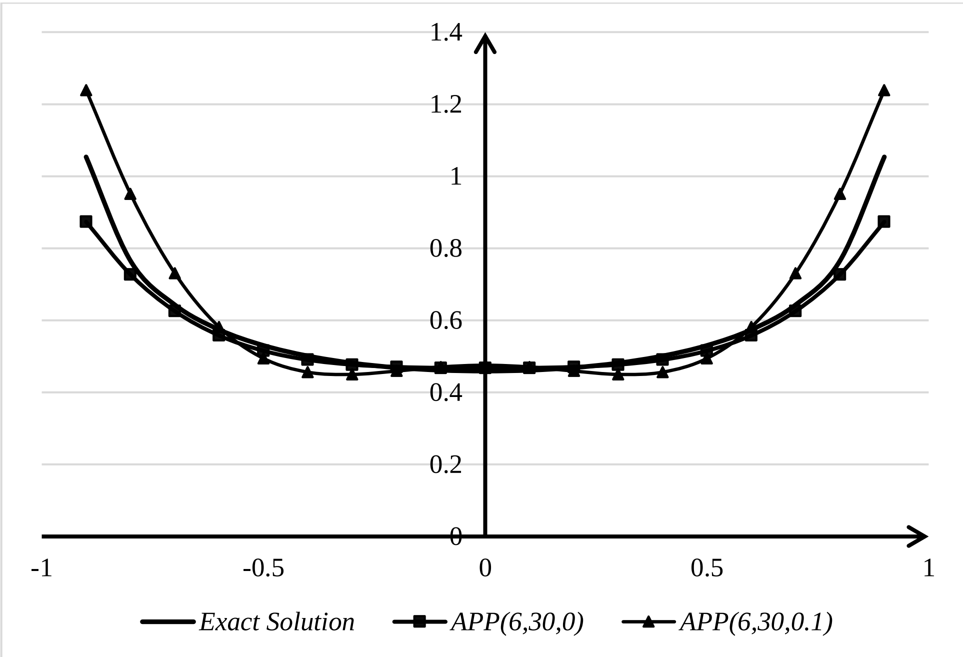

Fig. 2. A comparison of the exact solution $\tau(t_i)$ at the set op points $t_i = -0.9{:}0.1{:}0.9$ with the numerical solutions $APP(6,30,0)$ and $APP(6,30,0.1)$

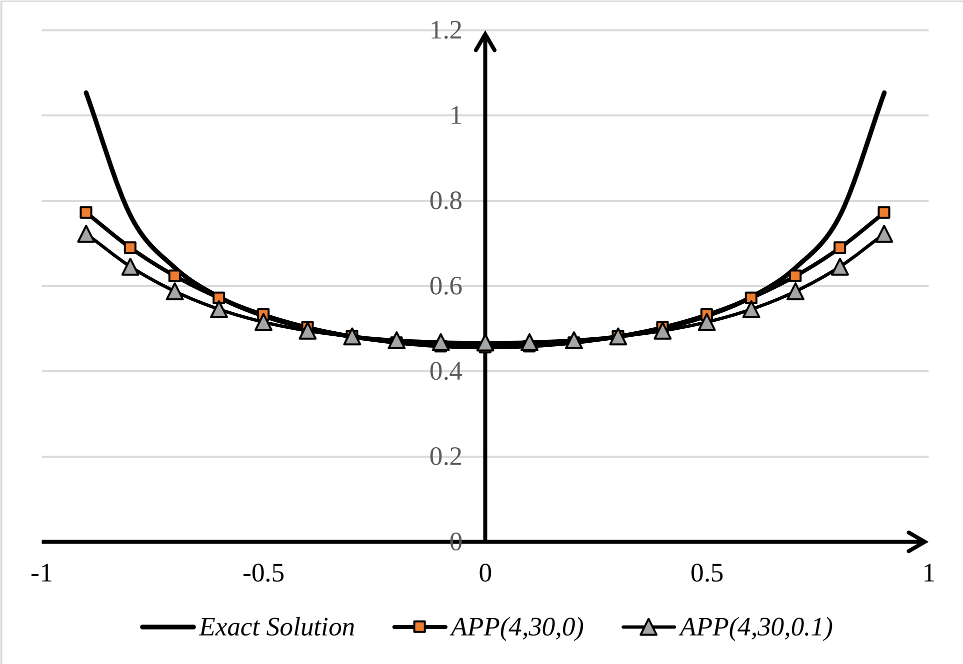

Fig. 3. A comparison of the exact solution $\tau(t_i)$ at the set op points $t_i = -0.9{:}0.1{:}0.9$ with the numerical solutions $APP(4,30,0)$ and $APP(4,30,0.1)$

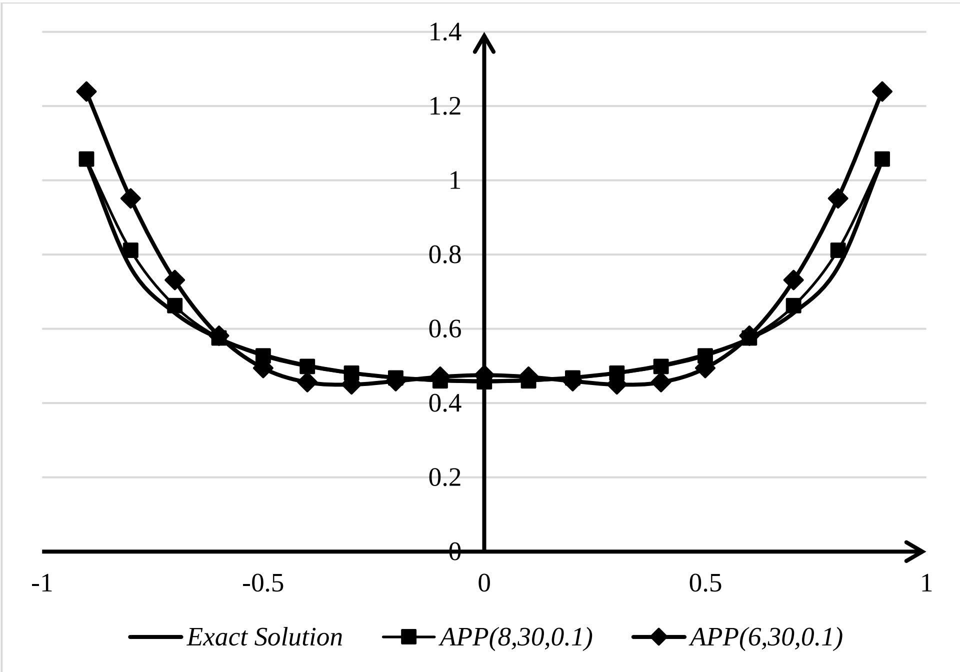

Fig. 4. A comparison of the exact solution $\tau(t_i)$ at the set op points $t_i = -0.9{:}0.1{:}0.9$ with the numerical solutions $APP(8,30,0.1)$ and $APP(6,30,0.1)$

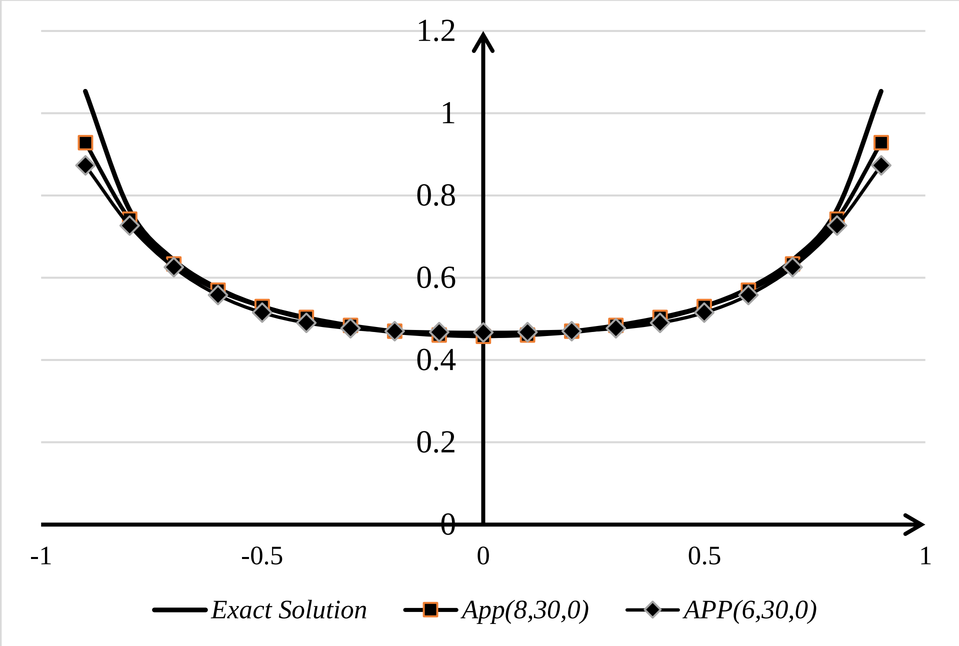

Fig. 5. A comparison of the exact solution $\tau(t_i)$ at the set op points $t_i = -0.9{:}0.1{:}0.9$ with the numerical solutions $APP(8,30,0)$ and $APP(6,30,0)$