The Division Ring, Similarity and Consimilarity over Conjugate Product

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***Abstract*—Recently Wu, Wang and Teng introduced the division ring over conjugate product as a tool to investigate antilinear systems. In this paper we show that the division ring is a special case of a known construction of a right ring of fractions of a right Ore domain. We also investigate similarity and consimilarity of complex polynomials over conjugate product and characterize all the polynomials which are similar to a given polynomial of degree less than 3, solving partially a problem posed by Wu, Wang and Teng.**

***Index Terms*—Sylvester-polynomial-conjugate matrix equation, division ring, right Ore domain, right ring of fractions, skew polynomial ring, similarity, consimilarity.**

I. Introduction

The Sylvester matrix equation is the equation $AX - XB = C$, where all matrices are complex of appropriate sizes, the matrices A, B, C are given, and the problem is to find the matrices X that satisfy this equation. The matrix equation $AX - \overline{X}B = C$, where $\overline{X}$ denotes the matrix obtained by taking the complex conjugate of each element of X, is called the Sylvester-conjugate matrix equation. These equations, their versions and generalizations (e.g., the well-known Lyapunov matrix equation $AX + XA^T = C$ and the Kalman-Yakubovich-conjugate matrix equation $X - A\overline{X}B = C$) have wide applications, among others in systems and control theory, signal processing, stability theory, differential equations, model reduction, filtering and image restoration. There is a vast literature on the existence, uniqueness and properties of solutions of these matrix equations; the reader is referred to the monographs [4], [5], [18] for more information on the theory and applications of these matrix equations.

In [15] a class of complex matrix equations was studied, the so-called Sylvester-polynomial-conjugate matrix equations

$$A(s) \overset{F}{\boxplus} X + B(s) \overset{F}{\boxplus} Y = C(s) \overset{F}{\boxplus} R, \tag{1}$$

where $\overset{F}{\boxplus}$ is the Sylvester-conjugate sum operator (introduced in [15]), the complex polynomial matrices $A(s), B(s), C(s)$ and complex matrices F, R are given, and complex matrices X, Y are unknown (the Sylvester-conjugate matrix equation and the Kalman-Yakubovich-conjugate matrix equations are special cases of (1)).

In the same paper [15], in order to establish a unified approach for solving the Sylvester-polynomial-conjugate matrix equations (1), the concept of conjugate product $\circledast$ of complex polynomial matrices was proposed. The essence of the conjugate product $\circledast$ is that, to multiply two complex polynomials in a variable s, the variable s is multiplied by a complex number z according to the rule $s \circledast z = \overline{z}s$, where $\overline{z}$ is the complex conjugate of z. The conjugate product of complex polynomials was investigated in [14], where, among other results, it was shown that the set of complex polynomials with the usual addition and the multiplication $\circledast$ is a ring, denoted by $(\mathbb{C}[s], +, \circledast)$ and called the polynomial ring in the framework of conjugate product.

In [17], a polynomial description in the framework of conjugate product was given for antilinear systems. With the aim of getting tools to investigate antilinear systems, in [16] Wu, Wang and Teng introduced a construction which extends the ring $(\mathbb{C}[s], +, \circledast)$ to a division ring, which in [16] is denoted by $(\mathbb{C}(s), +, \circledast)$ and called the division ring of rational fractions in the framework of conjugate product.

In [16, Section I] the authors assert that the conjugate product $\circledast$ is a new concept and that rational fractions over the conjugate product have not been investigated so far. In this paper we show, however, that the division ring $(\mathbb{C}(s), +, \circledast)$ of rational fractions introduced in [16] is a special case of a known general construction of a right ring of fractions, introduced by Ore in 1931 in [11]. To show this, we start by recalling some necessary definitions from ring theory in a later part of this Section. Next, in Section II, we show that the ring $(\mathbb{C}[s], +, \circledast)$, which in [16] is a basis for constructing the division ring of rational fractions $(\mathbb{C}(s), +, \circledast)$, is a case of another known algebraic construction called the skew polynomial ring. In Section III we briefly present the Ore construction, and in Section IV we show that the division ring of rational fractions $(\mathbb{C}(s), +, \circledast)$ from [16] is indeed an immediate outcome of Ore's general construction.

Recall that complex matrices A and B are said to be similar (resp. consimilar) if there exists an invertible complex matrix X such that $X^{-1}AX = B$ (resp. $\overline{X}^{-1}AX = B$), or in other words, if the homogeneous Sylvester (resp. Sylvester-conjugate) matrix equation $AX - XB = 0$ (resp. $AX - \overline{X}B = 0$) has a solution in invertible matrices. Similarly to the case of complex matrices, in [16] the concepts of similarity and consimilarity were proposed for rational fractions in the framework of conjugate product. In [16, Subsection VI-A] the authors asked for necessary and sufficient conditions for similarity of given rational fractions A, B from the division ring $(\mathbb{C}(s), +, \circledast)$, i.e., the problem is to characterize when there exists a nonzero rational fraction P such that $AP = PB$. In Subsection V-A we solve the problem in the case where the rational fractions A, B, P are assumed to belong to the polynomial ring $(\mathbb{C}[s], +, \circledast)$ and the polynomial B is either constant, linear or quadratic. In the same subsection we also show that similarity is an

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equivalence relation on the ring $(\mathbb{C}[s], +, \circledast)$. In Subsection V-B we prove that rational fractions A and B are consimilar if and only if As and Bs are similar, which shows that the notions of similarity and consimilarity of rational fractions are, in a sense, parallel.

As promised, below we recall some necessary definitions from ring theory. It is worthy of mention that rings often appear in systems and control theory (see, e.g., [1], [8], [9], [10], [13], [14], [16]).

A ring is a set R equipped with two binary operations, denoted by $+$ and $\cdot$ (called addition and multiplication, respectively), which satisfy the following axioms:

1) $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$ (that is, addition is associative).
2) $a+b = b+a$ for all $a, b \in R$ (that is, addition is commutative).
3) There exists an element $0 \in R$ such that $a+0 = a$ for all $a \in R$ (that is, 0 is the additive identity).
4) For every $a \in R$ there exists $-a \in R$ such that $a+(-a) = 0$ (that is, $-a$ is the additive inverse of a).
5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$ (that is, multiplication is associative).
6) There exists an element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$ (that is, 1 is the multiplicative identity).
7) $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$ and $(b+c) \cdot a = (b \cdot a)+(c \cdot a)$ for all $a, b, c \in R$ (that is, multiplication is distributive over addition).

The multiplication symbol $\cdot$ is often omitted: we will frequently write ab rather than $a \cdot b$. If multiplication is commutative (that is, $ab = ba$ for all $a, b \in R$), then the ring R is said to be commutative. If $ab \neq 0$ for all $a, b \in R \setminus \{0\}$, then R is called a domain.

Let R be a ring and a an element of R. If there exists $b \in R$ such that $ab = ba = 1$, then a is called an invertible element of R, and b is called the multiplicative inverse of a and denoted by a^{-1}. If all elements of the set $R \setminus \{0\}$ are invertible, then R is called a division ring. A commutative division ring is called a field.

The set $\mathbb{R}^{n \times n}$ of $n \times n$ real matrices (which very often appears in applied mathematics), with usual addition and multiplication of matrices, is an example of a ring that is neither commutative nor a domain for $n > 1$. The set $\mathbb{Z}$ of integers, with usual addition and multiplication, is an example of a commutative domain which is not a field. The set $\mathbb{R}$ of real numbers as well as the set $\mathbb{C}$ of complex numbers, with usual addition and multiplication, are examples of fields. Examples of division rings that are not fields will be presented in Section IV.

Let R with addition $+$ and multiplication $\cdot$ be a ring. A subset T of R is called a subring of R if T is itself a ring under the operations $+$ and $\cdot$ restricted to T. If a ring T is a subring of R, then R is called an overring of T. For example, the ring $\mathbb{Z}$ of integers is a subring of the polynomial ring $\mathbb{Z}[s]$ (and $\mathbb{Z}[s]$ is an overring of $\mathbb{Z}$).

II. Skew polynomial rings

Given a ring R, a map $\sigma : R \to R$ is called an automorphism of R if σ is bijective and for all $a, b \in R$,

$$\sigma(a+b) = \sigma(a) + \sigma(b) \text{ and } \sigma(ab) = \sigma(a)\sigma(b).$$

For example, the map $\sigma : \mathbb{C} \to \mathbb{C}$ defined for all $z \in \mathbb{C}$ by $\sigma(z) = \overline{z}$ (the complex conjugate of z) is an automorphism of $\mathbb{C}$.

Let R be a ring and let σ be an automorphism of R. Let $\mathcal{P}$ be the set of polynomials over R in one variable s, i.e., $\mathcal{P}$ consists of all formal sums $\sum_{i=0}^{n} a_i s^i$, where n is a nonnegative integer and $a_i \in R$ for $i = 0, 1, \ldots, n$. These polynomials can be added in an obvious way:

$$\sum_{i=0}^{n} a_i s^i + \sum_{i=0}^{n} b_i s^i = \sum_{i=0}^{n} (a_i + b_i) s^i, \tag{2}$$

and multiplied formally, assuming that the elements of R commute with the variable s, i.e.,

$$sa = as \quad \text{for any } a \in R. \tag{3}$$

Thus the multiplication is as follows:

$$\sum_{i=0}^{m} a_i s^i \cdot \sum_{j=0}^{n} b_j s^j = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j s^{i+j}.$$

With these operations of addition and multiplication, $\mathcal{P}$ is a ring, denoted by $R[s]$ and called the ring of polynomials over R. We can use the automorphism σ of R to define another multiplication in $\mathcal{P}$, by replacing the rule (3) with the rule

$$sa = \sigma(a)s \quad \text{for any } a \in R. \tag{4}$$

Assuming (4) and the associativity of multiplication, we obtain

$$\begin{aligned} s^2 a &= (ss)a = s(sa) = s\big(\sigma(a)s\big) = \big(s\sigma(a)\big)s \\ &= \Big(\sigma\big(\sigma(a)\big)s\Big)s = \big(\sigma^2(a)s\big)s = \sigma^2(a)\big(ss\big) = \sigma^2(a)s^2, \end{aligned}$$

and by induction it follows that for any $a \in R$ and nonnegative integer i we have

$$s^i a = \sigma^i(a) s^i, \tag{5}$$

where σ^i is the composition $\sigma \circ \sigma \circ \cdots \circ \sigma$ with σ repeated i times. Hence the "new" multiplication in $\mathcal{P}$ is as follows:

$$\sum_{i=0}^{m} a_i s^i \cdot \sum_{j=0}^{n} b_j s^j = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i \sigma^i(b_j) s^{i+j}. \tag{6}$$

It is well known (see, e.g., [12, §1.6]) that the set $\mathcal{P}$ with addition (2) and multiplication (6) is a ring. The ring is called the skew polynomial ring and denoted by $R[s;\sigma]$. We will denote polynomials by capital letters, and to save space, without indicating the indeterminate s (e.g., A instead of $A(s)$).

In conclusion, the "usual" polynomial ring $R[s]$ and the skew polynomial ring $R[s;\sigma]$ have the same set $\mathcal{P}$ of polynomials over R as the underlying set, and the same addition, but the multiplication in $R[s;\sigma]$ is "skewed" by the rule (4). Let us note that the identity map id_R of R is an automorphism of R and for $\sigma = id_R$ the rings $R[s]$ and $R[s;\sigma]$ coincide. Hence the polynomial ring $R[s]$ is a special case of the skew polynomial ring construction, i.e., the notion of a skew polynomial ring generalizes the notion of a polynomial ring.

In [16], for complex polynomials (i.e., polynomials with coefficients in $\mathbb{C}$) the concept of conjugate product, denoted by $\circledast$, is considered. First, in [16], for a complex number c the authors define $c^{*0} = c$, and

$$c^{*k} = \overline{c^{*(k-1)}} \text{ for any positive integer } k.$$

Let us observe that if σ is the complex conjugation, then, simply, $c^{*k} = \sigma^k(c)$. Next, in [16, Definition 1] (see also [14]), the conjugate product $\circledast$ of complex polynomials $A = \sum_{i=0}^{m} a_i s^i$ and $B = \sum_{j=0}^{n} b_j s^j$ is defined as

$$A \circledast B = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j^{*i} s^{i+j}. \tag{7}$$

In [14, Theorem 1] it is shown that the set of complex polynomials with usual addition (2) and the multiplication $\circledast$ is a ring; we refer to the ring as $(\mathbb{C}[s], +, \circledast)$, adopting the notation from [14]. By comparing (7) and (6) it is obvious that the ring $(\mathbb{C}[s], +, \circledast)$ is just the skew polynomial ring $\mathbb{C}[s;\sigma]$, where σ is the complex conjugation. The advantage of this observation is that known general results on skew polynomial rings can be applied to the concrete ring $(\mathbb{C}[s], +, \circledast)$ considered in [16], as we will see in Section IV. Since in the ring $(\mathbb{C}[s], +, \circledast) = \mathbb{C}[s;\sigma]$ we have $si = \sigma(i)s = -is \neq is$, the ring is not commutative. The ring $(\mathbb{C}[s], +, \circledast)$ is well known in the literature; e.g., in [2, p. 54] the ring is called the complex-skew polynomial ring and denoted by $\mathbb{C}[s; -]$.

III. Right ring of fractions

A. Right Ore domains

One of basic ring constructions is the field of fractions of a commutative domain R, constructed as a set of fractions, that is, expressions $\frac{a}{b}$ with $a \in R$ and $b \in R \setminus \{0\}$ subject to an obvious equivalence relation (see, e.g., [6, Section III.4]). For example, the field $\mathbb{Q}$ of rationals is the field of fractions of the domain $\mathbb{Z}$ of integers. In the non-commutative case, it is not always possible to pass from a domain to a division ring built from fractions. Below we explain when such a division ring exists.

Let R be a domain and $R^* = R \setminus \{0\}$ the set of non-zero elements of R. A right ring of fractions of R is defined to be any overring Q of R satisfying the following two conditions:

1) Every element of R^* is invertible in Q.
2) Every element of Q can be written in the form as^{-1} for some $a \in R$ and $s \in R^*$.

Let R be a domain and $R \times R^*$ the Cartesian product of R and R^* (i.e., the set of all ordered pairs (a, s) where $a \in R$ and $s \in R^*$). In 1931, in [11] Ore proved that a right ring of fractions of R exists if and only if

$$\begin{gathered}\text{for every } (a,s) \in R \times R^* \text{ there exists} \\ (a_1, s_1) \in R \times R^* \text{ such that } as_1 = sa_1.\end{gathered} \tag{8}$$

Nowadays, if R satisfies condition (8), then R is called a right Ore domain. Hence Ore's theorem can be stated as follows.

Theorem 1: (Ore, 1931) A domain R has a right ring of fractions if and only if R is a right Ore domain.

In Subsection III-B we will outline Ore's construction of a right ring of fractions. In the context of the paper [16] it is worth recalling the following property of skew polynomial rings (see, e.g., [7, Theorem 10.28]).

Theorem 2: If R is a right Ore domain and σ is an automorphism of R, then $R[s;\sigma]$ is a right Ore domain.

As we have already observed in the last paragraph of Section II, the ring $(\mathbb{C}[s], +, \circledast)$ of polynomials in the framework of conjugate product considered in [16] is just the skew polynomial ring $\mathbb{C}[s;\sigma]$, where σ is the complex conjugation. Since clearly the field $\mathbb{C}$ is a right Ore domain, Theorem 2 implies that so is the ring $(\mathbb{C}[s], +, \circledast)$. Some important consequences of this observation will be presented in Section IV.

B. The Ore construction of a right ring of fractions

In this subsection we outline the Ore construction of a right ring of fractions of a right Ore domain. The construction is presented in detail in many ring theory textbooks (e.g., [3, Section 3.8], [7, §10A], [12, §3.1]).

Let R be a right Ore domain and let $R^* = R \setminus \{0\}$. The following relation $\sim$ on $R \times R^*$ is an equivalence relation:

$$\begin{gathered}(a,s) \sim (a',s') \text{ if and only if there exist } b, b' \in R^* \\ \text{such that } ab = a'b' \text{ and } sb = s'b'.\end{gathered}$$

The equivalence class of $(a, s) \in R \times R^*$ is denoted by $\frac{a}{s}$.

Let Q be the set of all equivalence classes and let $\frac{a_1}{s_1}, \frac{a_2}{s_2} \in Q$. By (8) there exist $r \in R$ and $s \in R^*$ such that $s_1 s = s_2 r \in R^*$. The sum of $\frac{a_1}{s_1}$ and $\frac{a_2}{s_2}$ is defined by

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s + a_2 r}{s_1 s}$$

(see Example 1 for an illustration of the operation). This is a well-defined binary operation on Q satisfying axioms 1–4 of the definition of a ring, with the additive identity $\frac{0}{1}$. In order to multiply $\frac{a_1}{s_1}$ with $\frac{a_2}{s_2}$, we use (8) to find $r \in R$ and $s \in R^*$ such that $s_1 r = a_2 s$. Then we define

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 r}{s_2 s}. \tag{9}$$

It can be checked that (9) gives a well-defined multiplication on Q, and finally that Q, with the addition and multiplication just defined, is a ring. Note that $\frac{1}{1}$ is the multiplicative identity in Q. Furthermore, if $\frac{a}{s} \neq \frac{0}{1}$, then $a \neq 0$ and thus $\frac{s}{a} \in Q$ and $\frac{a}{s} \cdot \frac{s}{a} = \frac{s}{a} \cdot \frac{a}{s} = \frac{1}{1}$, which shows that Q is a division ring.

By identifying an element $a \in R$ with the fraction $\frac{a}{1} \in Q$, we get the containment $R \subseteq Q$ with $R \ni a = \frac{a}{1} \in Q$. Since addition and multiplication in R agree with these in Q, Q is an overring of R. Furthermore,

$$\frac{a}{s} = \frac{a}{1} \cdot \frac{1}{s} = \frac{a}{1} \cdot \left(\frac{s}{1}\right)^{-1} = as^{-1}$$

and thus the division ring Q is a right ring of fractions of R.

Example 1: As we will show in Section IV, the Ore construction of a right ring of fractions can be applied to the skew polynomial ring $\mathbb{C}[s;\sigma]$, where σ is the complex conjugation. To illustrate this construction, let us consider the following four complex polynomials:

$$\begin{gathered}A = (1-i)s + 2 + i, \; B = s^2 + (1+3i)s + 2 - i, \\ C = s + 2 - 2i, \; D = (1+i)s + 3 + i \in \mathbb{C}[s;\sigma].\end{gathered}$$

To calculate the sum $\frac{A}{B} + \frac{C}{D}$ in the right ring of fractions of $\mathbb{C}[s;\sigma]$, we first make the denominators of the fractions the same, that is, we find nonzero polynomials R, T such that $BR = DT$. It can be verified that for $R = (1+i)s + 2i$ and $T = s^2 + (2-i)s + 1 + i$ in the ring $\mathbb{C}[s;\sigma]$ we have

$$BR = DT = (1+i)s^3 + (4+4i)s^2 + (9-i)s + 2 + 4i \quad (10)$$

and thus

$$\frac{A}{B} + \frac{C}{D} = \frac{AR}{BR} + \frac{CT}{DT} = \frac{AR}{BR} + \frac{CT}{BR} = \frac{AR+CT}{BR}$$
$$= \frac{s^3 + (4-3i)s^2 + (2-6i)s + 2 + 4i}{(1+i)s^3 + (4+4i)s^2 + (9-i)s + 2 + 4i}.$$

We can also use (10) to calculate the product $\frac{A}{B} \cdot \frac{D}{C}$:

$$\frac{A}{B} \cdot \frac{D}{C} = \frac{AR}{BR} \cdot \frac{DT}{CT} = \frac{AR}{BR} \cdot \frac{BR}{CT} = \frac{AR}{CT}$$
$$= \frac{-2is^2 + (-1+i)s - 2 + 4i}{s^3 + (4-i)s^2 + (3-7i)s + 4}.$$

IV. An application: the division ring over conjugate product

Let D be a commutative domain and $\sigma : D \to D$ an automorphism of D. Then obviously D is a right Ore domain and thus, by Theorem 2, the skew polynomial ring $D[s;\sigma]$ is a right Ore domain. Hence by Theorem 1, a right ring of fractions of $D[s;\sigma]$ exists, automatically being a division ring, and can be constructed by the Ore method as described in Subsection III-B. This path from a commutative domain D with an automorphism σ to the division ring is presented in Figure 1.

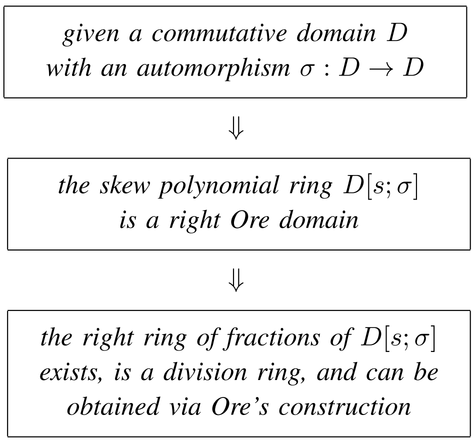

Figure 1. Three-step construction of a division ring

As we have observed in Section II, the ring $(\mathbb{C}[s], +, \circledast)$ from [16] is simply the skew polynomial ring $\mathbb{C}[s;\sigma]$, where σ is the complex conjugation. Since σ is an automorphism of the field $\mathbb{C}$, according to the scheme presented in Figure 1 the ring $R = \mathbb{C}[s;\sigma]$ is a right Ore domain and a right ring of fractions of R can be constructed via the Ore method outlined in Subsection III-B. To sum up, it follows from general results of the theory of rings that Ore's construction can be applied to the ring $(\mathbb{C}[s], +, \circledast) = \mathbb{C}[s;\sigma]$, giving as the result a division ring, which is just the right ring of fractions of the ring $(\mathbb{C}[s], +, \circledast)$.

In [16] a construction of a division ring is presented, called the division ring of rational fractions in the framework of conjugate product and denoted by $(\mathbb{C}(s), +, \circledast)$. The construction starts from the polynomial ring in the framework of conjugate product $(\mathbb{C}[s], +, \circledast)$ and follows the same path as the general Ore construction recalled in Subsection III-B (in fact it is well known that the only way to pass from a right Ore domain to its right ring of fractions is to follow Ore's construction; see, e.g., [3, p. 71]). Hence the division ring over conjugate product from [16] is just the right ring of fractions of the ring $(\mathbb{C}[s], +, \circledast)$. The detailed verifications of correctness of the construction of the division ring $(\mathbb{C}(s), +, \circledast)$, which form a major part of [16], are unnecessary, since the Ore machinery works for any right Ore domain, so in particular for the ring $(\mathbb{C}[s], +, \circledast)$.

Now it is easy to find other examples of rings for which the method presented in Figure 1 works. For instance, one can consider the ring $D = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ (with usual addition and multiplication of numbers) and its automorphism $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. Since D is a commutative domain, the right ring of fractions of the skew polynomial ring $D[s;\sigma]$ exists, is a division ring (but not a field) and can be obtained via the Ore construction.

V. Similarity and consimilarity of complex-skew polynomials

Throughout this section, σ denotes the complex conjugate operator, i.e., $\sigma(z) = \overline{z}$ for any $z \in \mathbb{C}$.

According to [16, Definition 8], for a polynomial $A = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s;\sigma]$, its conjugate is defined as

$$\overline{A} = \sum_{i=0}^{n} \overline{a_i} s^i.$$

Furthermore, if Q is the right ring of fractions of $\mathbb{C}[s;\sigma]$ and $F = AB^{-1}$ is an element of Q (where $A, B \in \mathbb{C}[s;\sigma]$), then according to [16, Definition 9] the conjugate of F is defined to be $\overline{F} = \overline{A}\,\overline{B}^{-1}$.

In [16, Section VI], two similarity concepts for elements of the right ring of fractions Q are considered: an element $A \in Q$ is said to be similar (resp. consimilar) to an element $B \in Q$ if there exists a nonzero element $P \in Q$ such that

$$AP = PB \text{ (resp. } A\overline{P} = PB). \quad (11)$$

In [16, Subsection VI-A] the authors left open the problem of giving necessary and sufficient conditions for similarity of elements of Q, describing it as nontrivial. Below (in Theorem 4) we solve the problem in the case where A, B, P appearing in (11) are assumed to belong to $\mathbb{C}[s;\sigma]$ and the polynomial B is either constant, linear or quadratic.

A. Similarity

Definition 1: Let $A, B \in \mathbb{C}[s;\sigma]$. We say that the polynomial A is similar to the polynomial B if there exists a nonzero polynomial $P \in \mathbb{C}[s;\sigma]$ such that $AP = PB$.

Our first aim in this subsection is to show that the similarity relation defined above is an equivalence relation on $\mathbb{C}[s;\sigma]$. For that we introduce the following operator (denoted by tilde) on complex-skew polynomials.

Definition 2: For a polynomial $A = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s;\sigma]$ we denote

$$\tilde{A} = \sum_{i=0}^{n} (-1)^{i+1} \sigma^{n+1}(a_i) s^i,$$

i.e., to get $\tilde{A}$, for every even (resp. odd) i the s^i-coefficient of A is replaced with the additive inverse of its complex conjugate (resp. remains unchanged).

For example, if $A = a_3s^3 + a_2s^2 + a_1s + a_0$, then $\tilde{A} = a_3s^3 - \overline{a_2}s^2 + a_1s - \overline{a_0}$.

Lemma 1: In the ring $\mathbb{C}[s;\sigma]$, for any polynomial $A \in \mathbb{C}[s;\sigma]$ we have $A\tilde{A} \in \mathbb{R}[s^2]$, i.e., $A\tilde{A}$ is a polynomial with real coefficients such that for any odd k the s^k-coefficient of $A\tilde{A}$ is equal to zero.

Proof: Let $A = \sum_{i=0}^{n} a_is^i \in \mathbb{C}[s;\sigma]$, let k be a given nonnegative integer, and let b_k be the s^k-coefficient of $A\tilde{A}$. Denote by S the set of all ordered pairs (i,j) of nonnegative integers such that $i+j=k$, i.e.,

$$S = \{(0,k),(1,k-1),\dots,(k-1,1),(k,0)\}.$$

Then it follows from Definition 2 and (6) that

$$b_k = \sum_{(i,j)\in S} a_i\sigma^i((-1)^{j+1}\sigma^{j+1}(a_j)). \tag{12}$$

To complete the proof, it suffices to show that if k is odd, then $b_k = 0$, and if k is even, then $b_k \in \mathbb{R}$.

Assume that k is odd. Then the number of elements of S is equal to $k+1$, so it is even and each element (i,j) of S can be paired with the element (j,i), that is, S splits into disjoint 2-element subsets of the form $\{(i,j),(j,i)\}$. Since $k+1$ is even, $\sigma^{k+1}(z) = z$ for any $z \in \mathbb{C}$ and thus each such a 2-element subset $\{(i,j),(j,i)\}$ of S contributes the following value to the total sum (12):

$$\begin{aligned} &a_i\sigma^i((-1)^{j+1}\sigma^{j+1}(a_j)) + a_j\sigma^j((-1)^{i+1}\sigma^{i+1}(a_i)) \\ &\quad = a_i(-1)^{j+1}\sigma^{k+1}(a_j) + a_j(-1)^{i+1}\sigma^{k+1}(a_i) \qquad (13) \\ &= a_i(-1)^{j+1}a_j + a_j(-1)^{i+1}a_i = ((-1)^{j+1}+(-1)^{i+1})a_ia_j. \end{aligned}$$

Since $i+j=k$ is odd, $j+1$ and $i+1$ are of different parity and thus $(-1)^{j+1}+(-1)^{i+1} = 0$. Hence (13) is equal to zero, and consequently so is b_k.

We are left with the case where k is even, say $k = 2m$. Then S is a disjoint sum of 2-element subsets of the form $\{(i,j),(j,i)\}$ and the singleton $\{(m,m)\}$. Since $k+1$ is odd, $\sigma^{k+1}(z) = \overline{z}$ for any $z \in \mathbb{C}$ and thus each 2-element subset $\{(i,j),(j,i)\}$ contributes to (12) the following value:

$$\begin{aligned} &a_i(-1)^{j+1}\sigma^{k+1}(a_j) + a_j(-1)^{i+1}\sigma^{k+1}(a_i) \\ &= a_i(-1)^{j+1}\overline{a_j} + a_j(-1)^{i+1}\overline{a_i}. \end{aligned} \tag{14}$$

Since $i+j=k$ is even, $j+1$ and $i+1$ are of the same parity and thus $(-1)^{j+1} = (-1)^{i+1}$. Hence (14) is equal to $(-1)^{i+1}(a_i\overline{a_j} + \overline{a_i}a_j) = (-1)^{i+1}(a_i\overline{a_j} + \overline{a_i\overline{a_j}})$. Since $z + \overline{z} \in \mathbb{R}$ for any $z \in \mathbb{C}$, it follows that (14) is a real number. Furthermore, the singleton $\{(m,m)\}$ contributes to (12) the value

$$a_m(-1)^{m+1}\sigma^{2m+1}(a_m) = (-1)^{m+1}a_m\overline{a_m} = (-1)^{m+1}|a_m|^2,$$

which is a real number, since the modulus $|a_m|$ of a_m is real. Hence $b_k \in \mathbb{R}$, as desired. ∎

Since in the ring $\mathbb{C}[s;\sigma]$ we have $sr = rs$ and $s^2a = as^2$ for any $r \in \mathbb{R}$ and $a \in \mathbb{C}$, it follows from Lemma 1 that for any $A \in \mathbb{C}[s;\sigma]$, $A\tilde{A}$ commutes with all $B \in \mathbb{C}[s;\sigma]$, i.e., in $\mathbb{C}[s;\sigma]$ we have that $(A\tilde{A})B = B(A\tilde{A})$. We will use this observation in the proof of the following theorem.

Theorem 3: The similarity of complex-skew polynomials (as defined in Definition 1) is an equivalence relation, i.e., the following properties hold.

(a) Reflexivity: for any $A \in \mathbb{C}[s;\sigma]$, A is similar to A;
(b) Symmetry: for any $A, B \in \mathbb{C}[s;\sigma]$, if A is similar to B, then B is similar to A;
(c) Transivity: for any $A, B, C \in \mathbb{C}[s;\sigma]$, if A is similar to B and B is similar to C, then A is similar to C.

Proof: Since for $P = 1$ we have that $AP = PA$, (a) holds. To prove (b), assume that $AP = PB$ for some $A, B, P \in \mathbb{C}[s;\sigma]$ with $P \neq 0$. Then $AP\tilde{P} = PB\tilde{P}$. From Lemma 1 it follows that A and $P\tilde{P}$ commute and thus

$$P\tilde{P}A = PB\tilde{P}. \tag{15}$$

Since $\mathbb{C}[s;\sigma]$ is a domain and $P \neq 0$, (15) implies that $B\tilde{P} = \tilde{P}A$ with $\tilde{P} \neq 0$. Hence B is similar to A.

To prove (c), assume that $AP = PB$ and $BQ = QC$ for some $A, B, C, P, Q \in \mathbb{C}[s;\sigma]$ with $P \neq 0$ and $Q \neq 0$. Then $APQ = PBQ = PQC$ and since $\mathbb{C}[s;\sigma]$ is a domain, $PQ \neq 0$. Hence A is similar to C. ∎

In the theorem below we characterize all pairs of polynomials $A, B \in \mathbb{C}[s;\sigma]$ such that the polynomial B is either constant, linear or quadratic (i.e., $B = \alpha s^2 + \beta s + \gamma$ for some $\alpha, \beta, \gamma \in \mathbb{C}$) and A is similar to B.

Theorem 4: Let $B = \alpha s^2 + \beta s + \gamma \in \mathbb{C}[s;\sigma]$. Then for any polynomial $A \in \mathbb{C}[s;\sigma]$ the following conditions are equivalent.

(i) The polynomial A is similar to the polynomial B.
(ii) $A = as^2 + bs + c$, where $a \in \{\alpha, \overline{\alpha}\}$ and for some $k \in \{0,1\}$, $c = \sigma^k(\gamma)$ and

$$(a - \sigma^k(\alpha))(c - \overline{c}) = |\beta|^2 - |b|^2. \tag{16}$$

Proof: (i) ⇒ (ii): Assume (i). Hence there exists a polynomial $P = p_ns^n + p_{n-1}s^{n-1} + \cdots + p_ks^k \in \mathbb{C}[s;\sigma]$ with $p_n \neq 0$ and $p_k \neq 0$, such that $AP = PB$. In particular, the degrees of A and B satisfy $\deg A = \deg B \leq 2$, and thus $A = as^2 + bs + c$ for some $a, b, c \in \mathbb{C}$. Let us observe that $Bs^m = s^mB$ for any even positive integer m. Thus, from the equation $AP = PB$ it follows that $A(Ps^m) = (Ps^m)B$ for any even m, and thus $A(Ps^m + P) = (Ps^m + P)B$ as well. Hence we can assume that $n > k+2$. Therefore, we have

$$\begin{aligned} &(as^2 + bs + c)(p_ns^n + p_{n-1}s^{n-1} + \cdots + p_ks^k) \\ &\quad = (p_ns^n + p_{n-1}s^{n-1} + \cdots + p_ks^k)(\alpha s^2 + \beta s + \gamma) \end{aligned} \tag{17}$$

with $p_n \neq 0$, $p_k \neq 0$ and $n > k+2$. Looking at, respectively, the $s^{n+2}, s^{k+2}, s^{k+1}, s^k$-coefficients of both sides of (17) we obtain the following four equations:

$$ap_n = p_n\sigma^n(\alpha) \tag{18}$$

$$\begin{aligned} &ap_k + b\overline{p_{k+1}} + cp_{k+2} \\ &\quad = p_{k+2}\sigma^{k+2}(\gamma) + p_{k+1}\sigma^{k+1}(\beta) + p_k\sigma^k(\alpha) \end{aligned} \tag{19}$$

$$b\overline{p_k} + cp_{k+1} = p_{k+1}\sigma^{k+1}(\gamma) + p_k\sigma^k(\beta) \tag{20}$$

$$cp_k = p_k\sigma^k(\gamma) \tag{21}$$

Since p_n and p_k are both nonzero, (18) and (21) imply that

$$a = \sigma^n(\alpha) \in \{\alpha, \overline{\alpha}\} \text{ and } c = \sigma^k(\gamma).$$

Since $c = \sigma^k(\gamma)$, it follows that $\sigma^{k+1}(\gamma) = \overline{c}$ and $\sigma^{k+2}(\gamma) = c$, and thus (19) and (20) are equivalent to, respectively,

$$p_k(a - \sigma^k(\alpha)) = p_{k+1}\sigma^{k+1}(\beta) - b\overline{p_{k+1}} \quad (22)$$

$$p_{k+1}(c - \overline{c}) = p_k\sigma^k(\beta) - b\overline{p_k} \quad (23)$$

By conjugating both sides of (23) we obtain

$$\overline{p_{k+1}}(c - \overline{c}) = \overline{b}p_k - \overline{p_k}\sigma^{k+1}(\beta). \quad (24)$$

Now by multiplying (22) by $c - \overline{c}$ and using (23) and (24), we obtain

$$p_k(a - \sigma^k(\alpha))(c - \overline{c}) = p_k(|\beta|^2 - |b|^2),$$

and since $p_k \neq 0$, the desired equation (16) follows. Obviously, if k is even, then $\sigma^k = \sigma^0$, and if k is odd, then $\sigma^k = \sigma^1$. Thus, in the equations $c = \sigma^k(\gamma)$ and (16), we can assume that $k \in \{0, 1\}$.

(ii) ⇒ (i): Assume (ii). To show that the polynomial A is similar to B, we consider two cases.

Case 1: $a = \sigma^k(\alpha)$. Then (16) implies $|b| = |\beta|$. We claim that

$$\text{there exists } v \in \mathbb{C} \setminus \{0\} \text{ such that } b\overline{v} = v\sigma^k(\beta). \quad (25)$$

Indeed, if $\beta = 0$, then $|b| = |\beta| = 0$, so also $b = 0$ and we can put $v = 1$. Assume $\beta \neq 0$ and let v be a square root of $\frac{b}{\sigma^k(\beta)}$, i.e., $v \in \mathbb{C}$ and $v^2 = \frac{b}{\sigma^k(\beta)}$. Since

$$|v|^2 = \left|\frac{b}{\sigma^k(\beta)}\right| = \frac{|b|}{|\sigma^k(\beta)|} = \frac{|b|}{|\beta|} = 1,$$

it follows that $\frac{v}{\overline{v}} = v^2$ and thus $\frac{v}{\overline{v}} = \frac{b}{\sigma^k(\beta)}$, i.e., $b\overline{v} = v\sigma^k(\beta)$, which proves (25).

Now, by using (25), for the nonzero polynomial $P = vs^k$ we obtain

$$\begin{aligned} AP &= (as^2 + bs + c)vs^k = \big(\sigma^k(\alpha)s^2 + bs + \sigma^k(\gamma)\big)vs^k \\ &= v\sigma^k(\alpha)s^{k+2} + b\overline{v}s^{k+1} + v\sigma^k(\gamma)s^k \\ &= v\sigma^k(\alpha)s^{k+2} + v\sigma^k(\beta)s^{k+1} + v\sigma^k(\gamma)s^k \\ &= vs^k\alpha s^2 + vs^k\beta s + vs^k\gamma = vs^k(\alpha s^2 + \beta s + \gamma) \\ &= PB, \end{aligned}$$

which shows that A is similar to B.

Case 2: $a \neq \sigma^k(\alpha)$. Since $a \in \{\alpha, \overline{\alpha}\} = \{\sigma^k(\alpha), \sigma^{k+1}(\alpha)\}$, it follows that $a = \sigma^{k+1}(\alpha)$ and $\overline{a} = \sigma^{k+2}(\alpha) = \sigma^k(\alpha) \neq a$. Let

$$d = \frac{\sigma^{k+1}(\beta) - b}{a - \overline{a}} \quad \text{and} \quad P = s^{k+1} + ds^k.$$

Then $AP = as^{k+3} + (ad + b)s^{k+2} + (b\overline{d} + c)s^{k+1} + cds^k$ and $PB = \sigma^{k+1}(\alpha)s^{k+3} + \big(\sigma^{k+1}(\beta) + d\sigma^k(\alpha)\big)s^{k+2} + \big(\sigma^{k+1}(\gamma) + d\sigma^k(\beta)\big)s^{k+1} + d\sigma^k(\gamma)s^k$.

We show that $AP = PB$. Since $c = \sigma^k(\gamma)$ and $a = \sigma^{k+1}(\alpha)$, the s^k-coefficients of AP and PB are equal, and so are the s^{k+3}-coefficients. To show that the s^{k+2}-coefficients of AP and PB are equal, we show that their difference is equal to 0:

$$\begin{aligned} &ad + b - (\sigma^{k+1}(\beta) + d\sigma^k(\alpha)) = d(a - \overline{a}) + b - \sigma^{k+1}(\beta) \\ &= (\sigma^{k+1}(\beta) - b) + b - \sigma^{k+1}(\beta) = 0. \end{aligned}$$

To show that also the s^{k+1}-coefficients of AP and PB are equal, we consider their difference:

$$b\overline{d} + c - (\sigma^{k+1}(\gamma) + d\sigma^k(\beta)) = (b\overline{d} - d\sigma^k(\beta)) + (c - \overline{c}).$$

Since $\overline{d} = \frac{\overline{b} - \sigma^k(\beta)}{a - \overline{a}}$, using (16) we obtain

$$\begin{aligned} b\overline{d} - d\sigma^k(\beta) &= b\,\frac{\overline{b} - \sigma^k(\beta)}{a - \overline{a}} - \frac{\sigma^{k+1}(\beta) - b}{a - \overline{a}}\sigma^k(\beta) \\ &= \frac{b\overline{b} - \sigma^{k+1}(\beta)\sigma^k(\beta)}{a - \overline{a}} = \frac{|b|^2 - |\beta|^2}{a - \overline{a}} \\ &= \overline{c} - c. \end{aligned}$$

Hence the difference of s^{k+1}-coefficients of AP and PB is equal to 0, which completes the proof that $AP = PB$. Therefore, A is similar to B. ■

Example 2: Using Theorem 4 it is easy to verify that the polynomial

$$A = (5 - 2i)s^2 + (4 + 5i)s + 4i \in \mathbb{C}[s; \sigma]$$

is similar to the polynomial

$$B = (5 + 2i)s^2 + (3 + 8i)s + 4i \in \mathbb{C}[s; \sigma].$$

As an immediate consequence of Theorem 4 we obtain the following characterizations of polynomials that are similar to a given polynomial of degree, respectively, 0, 1, or 2.

Corollary 2: Let $\gamma \in \mathbb{C}$. A polynomial $A \in \mathbb{C}[s; \sigma]$ is similar to the polynomial $B = \gamma$ if and only if $A = \gamma$ or $A = \overline{\gamma}$.

Corollary 3: Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$. A polynomial $A \in \mathbb{C}[s; \sigma]$ is similar to the polynomial $B = \beta s + \gamma$ if and only if $A = bs + c$ with $|b| = |\beta|$ and $c \in \{\gamma, \overline{\gamma}\}$.

Example 3: (cf. [16, Example 3]) As an application of Corollary 3, we can see that the polynomial $A = (1+i)s+1$ is similar to the polynomial $B = \frac{7+17i}{13}s+1$.

Corollary 4: Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$. A polynomial $A \in \mathbb{C}[s; \sigma]$ is similar to the polynomial $B = \alpha s^2 + \beta s + \gamma$ if and only if $A = as^2 + bs + c$, where $a \in \{\alpha, \overline{\alpha}\}$ and for some $k \in \{0, 1\}$, $c = \sigma^k(\gamma)$ and $(a - \sigma^k(\alpha))(c - \overline{c}) = |\beta|^2 - |b|^2$.

We make the following remark on similarity of polynomials of degree 4.

Remark 1: Analogously as in the first part of the proof of Theorem 4 one can show that if a polynomial

$$A = as^4 + bs^3 + cs^2 + ds + e \in \mathbb{C}[s; \sigma]$$

is similar to the polynomial

$$B = \alpha s^4 + \beta s^3 + \gamma s^2 + \delta s + \varepsilon \in \mathbb{C}[s; \sigma],$$

then for some $k, n \in \{0, 1\}$ we have

$$a = \sigma^n(\alpha) \quad \text{and} \quad (c - \sigma^n(\gamma))(a - \overline{a}) = |\beta|^2 - |b|^2, \quad (26)$$

and

$$e = \sigma^k(\varepsilon) \quad \text{and} \quad (c - \sigma^k(\gamma))(e - \overline{e}) = |\delta|^2 - |d|^2. \quad (27)$$

Hence (26) and (27) are necessary conditions for A to be similar to B. These conditions, however, are not sufficient. Indeed, for $A = -is^4+i$ and $B = is^4+i$ both the conditions (26) and (27) are satisfied but, as we show below, A is not similar to B. Suppose for a contradiction that the polynomial A is similar to the polynomial B. Then there exists a polynomial $P = p_n s^n + p_{n-1}s^{n-1} + \cdots + p_0 \in \mathbb{C}[s;\sigma]$ with $p_n \neq 0$ such that $AP = PB$. Without loss of generality we can assume that $n \geq 4$. By comparing s^{n+4}-coefficients of AP and PB we obtain $-ip_n = p_n\sigma^n(i)$, and thus $-i = \sigma^n(i)$, which implies that n is odd. Hence comparison of s^n-coefficients of AP and PB yields $i = \sigma^n(i)$. Thus $i = \sigma^n(i) = -i$, and this contradiction shows that $A = -is^4 + i$ is not similar to $B = is^4 + i$.

Given polynomials $A, B \in \mathbb{C}[s;\sigma]$, Theorem 4 characterizes, in terms of coefficients of A and B, when A is similar to B in the case where the degree of B is less than 3. We leave as an open problem to characterize, in terms of coefficients of A and B, when A is similar to B in the general case, i.e., with no assumption on the degree of B (the problem is very closely related to the questions asked in [16, p. 64025]).

B. Consimilarity

Definition 3: We say that a polynomial $A \in \mathbb{C}[s;\sigma]$ is consimilar to a polynomial $B \in \mathbb{C}[s;\sigma]$ if there exists a nonzero polynomial $P \in \mathbb{C}[s;\sigma]$ such that $A\overline{P} = PB$.

Theorem 5: For any polynomials $A, B \in \mathbb{C}[s;\sigma]$ the following conditions are equivalent.

(i) A is consimilar to B.
(ii) As is similar to Bs.
(iii) sA is similar to sB.
(iv) sAs is consimilar to sBs.

Proof: Let $P \in \mathbb{C}[s;\sigma]$. Since $\mathbb{C}[s;\sigma]$ is a domain and $\overline{P}s = sP$, the following equivalences hold

$$A\overline{P} = PB \Leftrightarrow A\overline{P}s = PBs \Leftrightarrow (As)P = P(Bs) \quad (28)$$

$$A\overline{P} = PB \Leftrightarrow sA\overline{P} = sPB \Leftrightarrow (sA)\overline{P} = \overline{P}(sB) \quad (29)$$

$$A\overline{P}=PB \Leftrightarrow sA\overline{P}s=sPBs \Leftrightarrow (sAs)P=\overline{P}(sBs) \quad (30)$$

Now it is clear that conditions (i) and (ii) are equivalent by (28), (i) and (iii) are equivalent by (29), and (i) and (iv) are equivalent by (30). ∎

As a consequence of Theorems 3 and 5 we obtain the following corollary.

Corollary 5: The consimilarity of complex-skew polynomials (as defined in Definition 3) is an equivalence relation.

Proof: Obviously, consimilarity is reflexive. To prove that it is symmetric, assume that $A, B \in \mathbb{C}[s;\sigma]$ and A is consimilar to B. Then by Theorem 5, As is similar to Bs, and thus by Theorem 3, Bs is similar to As. Now Theorem 5 implies that B is consimilar to A.

To prove that consimilarity is transitive, assume that $A, B, C \in \mathbb{C}[s;\sigma]$ are such that A is consimilar to B, and B is consimilar to C. Then Theorem 5 implies that As is similar to Bs, and Bs is similar to Cs. Hence it follows from Theorem 3 that As is similar to Cs, and thus by Theorem 5, A is consimilar to C. ∎

By combining Theorem 5 with Corollary 3 we obtain the following characterization of polynomials that are consimilar to a given constant polynomial.

Corollary 6: Let $\beta \in \mathbb{C}$. A polynomial $A \in \mathbb{C}[s;\sigma]$ is consimilar to the polynomial $B = \beta$ if and only if $A = b$ with $|b| = |\beta|$.

The following characterization of polynomials that are consimilar to a given linear polynomial is an immediate consequence of Theorem 5 and Corollary 4.

Corollary 7: Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. A polynomial $A \in \mathbb{C}[s;\sigma]$ is consimilar to the polynomial $B = \alpha s + \beta$ if and only if $A = as + b$ with $a \in \{\alpha, \overline{\alpha}\}$ and $|b| = |\beta|$.

Example 4: (cf. [16, Example 4]) It is an immediate consequence of Corollary 7 that the polynomial $A = (1 - i)s + 2$ is consimilar to the polynomial $B = (1 + i)s + 2i$.

Let Q be the right ring of fractions of the ring $\mathbb{C}[s;\sigma]$ (i.e., Q is the division ring $(\mathbb{C}(s), +, \circledast)$ considered in [16]). We close this subsection with the following remark on the consimilarity relation in the ring Q.

Remark 2: Let Q be the right ring of fractions of the ring $\mathbb{C}[s;\sigma]$. As in [16], elements of Q are called rational fractions. Let us recall that according to [16, Definitions 10, 11], a rational fraction $A \in Q$ is similar (resp. consimilar) to a rational fraction $B \in Q$ if there exists a nonzero rational fraction $P \in Q$ such that $AP = PB$ (resp. $A\overline{P} = PB$). We claim that

$$\overline{P}s = sP \text{ for any } P \in Q. \quad (31)$$

Indeed, let $P = CD^{-1}$, where $C, D \in \mathbb{C}[s;\sigma]$ and $D \neq 0$. Since obviously $\overline{T}s = sT$ for any $T \in \mathbb{C}[s;\sigma]$, we obtain $\overline{D}^{-1}s = sD^{-1}$ (observe that since $D \neq 0$, $\overline{D}^{-1}$ and D^{-1} exist) and $\overline{C}s = sC$, and thus

$$\begin{aligned}\overline{P}s &= (\overline{C}\,\overline{D}^{-1})s = \overline{C}(\overline{D}^{-1}s) = \overline{C}(sD^{-1}) \\ &= (\overline{C}s)D^{-1} = (sC)D^{-1} = s(CD^{-1}) = sP,\end{aligned}$$

which proves (31). Hence the same argument as in the proof of Theorem 5 shows that for any rational fractions $A, B \in Q$ the following conditions are equivalent:

(i) A is consimilar to B.
(ii) As is similar to Bs.
(iii) sA is similar to sB.
(iv) sAs is consimilar to sBs.

This observation shows that the notions of similarity and consimilarity (considered in [16]) are, in a sense, parallel.

C. Some further properties of similarity

In this subsection, similarity is meant in the sense of Definition 1.

As we have already noted, for any polynomial $A \in \mathbb{C}[s;\sigma]$ we have $As = s\overline{A}$ and thus A is always similar to $\overline{A}$. Below we consider another operator (denoted by hat) on complex-skew polynomials which for any polynomial $A \in \mathbb{C}[s;\sigma]$ gives a polynomial similar to A.

Definition 4: For a polynomial $A = \sum_{i=0}^{n} a_i s^i \in \mathbb{C}[s;\sigma]$ we denote

$$\widehat{A} = \sum_{i=0}^{n} (-1)^i a_i s^i,$$

i.e., for every odd (resp. even) i the s^i-coefficients of $\widehat{A}$ and A are opposite in sign (resp. are equal).

For example, if $A = a_3s^3 + a_2s^2 + a_1s + a_0$, then $\widehat{A} = -a_3s^3 + a_2s^2 - a_1s + a_0$.

Parts (a), (b) and (c) of the following lemma show that the "hat" operator is an automorphism of the ring $\mathbb{C}[s;\sigma]$.

Lemma 8: For any polynomials $A, B \in \mathbb{C}[s;\sigma]$ the following equations hold.

(a) $\widehat{\widehat{A}} = A$.
(b) $\widehat{A+B} = \widehat{A} + \widehat{B}$.
(c) $\widehat{AB} = \widehat{A}\widehat{B}$.
(d) $Ai = i\widehat{A}$.

Proof: (a) and (b) are obvious. To prove (c), let $A = \sum_{i=0}^{m} a_i s^i$ and $B = \sum_{j=0}^{n} b_j s^j$. Then using (6) we obtain

$$\begin{aligned}\widehat{A}\widehat{B} &= \sum_{i=0}^{m}(-1)^i a_i s^i \cdot \sum_{j=0}^{n}(-1)^j b_j s^j \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^i a_i \sigma^i\big((-1)^j b_j\big)s^{i+j} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} a_i \sigma^i(b_j) s^{i+j} = \widehat{AB},\end{aligned}$$

which proves (c). Finally, (d) follows from the observation that, by (5), for any nonnegative integer k we have $s^k i = \sigma^k(i)s^k = (-1)^k i s^k$. ■

From Lemma 8 it follows that for any $A \in \mathbb{C}[s;\sigma]$ the polynomial A is similar to $\widehat{A}$. We will use Lemma 8 to prove one more result on similarity of complex-skew polynomials. For that, we need the following observation.

Let $A = \sum_{k=0}^{n} a_k s^k$ be a complex polynomial. Then each a_k, being a complex number, can be uniquely written in the form $a_k = b_k + ic_k$, where b_k and c_k are real numbers. Hence

$$A = \sum_{k=0}^{n}(b_k + ic_k)s^k = \sum_{k=0}^{n} b_k s^k + i\sum_{k=0}^{n} c_k s^k,$$

and thus A can be uniquely written in the form $A = A_1 + iA_2$, where A_1, A_2 are real polynomials (i.e., polynomials with real coefficients). Conversely, if $A_1 = \sum_{k=0}^{n} b_k s^k$ and $A_2 = \sum_{k=0}^{n} c_k s^k$ are real polynomials, then

$$A_1 + iA_2 = \sum_{k=0}^{n}(b_k + ic_k)s^k$$

is a complex polynomial and thus $A_1 + iA_2$ is an element of the ring $\mathbb{C}[s;\sigma]$.

Theorem 6: For any real polynomials A_1, A_2, B_1, B_2 the following conditions are equivalent in the ring $\mathbb{C}[s;\sigma]$.

(i) $A_1 + iA_2$ is similar to $B_1 + iB_2$.
(ii) $B_1 + i\widehat{B_2}$ is similar to $\widehat{A_1} + iA_2$.

Proof: Obviously (i) holds if and only if there exist real polynomials P_1, P_2, not both equal to zero, such that

$$(A_1 + iA_2)(P_1 + iP_2) = (P_1 + iP_2)(B_1 + iB_2). \quad (32)$$

By performing the multiplication on both sides of (32) and applying Lemma 8(d), we can see that (32) is equivalent to

$$\begin{aligned}&A_1P_1 - \widehat{A_2}P_2 + i(\widehat{A_1}P_2 + A_2P_1) \\ &= P_1B_1 - \widehat{P_2}B_2 + i(\widehat{P_1}B_2 + P_2B_1),\end{aligned}$$

i.e., to the system of equations

$$\begin{aligned}A_1P_1 - \widehat{A_2}P_2 &= P_1B_1 - \widehat{P_2}B_2 \\ \widehat{A_1}P_2 + A_2P_1 &= \widehat{P_1}B_2 + P_2B_1.\end{aligned}$$

Since all the polynomials appearing in the second equation of the system are real, they commute in the ring $\mathbb{C}[s;\sigma]$ and thus we can rewrite the second equation in the equivalent form

$$B_1P_2 + B_2\widehat{P_1} = P_2\widehat{A_1} + P_1A_2. \quad (33)$$

Furthermore, by applying the "hat" operator to both sides of the first equation of the system and using Lemma 8 we can see that the first equation is equivalent to the equation

$$-\widehat{B_1}\widehat{P_1} + \widehat{B_2}P_2 = \widehat{P_2}A_2 - \widehat{P_1}\widehat{A_1}. \quad (34)$$

Now it is easy to verify that the system of equations (33) and (34) is equivalent to the equation

$$(B_1 + i\,\widehat{B_2})(P_2 - i\widehat{P_1}) = (P_2 - i\widehat{P_1})(\widehat{A_1} + iA_2). \quad (35)$$

We proved that equations (32) and (35) are equivalent, which implies that conditions (i) and (ii) are equivalent as well. ■

VI. Conclusion

In this paper, we have shown that the polynomial ring $(\mathbb{C}[s], +, \circledast)$ in the framework of conjugate product, considered in [14], [16] and [18], is a special case of the known skew polynomial ring construction. The advantage of this observation is that known general results of the theory of rings can be applied to the concrete ring $(\mathbb{C}[s], +, \circledast)$. In particular, the division ring over conjugate product, which was introduced in [16] as a tool for investigating antilinear systems, can be obtained via the known ring-theoretic Ore construction dating from 1930s. We have also shown that the similarity and consimilarity over conjugate product are parallel notions, in the sense that a complex polynomial (resp. rational fraction) A is consimilar to a complex polynomial (resp. rational fraction) B if and only if As is similar to Bs. We have proved that the similarity and consimilarity over conjugate product are equivalence relations on the set of complex polynomials. Furthermore, we have characterized all the complex polynomials which are similar over conjugate product to a given complex polynomial of degree less than 3, which gives a partial solution to a problem posed in [16]. We believe that this partial solution will help to solve the problem in full generality.

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