Inexact Modified QHSS Iteration Methods for Complex Symmetric Linear Systems of Strong Skew-Hermitian Parts

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***Abstract*—In order to improve the efficiency of the modified QHSS (MQHSS) iteration method by Chen and the co-authors (Appl. Numer. Math. 2020; doi.org/10.1016/j.apnum.2020.01.018), we employ the iterative methods in the inner iteration process to accelerate the convergence speed and then obtain the inexact MQHSS (IMQHSS) iteration method. The asymptotically convergence conditions of the IMQHSS iteration method are analyzed in detail. Numerical experiments are performed to test the efficiency and robustness of the IMQHSS iteration method and the corresponding preconditioned Krylov subspace method.**

***Index Terms*—Complex symmetric linear systems; MQHSS iteration method; asymptotically convergence; strong skew-Hermitian matrix.**

I. Introduction

WE are considering the fast solvers for the following nonsingular complex symmetric linear system of equations

$$Ax := (W + \mathbf{i}T)x = b := f + \mathbf{i}g, \tag{I.1}$$

where the real parts $W \in \mathbb{R}^{n\times n}$ and the imaginary parts $T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite (SPSD) matrix and symmetric positive definite (SPD) matrix, respectively. $f, g \in \mathbb{R}^n$ are known vectors, $x \in \mathbb{C}^n$ is an unknown vector and $\mathbf{i} = \sqrt{-1}$ is the imaginary unit. Here, we assume that T is dominant over W, i.e., in some matrix norm $\|\cdot\|_*$ holds [21]

$$\|T\|_* \gg \|W\|_*.$$

Complex linear systems of the form (I.1) arise in a lot of applications, for example, quantum mechanics [25], lattice quantum chromody-namics [18], nonlinear waves [1], FFT-based solutions of certain time-dependent PDEs [15], chemical oscillations, structural dynamics [17] and so on. For more applications, we refer to [2], [6], [7], [14] and references therein.

As is known, non-Hermitian matrices possess the Hermitian and skew-Hermitian (HS) splitting. Hence, for any complex symmetric matrix A, it holds $A = H + S$, where $H = \frac{1}{2}(A + A^*) = W$ and $S = \frac{1}{2}(A - A^*) = \mathbf{i}T$. Based on the HS splitting, Bai, Golub and Ng [10] first constructed the Hermitian and skew-Hermitian splitting (HSS) iteration method for solving the nonsingular linear systems of non-Hermitian positive definite coefficient matrix. Because of the unconditionally convergent property and the elegant mathematical properties, a variety of considerable attentions and results based on HSS iteration method have been proposed in many papers. Some classical iteration methods can be found in existed papers, e.g., the preconditioned HSS (PHSS) iteration method [12], the parameterized SHSS (P-SHSS) iteration method [29], the lopsided PMHSS iteration method [23], the accelerated HSS (AHSS) iteration method [8], the generalization of HSS (GHSS) iteration method [13], the modified HSS (MHSS) iteration method [6], the modified GHSS iteration method [22], the new single-step method [28] and the shift-splitting based C-to-R method [31]. For more details about the generalization and comprehensive survey on the HSS iteration method can be seen in [9], [4], [3], [27], [19], [30].

When T is dominant over W, we rewrite the linear system (I.1) as a quasi-normal equation equivalently by multiplying $I - \mathbf{i}\omega T$ to both sides of the equation by left to obtain $(I - \mathbf{i}\omega T)Ax = (I - \mathbf{i}\omega T)b$, i.e.,

$$(I - \mathbf{i}\omega T)(W + \mathbf{i}T)x = (W + \omega T^2 + \mathbf{i}T - \mathbf{i}\omega TW)x, \tag{I.2}$$

where I is an identity matrix and ω is a chosen reasonably small constant such that the matrix $I - \mathbf{i}\omega T$ is as far as well-conditioned. As a matter of fact, if the parameter ω is small enough, the normalizing effect of the quasi-normal equation (I.2) is much more significant than the classical normal equation [5]. By making use of the alternatively iterative technique, the quasi-HSS (QHSS) iteration method for solving the quasi-normal linear system (I.2) given in [5] can be described as follows.

Algorithm I.1. *(The QHSS iteration method)*

For any initial guess $x^{(0)} \in \mathbb{C}^n$ and $k = 0, 1, 2, \ldots$, use the following procedure to compute the next iterate $x^{(k+1)}$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges:

$$\begin{cases} (\alpha I + \mathbf{i}T)x^{(k+\frac{1}{2})} = (\alpha I - H_\omega + \mathbf{i}\omega TW)x^{(k)} + (I - \mathbf{i}\omega T)b, \\ (\alpha I + H_\omega)x^{(k+1)} = (\alpha I - \mathbf{i}T)x^{(k+\frac{1}{2})} + \mathbf{i}\omega TWx^{(k)} + (I - \mathbf{i}\omega T)b, \end{cases}$$

where ω is a given nonnegative constant, α is a positive constant and $H_\omega = W + \omega T^2$.

It can be easy seen from Algorithm I.1 that, when we use the QHSS iteration method, we have to solve two subsystems of linear equations with respect to the coefficient matrices $\alpha I + \mathbf{i}T$ and $\alpha I + H_\omega$. In general, because of the

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SPD property, the linear system of the coefficient matrix $\alpha I + H_\omega$ can be solved effectively either exactly by the Cholesky factorization or inexactly by iterative methods (e.g., Krylov subspace methods or multigrid methods). However, if T is dominant over W, then the solution of the shifted linear system with the coefficient matrix $\alpha I + \mathbf{i}T$ is as difficult as that of the original linear system (I.1).

As the complex linear system (I.2) can be equivalently rewritten as

$$\mathbf{i}Tx = \mathbf{i}\omega TWx - (W + \omega T^2)x + (I - \mathbf{i}\omega T)b.$$

Multiplying $-\mathbf{i}$ and then adding αI by both sides of the above equality, it follows

$$(\alpha I + T)x = (\alpha I + \mathbf{i}(W + \omega T^2))x + \omega TWx - \mathbf{i}(I - \mathbf{i}\omega T)b.$$

Therefore, the MQHSS iteration method in [16] can be described as the following algorithm.

Algorithm I.2. *(The MQHSS iteration method)*

For any initial guess $x^{(0)} \in \mathbb{C}^n$ and $k = 0, 1, 2, \ldots$, use the following procedure to compute the next iterate $x^{(k+1)}$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges:

$$\begin{cases} (\alpha I + T)x^{(k+\frac{1}{2})} = (\alpha I + \mathbf{i}H_\omega)x^{(k)} + \omega TWx^{(k)} - \mathbf{i}(I - \mathbf{i}\omega T)b, \\ (\alpha I + H_\omega)x^{(k+1)} = (\alpha I - \mathbf{i}T)x^{(k+\frac{1}{2})} + \mathbf{i}\omega TWx^{(k)} + (I - \mathbf{i}\omega T)b, \end{cases}$$

where ω is a given nonnegative constant, α is a positive constant and $H_\omega = W + \omega T^2$.

We can see from Algorithm I.2 that, if $\omega = 0$, then the MQHSS iteration method reduces to the MHSS iteration method [6]. Besides, because the coefficient matrices of the two linear subsystems are SPD matrices, then we can use the symmetric Krylov subspace methods to solve the linear subsystems at each iterative step. Hence, the main target of this paper is to further improve the efficiency of the MQHSS iteration method by focusing on the inexact inner solvers of the shifted subsystems with the coefficient matrices being $\alpha I + T$ and $\alpha I + H_\omega$.

The rest of this paper is organized as follows. In Section II, we describe the inexact MQHSS (IMQHSS) iteration method. Section III is devoted to the detailed asymptotically convergence conditions analysis. In Section IV, we examine the feasibility and efficiency of the IMQHSS iteration method and the IMQHSS preconditioned GMRES method by numerical experiments. Finally, a brief conclusion will be given in Section V to end this work.

Throughout the paper, we use $\sigma(M)$ and $\rho(M)$ to denote the eigenvalues set and the spectral radius of the matrix M, respectively. $\|\cdot\|$ denotes the norm of either a vector or a matrix and $\|\cdot\|_2$ denotes the Euclidean norm.

II. The IMQHSS iteration method

In this section, we firstly reformulate the MQHSS iteration scheme as

$$x^{(k+1)} = L(\omega,\alpha)x^{(k)} + M(\omega,\alpha)^{-1}b, k = 0, 1, 2, \cdots, \tag{II.1}$$

where

$$L(\omega,\alpha) = (\alpha I + H_\omega)^{-1}(\alpha I + T)^{-1}((\alpha I - \mathbf{i}T)(\alpha I + \mathbf{i}H_\omega) + (1 + \mathbf{i})\alpha\omega TW).$$

Rewrite $L(\omega,\alpha)$ as $M(\omega,\alpha)^{-1}N(\omega,\alpha)$, then we have

$$\begin{cases} M(\omega,\alpha) = \dfrac{1+\mathbf{i}}{2\alpha}(I - \mathbf{i}\omega T)^{-1}(\alpha I + T)(\alpha I + H_\omega), \\ N(\omega,\alpha) = \dfrac{1+\mathbf{i}}{2\alpha}(I - \mathbf{i}\omega T)^{-1}((\alpha I - \mathbf{i}T)(\alpha I + \mathbf{i}H_\omega) \\ \qquad\qquad + (1+\mathbf{i})\alpha\omega TW). \end{cases} \tag{II.2}$$

Because both matrices $\alpha I + T$ and $\alpha I + H_\omega$ are SPD, then we can further improve the efficiency of the MQHSS iteration method at each iterate step by solving both of the two subsystems inexactly by multigrid method [20], [26] or Krylov subspace methods, e.g., the conjugate gradient method [24]. By using the same strategy in [11], the MQHSS iteration method can be equivalently reformulated as

$$\begin{cases} (\alpha I + T)x^{(k+\frac{1}{2})} = (\alpha I + T)x^{(k)} - \mathbf{i}(I - \mathbf{i}\omega T)(b - Ax^{(k)}), \\ (\alpha I + H_\omega)x^{(k+1)} = (\alpha I + H_\omega)x^{(k+\frac{1}{2})} + (I - \mathbf{i}\omega T)(b - Ax^{(k+\frac{1}{2})}) \\ \qquad\qquad + \mathbf{i}\omega TW(x^{(k)} - x^{(k+\frac{1}{2})}). \end{cases} \tag{II.3}$$

Hence, we describe the inexact MQHSS (IMQHSS) iteration method in the following algorithm.

Algorithm II.1. *(The IMQHSS iteration method)*

Given any initial guess $x^{(0)} \in \mathbb{C}^n$ and sequences of stopping tolerances ε_k and η_k for the inner iteration process. For $k = 0, 1, 2, \ldots$, we use the following procedure to compute the next iterate $x^{(k+1)}$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \subset \mathbb{C}^n$ converges:

Step 1. solve the shifted SPD residual equation $(\alpha I + T)z^{(k)} = \tilde{r}^{(k)}$, with $\tilde{r}^{(k)} = -\mathbf{i}(I - \mathbf{i}\omega T)(b - Ax^{(k)})$, by iterating until $z^{(k)}$ satisfying $\|p^{(k)}\| = \|\tilde{r}^{(k)} - (\alpha I + T)z^{(k)}\| \le \varepsilon_k \|\tilde{r}^{(k)}\|$, and then compute $x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}$;

Step 2. solve the shifted SPD residual equation $(\alpha I + H_\omega)z^{(k+\frac{1}{2})} = \tilde{s}^{(k+\frac{1}{2})}$, with

$$\tilde{s}^{(k+\frac{1}{2})} = (I - \mathbf{i}\omega T)(b - Ax^{(k+\frac{1}{2})}) + \mathbf{i}\omega TW(x^{(k)} - x^{(k+\frac{1}{2})}),$$

by iterating until $z^{(k+\frac{1}{2})}$ satisfying $\|q^{(k+\frac{1}{2})}\| = \|\tilde{s}^{(k+\frac{1}{2})} - (\alpha I + H_\omega)z^{(k+\frac{1}{2})}\| \le \eta_k \|\tilde{s}^{(k+\frac{1}{2})}\|$, and then compute $x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}$.

III. The asymptotically convergence analysis

In this section, we focus on the asymptotically convergence property of the IMQHSS iteration method. Here and in the sequence, we define a vector norm as $|||x||| = \|(\alpha I + H_\omega)x\|_2$ for any vector $x \in \mathbb{C}^n$ and then induce the corresponding matrix norm $|||X||| = \|(\alpha I + H_\omega)X(\alpha I + H_\omega)^{-1}\|_2$ for any matrix $X \in \mathbb{C}^{n\times n}$.

The first theorem describes the convergence conditions of the MQHSS iteration method.

Theorem III.1. *Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular complex symmetric matrix with the SPSD matrix W and the SPD matrix T be its real and imaginary parts, respectively. Let ω and α be a nonnegative constant and a positive constant, respectively. Denote by*

$$\theta(\omega,\alpha) = \max_{\lambda_i \in \sigma(T)} \frac{\sqrt{\alpha^2 + \lambda_i^2}}{\alpha + \lambda_i} \cdot \max_{\mu_j \in \sigma(H_\omega)} \frac{\sqrt{\alpha^2 + \mu_j^2}}{\alpha + \mu_j} + \frac{\sqrt{2}\omega\alpha\sqrt{\varsigma}\|T\|_2\|H_\omega\|_2}{(\alpha + \|T\|_2)(\alpha + \|H_\omega\|_2)},$$

where $\varsigma := \min\{\kappa(W),\kappa(H_\omega)\}$, *then once* $\theta(\omega,\alpha)<1$, *the MQHSS iteration method converges to the exact solution.*

Proof: From (II.1), we know $L(\omega,\alpha)=(\alpha I+H_\omega)^{-1}(\alpha I+T)^{-1}((\alpha I-\mathbf{i}T)(\alpha I+\mathbf{i}H_\omega)+(1+\mathbf{i})\alpha\omega TW)$. Denote by

$$L^{(1)}(\omega,\alpha)=(\alpha I+H_\omega)^{-1}(\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)(\alpha I+\mathbf{i}H_\omega)$$

and

$$L^{(2)}(\omega,\alpha)=(\alpha I+H_\omega)^{-1}(\alpha I+T)^{-1}TW,$$

then it follows

$$\begin{aligned}\rho(L(\omega,\alpha))&=\rho(L^{(1)}(\omega,\alpha)+(1+\mathbf{i})\alpha\omega L^{(2)}(\omega,\alpha))\\ &\le\|L^{(1)}(\omega,\alpha)+(1+\mathbf{i})\alpha\omega L^{(2)}(\omega,\alpha)\|_2\\ &\le\|L^{(1)}(\omega,\alpha)\|_2+\sqrt{2}\alpha\omega\|L^{(2)}(\omega,\alpha)\|_2\\ &\le\|(\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)\|_2\cdot\|(\alpha I\\ &\quad+H_\omega)^{-1}(\alpha I+\mathbf{i}H_\omega)\|_2\\ &\quad+\sqrt{2}\alpha\omega\|(\alpha I+T)^{-1}T\|_2\cdot\|WH_\omega^{-1}\|_2\\ &\quad\cdot\|H_\omega(\alpha I+H_\omega)^{-1}\|_2.\end{aligned}$$

After some simple algebra computations, we have

$$\|(\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)\|_2\le\max_{\lambda_i\in\sigma(T)}\frac{\sqrt{\alpha^2+\lambda_i^2}}{\alpha+\lambda_i},$$

$$\|(\alpha I+H_\omega)^{-1}(\alpha I+\mathbf{i}H_\omega)\|_2\le\max_{\mu_j\in\sigma(H_\omega)}\frac{\sqrt{\alpha^2+\mu_j^2}}{\alpha+\mu_j},$$

$$\|(\alpha I+T)^{-1}T\|_2\le\frac{\|T\|_2}{\alpha+\|T\|_2},\|H_\omega(\alpha I+H_\omega)^{-1}\|_2\le\frac{\|H_\omega\|_2}{\alpha+\|H_\omega\|_2}$$

and $\|WH_\omega^{-1}\|_2\le\min\{\sqrt{\kappa(W)},\sqrt{\kappa(H_\omega)}\}=\sqrt{\varsigma}$ [5]. Therefore, after substituting the above inequalities and by making use of the monotonically increasing property of the function $f(t)=\frac{t}{\alpha+t}(t\ge 0)$, it follows, $\rho(L(\omega,\alpha))\le\theta(\omega,\alpha)$. Furthermore, if $\theta(\omega,\alpha)<1$, then $\rho(L(\omega,\alpha))<1$, i.e., the MQHSS iteration method converges. ■

Theorem III.2. *Let* $A\in\mathbb{C}^{n\times n}$ *be a nonsingular complex symmetric matrix with the SPSD matrix* W *and the SPD matrix* T *being its real and imaginary parts, respectively. Let* ω *and* α *be a nonnegative constant and a positive constant, respectively. Denote by*

$$\phi(\omega,\alpha)=\|(I-\boldsymbol{i}\omega T)A(\alpha I+H_\omega)^{-1}\|_2,$$

$$\psi(\omega,\alpha)=\|(\alpha I+H_\omega)(\alpha I+T)^{-1}\|_2,$$

$$\varepsilon=\max_k\{\varepsilon_k\},\quad\eta=\max_k\{\eta_k\},\quad\textit{and}\quad\xi=\max\{\varepsilon,\eta\}.$$

If $\{x^{(k)}\}$ *is an iteration sequence generated by the IMQHSS iteration method, then it holds*

$$|||x^{(k+1)}-x_*|||\le(\theta(\omega,\alpha)+\xi(1+\xi)(1+\psi(\omega,\alpha))\phi(\omega,\alpha))$$

$$\cdot|||x^{(k)}-x_*|||,k=0,1,2,\ldots,$$

where x_* *is the exact solution of the linear system (I.1). Furthermore, if* $\theta(\omega,\alpha)+\xi(1+\xi)(1+\psi(\omega,\alpha))\phi(\omega,\alpha)<1$, *then the sequence* $\{x^{(k)}\}$ *converges to* $x_*\in\mathbb{C}^n$. *Here* $\theta(\omega,\alpha)$ *is defined in Theorem III.1.*

Proof: From Algorithm II.1 and the first equation of (II.3), we have

$$\begin{aligned}x^{(k+\frac12)}-x_*&=x^{(k)}-x_*+(\alpha I+T)^{-1}(\tilde r^{(k)}-p^{(k)})\\ &=x^{(k)}-x_*+(\alpha I+T)^{-1}(\mathbf{i}(I-\mathbf{i}\omega T)A(x^{(k)}-x_*)-p^{(k)})\\ &=(\alpha I+T)^{-1}(\alpha I+T+\omega TW-T+\mathbf{i}H_\omega)(x^{(k)}-x_*)\\ &\quad-(\alpha I+T)^{-1}p^{(k)})\\ &=(\alpha I+T)^{-1}(\alpha I+\omega TW+\mathbf{i}H_\omega)(x^{(k)}-x_*)-(\alpha I+T)^{-1}p^{(k)})\end{aligned}$$

and by making use of the results in [5], it follows

$$\begin{aligned}x^{(k+1)}-x_*&=x^{(k+\frac12)}-x_*+(\alpha I+H_\omega)^{-1}(\tilde s^{(k+\frac12)}-p^{(k)})\\ &=(\alpha I+H_\omega)^{-1}(\alpha I-\mathbf{i}T)(x^{(k+\frac12)}-x_*)\\ &\quad+\mathbf{i}\omega(\alpha I+H_\omega)^{-1}TW(x^{(k+\frac12)}-x_*)-(\alpha I+H_\omega)^{-1}q^{(k+\frac12)}.\end{aligned}$$

Hence, we can acquire

$$x^{(k+1)}-x_*=L(\omega,\alpha)(x^{(k)}-x_*)$$

$$-(\alpha I+H_\omega)^{-1}((\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)p^{(k)}+q^{(k+\frac12)}),$$

i.e.,

$$(\alpha I+H_\omega)(x^{(k+1)}-x_*)=(\alpha I+H_\omega)L(\omega,\alpha)(\alpha I+H_\omega)^{-1}$$

$$\cdot(\alpha I+H_\omega)(x^{(k)}-x_*)-(\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)p^{(k)}+q^{(k+\frac12)}.$$

By taking norms on both sides, we have

$$\begin{aligned}&|||x^{(k+1)}-x_*|||\le|||L(\omega,\alpha)|||\cdot|||x^{(k)}-x_*|||\\ &\quad+\|(\alpha I+T)^{-1}(\alpha I-\mathbf{i}T)\|_2\|p^{(k)}\|_2+\|q^{(k+\frac12)}\|_2\\ &\le|||L(\omega,\alpha)|||\cdot|||x^{(k)}-x_*|||+\|p^{(k)}\|_2+\|q^{(k+\frac12)}\|_2\end{aligned}$$

and

$$\begin{aligned}\|p^{(k)}\|_2\le\varepsilon_k\cdot\|\tilde r^{(k)}\|_2&\le\varepsilon_k\cdot\|(I-\mathbf{i}\omega T)A(\alpha I+H_\omega)^{-1}\|_2\cdot|||x^{(k)}-x_*|||\\ &=\varepsilon_k\phi(\omega,\alpha)|||x^{(k)}-x_*|||.\end{aligned}$$

Besides, it holds

$$\begin{aligned}\tilde s^{(k+\frac12)}&=-((I-\mathbf{i}\omega T)A+\mathbf{i}\omega TW)(x^{(k+\frac12)}-x_*)+\mathbf{i}\omega TW(x^{(k)}-x_*)\\ &=-(H_\omega+\mathbf{i}T)(x^{(k+\frac12)}-x_*)+\mathbf{i}\omega TW(x^{(k)}-x_*)\\ &=-((H_\omega+\mathbf{i}T)(\alpha I+T)^{-1}(\alpha I+\mathbf{i}H_\omega+\omega TW)\\ &\quad+\mathbf{i}\omega TW)(x^{(k)}-x_*)-(H_\omega+\mathbf{i}T)(\alpha I+T)^{-1}p^{(k)}\\ &=(\alpha I+\mathbf{i}H_\omega)(\alpha I+T)^{-1}(I-\mathbf{i}\omega T)A(x^{(k)}-x_*)\\ &\quad+(I-(\alpha I+\mathbf{i}H_\omega)(\alpha I+T)^{-1})p^{(k)}.\end{aligned}$$

According to the facts

$$\|(\alpha I+\mathbf{i}H_\omega)(\alpha I+H_\omega)^{-1}\|_2=\max_{\lambda\in\sigma(H_\omega)}\frac{\sqrt{\alpha^2+\lambda^2}}{\alpha+\lambda}<1$$

and

$$\|(\alpha I+\mathbf{i}H_\omega)(\alpha I+T)^{-1}\|_2\le\|(\alpha I+\mathbf{i}H_\omega)(\alpha I+H_\omega)^{-1}\|_2$$

$$\cdot\|(\alpha I+H_\omega)(\alpha I+T)^{-1}\|_2\le\psi(\omega,\alpha),$$

and by making use of the second equation of (II.3), it follows

$$\begin{aligned}&\|q^{(k+\frac12)}\|_2\le\eta_k\cdot\|\tilde s^{(k+\frac12)}\|_2\\ &\le\eta_k\|(\alpha I+\mathbf{i}H_\omega)(\alpha I+T)^{-1}\|_2\\ &\cdot\|(I-\mathbf{i}\omega T)A(\alpha I+H_\omega)^{-1}\|_2\cdot|||x^{(k)}-x_*|||\\ &\qquad+\eta_k\cdot(1+\|(\alpha I+\mathbf{i}H_\omega)(\alpha I+T)^{-1}\|_2)\cdot\|p^{(k)}\|_2\\ &\le\eta_k\psi(\omega,\alpha)\phi(\omega,\alpha)\cdot|||x^{(k)}-x_*|||+\eta_k(1+\psi(\omega,\alpha))\cdot\|p^{(k)}\|_2.\end{aligned}$$

Therefore, by substituting all of the above equations, we obtain

$$
\begin{aligned}
&|||x^{(k+1)}-x_*||| \le (\theta(\omega,\alpha)+\varepsilon_k\phi(\omega,\alpha)+\eta_k\phi(\omega,\alpha)\psi(\omega,\alpha)\\
&+\varepsilon_k\eta_k(1+\psi(\omega,\alpha))\phi(\omega,\alpha))|||x^{(k)}-x_*|||\\
&\le (\theta(\omega,\alpha)+\varepsilon\phi(\omega,\alpha)+\eta\phi(\omega,\alpha)\psi(\omega,\alpha)\\
&+\varepsilon\eta(1+\psi(\omega,\alpha))\phi(\omega,\alpha))|||x^{(k)}-x_*|||\\
&=(\theta(\omega,\alpha)+\xi(\xi+1)(1+\psi(\omega,\alpha))\phi(\omega,\alpha))\cdot|||x^{(k)}-x_*|||.
\end{aligned}
$$

Hence, if $\theta(\omega,\alpha)+\xi(\xi+1)(1+\psi(\omega,\alpha))\phi(\omega,\alpha)<1$, then the IMQHSS iteration method converges to the exact solution of (I.1). ∎

Theorem III.3. *Let $A=W+\boldsymbol{i}T\in\mathbb{C}^{n\times n}$ be a nonsingular complex symmetric matrix. The SPSD matrix W and the SPD matrix T are the real and imaginary parts of A, respectively. Suppose that $\tau_1(k)$ and $\tau_2(k)$ are nondecreasing and positive sequences of integers satisfying $\tau_1(k)\ge 1$, $\tau_2(k)\ge 1$, $\varepsilon_k\le c_1\cdot\delta_1^{\tau_1(k)}$, $\eta_k\le c_2\cdot\delta_2^{\tau_2(k)}$, where c_1 and c_2 are nonnegative constants. $\delta_1,\delta_2\in(0,1)$ are both real constants. ε_k and η_k are defined in Theorem III.2. Then*

$$|||x^{(k+1)}-x_*|||\le(\sqrt{\theta(\omega,\alpha)}+\gamma\delta^{\tau(k)})^2\cdot|||x^{(k)}-x_*|||,$$

where $\tau(k)=\min\{\tau_1(k),\tau_2(k)\}$, $\delta=\max\{\delta_1,\delta_2\}$, $c=\max\{c_1,c_2\}$ and

$$\gamma=c\cdot\max\{\frac{(1+\psi(\omega,\alpha))\phi(\omega,\alpha)}{2\sqrt{\theta(\omega,\alpha)}},\sqrt{(1+\psi(\omega,\alpha))\phi(\omega,\alpha)}\}.$$

Furthermore, the asymptotically convergence rate is

$$\lim_{k\to+\infty}\sup\frac{|||x^{(k+1)}-x_*|||}{|||x^{(k)}-x_*|||}\le\theta(\omega,\alpha).$$

Or equivalently, the convergence rate of the IMQHSS iteration method is asymptotically the same as that of the MQHSS iteration method.

Proof: According to the proof of Theorem III.2, we have

$$
\begin{aligned}
&|||x^{(k+1)}-x_*|||\le(\theta(\omega,\alpha)+\varepsilon_k\phi(\omega,\alpha)\\
&+\eta_k\phi(\omega,\alpha)\psi(\omega,\alpha)+\varepsilon_k\eta_k(1+\psi(\omega,\alpha))\phi(\omega,\alpha))|||x^{(k)}-x_*|||\\
&\le(\theta(\omega,\alpha)+c_1\delta^{\tau(k)}\phi(\omega,\alpha)+c_2\delta^{\tau(k)}\psi(\omega,\alpha)\phi(\omega,\alpha)\\
&\qquad+c_1c_2(\delta^{\tau(k)})^2(1+\psi(\omega,\alpha))\phi(\omega,\alpha))\cdot|||x^{(k)}-x_*|||\\
&\le(\theta(\omega,\alpha)+c\delta^{\tau(k)}(1+\psi(\omega,\alpha))\phi(\omega,\alpha)\\
&\qquad+(c\cdot\sqrt{(1+\psi(\omega,\alpha))\phi(\omega,\alpha)}\cdot\delta^{\tau(k)})^2)\cdot|||x^{(k)}-x_*|||\\
&\le(\theta(\omega,\alpha)+2\gamma\sqrt{\theta(\omega,\alpha)}\cdot\delta^{\tau(k)}+\gamma^2\cdot(\delta^{\tau(k)})^2)\cdot|||x^{(k)}-x_*|||\\
&=(\sqrt{\theta(\omega,\alpha)}+\gamma\cdot\delta^{\tau(k)})^2\cdot|||x^{(k)}-x_*|||.
\end{aligned}
$$

Divided both sides by $|||x^{(k)}-x_*|||$, it follows

$$\frac{|||x^{(k+1)}-x_*|||}{|||x^{(k)}-x_*|||}\le(\sqrt{\theta(\omega,\alpha)}+\gamma\cdot\delta^{\tau(k)})^2.$$

Furthermore, we have

$$\sup\frac{|||x^{(k+1)}-x_*|||}{|||x^{(k)}-x_*|||}\le(\sqrt{\theta(\omega,\alpha)}+\gamma\cdot\delta^{\tau(k)})^2.$$

Let $k\to+\infty$, then the results of the theorem can be obtained immediately. ∎

IV. Numerical experiments

In this section, we will test the effectiveness of the new methods with respect to both iteration counts (denoted as 'IT') and the computing time (in second, denoted as 'CPU') for solving the complex symmetric linear system of strong skew-Hermitian parts. The IQHSS iteration method and the IQHSS preconditioned GMRES method [5] are used to compare with the IMQHSS iteration method and the IMQHSS preconditioned method, respectively. We adopt the conjugate gradient method [24] as inner iteration process in the case of inexact implementations for all the proposed methods of SPD coefficient matrix. Otherwise, we use the LU deccomposition for other coefficient matrix in the case of inner iteration process. The corresponding inner stopping tolerance is fixed at 0.01, i.e., $\varepsilon_k=\eta_k$ for any k in Algorithm II.1. Besides, we employ the matrix $M(\omega,\alpha)$ shown in (II.2) and the matrix

$$\frac{1}{2\alpha}(I-\mathbf{i}\omega T)^{-1}(\alpha I+\mathbf{i}T)(\alpha I+H_\omega)$$

shown in [5] as the preconditioners for the IMQHSS preconditioned GMRES method and the IQHSS preconditioned GMRES method, respectively. The corresponding methods are denoted as IQHSS-GMRES and IMQHSS-GMRES. We perform all the experiments using MATLAB (version R2017a) on Intel(R) Core(TM) CPU 3.4Ghz and 8.00 GB of RAM, with machine precision 10^{-16}. In all the experiments, we choose the zero vector to be the initial guess. The iteration is terminated once the current iterate $u^{(k)}$ satisfies

$$\text{RES}=\frac{\|b-Au^{(k)}\|_2}{\|b\|_2}<10^{-6},$$

or the number of iteration counts reaches the maximum number 1000.

Example 1. *Consider the complex Helmholtz equation [15], [23], [29]*

$$-\triangle u+\sigma_1u+\boldsymbol{i}\sigma_2u=f,$$

where u satisfies Dirichlet boundary conditions in $D=[0,1]\times[0,1]$. σ_1 and σ_2 are real coefficient functions. Using the centered difference and discretizing the problem on an $m\times m$ grid with mesh size $h=1/(m+1)$, we will obtain a system of linear equations

$$((K+\sigma_1I)+\boldsymbol{i}\sigma_2I)x=b,$$

where $K=I\otimes V_m+V_m\otimes I$ is the discretizaton of $-\triangle$, where $V_m=h^{-2}\,tridiag(-1,2,-1)\in\mathbb{R}^{m\times m}$. We set $W=-\sigma_2I$ and $T=K+\sigma_1I$. For simplicity, the right-hand side vector b is taken to be $b=-\boldsymbol{i}(W+\boldsymbol{i}T)\boldsymbol{1}$, where $\boldsymbol{1}$ is the vector of all ones vector. Furthermore, after multiplying both sides by h^2, we can obtain a normalized coefficient matrix and the corresponding right-hand side vector.

As is pointed in [5] that ω should be chosen reasonably small so that the matrix $I-\mathbf{i}\omega T$ could be well-conditioned. Hence, we fix ω to be 0.01 in all our experiments. To guarantee the strong skew-Hermitian property, we let $\sigma_2=1$. The parameter σ_1 varies as 1, 10 and 100. The number of iteration counts and the computing time for all the proposed methods are listed in Table I. The corresponding experimental optimal parameters are listed in Table II for all the proposed methods.

From the results in Table I, we can see that when the scale of the problem increases, the numbers of iteration counts and the computing time of the QHSS method and the IQHSS method increase rapidly. The modified QHSS iteration method and the inexact modified QHSS iteration method are more efficient than the QHSS iteration method and the inexact QHSS iteration method, respectively. The main reason of these results is the complex calculating computation. The modified methods are more efficient because those methods have avoided the complex calculating.

Moreover, we observe that when the problem is small scale, the inexact QHSS iteration method and the inexact MQHSS iteration method are worse than the QHSS iteration method and the MQHSS iteration method, respectively. As the mesh grid increases, the numbers of the iteration counts of the IMQHSS method and the IQHSS method keep close to each other, but the inexact methods use much less CPU time. Because of the computation complexity of the complex shift matrix $\alpha I + \mathbf{i}T$ in the IQHSS iteration method and in the QHSS iteration method, the IQHSS iteration method and the QHSS iteration method are less efficient than the IMQHSS iteration method both in terms of the number of iteration counts and the CPU time. These results are also illustrated by the numerical results in Table I.

TABLE II
THE EXPERIMENTAL OPTIMAL PARAMETERS α FOR THE PROPOSED METHODS WITH RESPECT TO DIFFERENT σ_1 AND DIFFERENT m.

	$\sigma_1 \setminus m$:	16	32	64	128	256
QHSS	1	0.06	0.04	0.02	0.01	0.009
	10	0.06	0.04	0.02	0.01	0.009
	100	0.06	0.04	0.02	0.01	0.009
MQHSS	1	0.5	0.3	0.3	0.3	0.3
	10	0.5	0.3	0.3	0.3	0.3
	100	0.5	0.3	0.3	0.3	0.3
IQHSS	1	0.05	0.03	0.02	0.01	0.01
	10	0.09	0.03	0.02	0.02	0.01
	100	0.08	0.03	0.02	0.02	0.01
IMQHSS	1	0.5	0.4	0.3	0.3	0.2
	10	0.7	0.4	0.3	0.4	0.3
	100	0.7	0.7	0.6	0.5	0.5
IQHSS-GMRES	1	0.7	0.7	0.7	0.7	0.7
	10	0.7	0.7	0.7	0.7	0.7
	100	0.7	0.5	0.6	0.6	0.7
IMQHSS-GMRES	1	0.6	0.5	0.3	0.2	0.2
	10	0.6	0.5	0.3	0.2	0.2
	100	0.6	0.3	0.3	0.2	0.2

To further illustrate the efficiency of the IMQHSS iteration method, we plot the eigenvalues distribution in Fig. 1, where the X-axis and the Y-axis denote the real parts and the imaginary parts of the corresponding eigenvalues, respectively. Here the figures on the left top, on the right top, on the left lower and on the right lower plot the eigenvalues distribution of the original coefficient matrix, the equivalent quasi-normal coefficient matrix, the QHSS preconditioned matrix and the MQHSS preconditioned matrix with the experimental optimal parameter α for $n = 32$, $\sigma_1 = 10$, $\sigma_2 = 1$. From Fig. 1, we find that the eigenvalues of the MQHSS preconditioned method are the most clustered around 1.

Finally, we plot in Fig. 2 about the residual against the numbers of iteration counts for the IQHSS-GMRES method and the IMQHSS-GMRES method, respectively. From Fig. 2, we observe again that the IMQHSS-GMRES method is more efficient than the IQHSS-GMRES method.

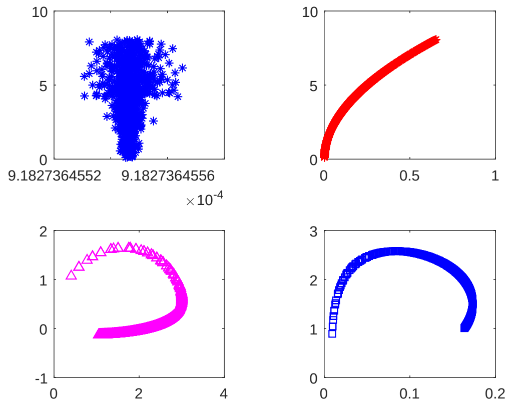

Fig. 1. Eigenvalues distribution. (Left top: original matrix. Right top: quasi-normal matrix. Left lower: the QHSS preconditioned matrix. Right lower: the MQHSS preconditioned matrix. $\sigma_1 = 100$, $m = 32$)

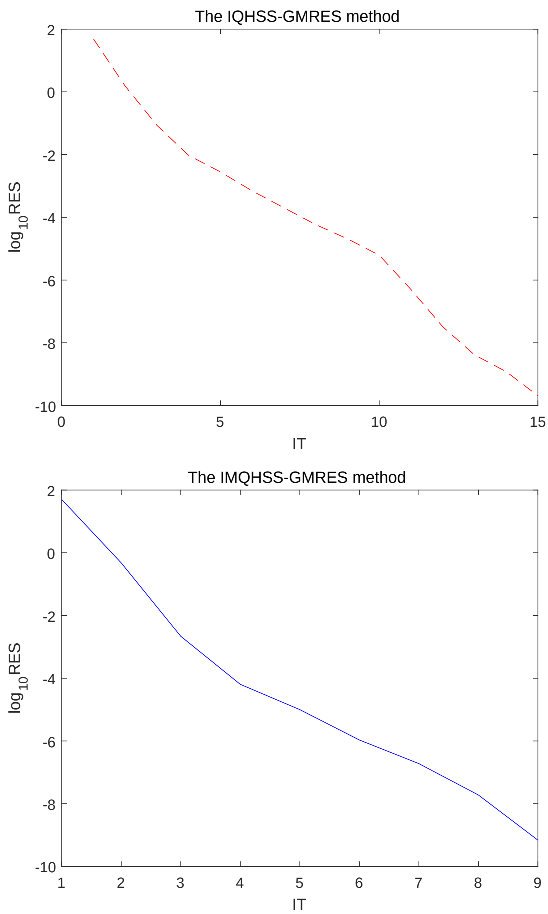

Fig. 2. $\log_{10}$RES against the number of iteration counts for the IQHSS-GMRES method (On the top) and the IMQHSS-GMRES method (At the bottom). ($\sigma_1 = 100$, $m = 32$)

Above all, we may draw a conclusion that when solving the complex symmetric linear systems of strong skew-

TABLE I
NUMERICAL RESULTS FOR THE PROPOSED METHODS CORRESPONDING TO THE EXPERIMENTAL OPTIMAL PARAMETERS.

σ_1	Method	m:	16	32	64	128	256
1	QHSS	IT	56	102	158	296	362
		CPU	0.01	0.35	1.45	22.17	136.75
	IQHSS	IT	69	135	183	339	401
		CPU	0.01	0.65	1.55	21.17	106.23
	IQHSS-GMRES	IT	13	20	32	37	56
		CPU	0.01	0.09	0.44	2.43	42.5
	MQHSS	IT	43	44	44	42	42
		CPU	0.01	0.14	0.31	3.05	35.85
	IMQHSS	IT	44	46	49	50	55
		CPU	0.01	0.14	0.31	3.11	33.14
	IMQHSS-GMRES	IT	**8**	**10**	**14**	**11**	**11**
		CPU	**0.003**	**0.03**	**0.08**	**0.74**	**6.55**
10	QHSS	IT	57	101	158	296	362
		CPU	0.01	0.34	1.47	22.19	135.73
	IQHSS	IT	68	135	183	341	404
		CPU	0.01	0.64	1.57	21.19	109.73
	IQHSS-GMRES	IT	12	20	31	37	55
		CPU	0.01	0.09	0.43	2.44	41.49
	MQHSS	IT	43	44	44	42	41
		CPU	0.01	0.14	0.31	3.05	35.84
	IMQHSS	IT	44	46	49	47	48
		CPU	0.01	0.14	0.31	2.99	26.04
	IMQHSS-GMRES	IT	**8**	**10**	**14**	**11**	**11**
		CPU	**0.003**	**0.03**	**0.08**	**0.74**	**6.55**
100	QHSS	IT	57	101	158	296	362
		CPU	0.01	0.64	1.47	22.19	135.73
	IQHSS	IT	67	135	178	321	396
		CPU	0.01	0.64	2.17	19.09	105.57
	IQHSS-GMRES	IT	10	15	23	25	56
		CPU	0.01	0.06	0.34	2.23	42.79
	MQHSS	IT	43	42	44	42	41
		CPU	0.01	0.14	0.31	3.05	35.84
	IMQHSS	IT	45	44	48	45	45
		CPU	0.01	0.13	0.31	2.92	25.76
	IMQHSS-GMRES	IT	**7**	**9**	**11**	**11**	**10**
		CPU	**0.003**	**0.03**	**0.07**	**0.74**	**6.54**

Hermitian parts, the IMQHSS-GMRES method should be a better choice than the IQHSS-GMRES method.

V. Concluding remarks

In this paper, we focus on the fast solvers for complex symmetric linear systems of strong skew-Hermitian parts. To further improve the efficiency of the MQHSS iteration method, we introduce an inexact MQHSS iteration method. The asymptotically convergence conditions are analyzed theoretically. Numerical experiments have shown that the inexact MQHSS iteration method and the corresponding preconditioned GMRES method are more efficient than the IQHSS iteration method and the IQHSS preconditioned GMRES method, respectively. Moreover, the IMQHSS-GMRES method keeps the most efficient among all the proposed methods.

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