Boundary Value Problems for a Coupled System of Hadamard-type Fractional Differential Equations

Suad Y. Al-Mayyahi, Mohammed S. Abdo, Saleh S. Redhwan, and Basim N. Abood,

***Abstract*—In this paper, we derive the equivalent fractional integral system to the nonlinear fractional differential system involving Hadamard fractional derivative subject to coupled boundary conditions. The existence and uniqueness results of solution for proposed system have been obtained. Moreover, we give some sufficient conditions to guarantee that the solutions to such system are Ulam-Hyers stable and Ulam-Hyers-Rassias stable. Our investigations based on the nonlinear analysis and fixed point theorems of Banach and Schaefer. To justify our results, we provide pertinent illustrative examples.**

***Index Terms*—fractional differential system; fixed point theorems; boundary conditions; Ulam-Hyers stability.**

I. Introduction

FRACTIONAL CALCULUS (FC) is one of the branches of mathematics which is a generalization of classical calculus that involves integrals and derivatives of arbitrary (non-integer) order.

FC has an important role in many fields of engineering, science, and economics, FC tools have been found to support significantly in the development of mathematical modeling which is more realistic to applied problems in terms of fractional differential equations (FDEs). Recently, the many versions of the fractional derivatives have been presented, such as Reimann-Liouville, Caputo, Hilfer, Hadamard, and Katugampola, etc. The most used are the Riemann-Liouville and Caputo fractional derivatives. Here we refer to [21], [22], [24], [26], [28], [30], [32].

Initial value problems (IVPs) and boundary value problems (BVPs) for FDEs have won large significance because of their many applications in applied sciences and engineering. Many authors have shown great interest in this topic and a variety of results have been obtained for FDEs involving different types of the conditions and the fractional operators, we refer the reader to previous studies [2], [4], [11], [15], [23], [25], [27], [31], [35], [36], [38], [40], [41] and the references cited therein.

Existence and uniqueness theory for FDEs involving Hadamard-type operator have been studied by several authors, see [1], [5], [10], [14], [17], [29], [33] and references therein.

In order to investigate the different kinds of stability in the Ulam sense for a coupled system of FDEs, we mention the works [3], [6], [7], [8], [18].

Fractional-order BVPs have been broadly concentrated by numerous researchers. Specifically, coupled systems of FDEs have pulled special attention in view of their occurrence in the mathematical modeling of physical phenomena. For some theoretical works on the coupled systems of FDEs, we refer the reader to some studied works [9], [12], [13], [34], [39].

Motivated by the research going on toward this path, in this article, we study existence, uniqueness and Ulam-Hyers (UH), generalized Ulam-Hyers (GUH), Ulam-Hyers-Rassias (UHR), and generalized Ulam-Hyers-Rassias (GUHR) of solutions for a coupled systems of Hadamard FDEs

$$\begin{cases} D_1^{\alpha}u(t) = f_1(t, u(t), v(t)), \ t \in [1, T], \ 0 < \alpha \leq 1, \\ D_1^{\beta}v(t) = f_2(t, u(t), v(t)), \ t \in [1, T], \ 0 < \beta \leq 1, \end{cases} \tag{I.1}$$

with the following coupled boundary conditions:

$$\begin{cases} u(1) = \delta_1 v(T), \\ v(1) = \delta_2 u(T), \end{cases} \tag{I.2}$$

where D_1^{θ} is the Hadamard fractional derivative of order $\theta \in \{\alpha, \beta\}$, $f_1, f_2 : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are appropriate functions, and δ_1, δ_2 are real numbers with $\delta_1\delta_2 \neq 1$.

This paper mainly investigates the existence, uniqueness and Ulam-Hyers (UH), generalized Ulam-Hyers (GUH), Ulam-Hyers-Rassias (UHR), and generalized Ulam-Hyers-Rassias (GUHR) of solutions of fractional coupled system involving Hadamard fractional derivative with coupled boundary conditions. Observe that the problems (I.1) and (I.2), which are covered in this paper, are new and are the first investigation of a fractional coupled system that includes the Hadamard fractional derivative. Moreover, the main results are obtained via fixed-point theorems of Banach and Schaefer.

The plan of the paper is as follows. In Section II, we give some basic definitions and known results related to fractional calculus. Section III is devoted to the existence and uniqueness of system (I.1)-(I.2). Moreover, Ulam-Hyers stability and Ulam-Hyers-Rassias stability have been discussed. Illustrative examples have provided to justify our obtained results in Section IV. At last, the paper is concluded in the last section.

II. Preliminaries

In this section, we recall some essential definitions and lemmas which are used throughout this paper.

Definition 2.1: [26] For a continuous function $u : [1, +\infty) \to \mathbb{R}$. The Hadamard fractional integral of order $\alpha > 0$ is defined by

$$I_1^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}u(s)\frac{ds}{s}.$$

provided the right-hand side is a pointwise defined on $[1, +\infty)$.

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Suad Y. Al-Mayyahi and Basim N. Abood are with the Department of mathematics, college of education of pure science, University of Wasit, Iraq; Emails: Suadluck7@gmail.com, basim.nasih@yahoo.com

Mohammed S. Abdo is with the Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen; Email: msabdo1977@gmail.com

Saleh S. Redhwan is with Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India; Email: saleh.redhwan909@gmail.com.

Definition 2.2: [22] Let $n-1<\alpha<n$, and $f(t)$ has an absolutely continuous derivative up to order $(n-1)$. Then the Caputo-Hadamard fractional derivative of order α is defined as

$$D_1^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t(\log\frac{t}{s})^{n-\alpha-1}\delta^n(u)(s)\frac{ds}{s},$$

where $\delta^n=(t\frac{d}{dt})^n$, and $n=[\alpha]+1$.

Lemma 2.3: [22] Let $\alpha>0$ and $u\in C^n[1,+\infty)$ such that $\delta^{(n)}(u)$ exists almost everywhere on any bounded interval of $[1,+\infty)$. Then we have

$$I_1^{\alpha}\left[D_1^{\alpha}u(t)\right]=u(t)-\sum_{k=0}^{n-1}\frac{\delta^{(k)}u(1)}{\Gamma(k+1)}(\log t)^k,$$

In particular, if $0<\alpha<1$, then we have

$$I_1^{\alpha}\left[D_1^{\alpha}u(t)\right]=u(t)-u(1).$$

Lemma 2.4: [26] For all $\alpha>0$ and $\beta>-1$, we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}(\log s)^{\beta-1}\frac{ds}{s}\\ =\;&\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\log t)^{\alpha+\beta-1}.\end{aligned}$$

Theorem 2.5: [16] (Banach fixed point theorem) Let (U,d) be a nonempty complete metric space with $\mathcal{N}:U\to U$ is a contraction mapping. Then the mapping $\mathcal{N}$ has a fixed point.

Theorem 2.6: [16] (Schaefer's fixed point theorem) Let U be a Banach space and let $T:U\longrightarrow U$ be continuous and compact mapping (completely continuous mapping). Moreover, suppose

$$\mathcal{S}=\{u\in U:u=\lambda Tu,\ \text{for some }\lambda\in(0,1)\}$$

be a bounded set. Then T has at least one fixed point in U.

Lemma 2.7: Let $\phi,h:[1,T]\to\mathbb{R})$ are continuous functions and $\delta_1\delta_2\neq 1$. Then the following linear coupled system

$$\begin{cases}D_1^{\alpha}u(t)=\phi(t), & t\in[1,T],\quad 0<\alpha\le 1,\\ D_1^{\beta}v(t)=h(t), & t\in[1,T],\quad 0<\beta\le 1,\\ u(1)=\delta_1 v(T), & \\ v(1)=\delta_2 u(T), & \end{cases}\tag{II.1}$$

is equivalent to the following integral system

$$\begin{aligned}u(t)&=\frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\&\left.+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\&+\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\\&+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\phi(s)\frac{ds}{s},\end{aligned}\tag{II.2}$$

$$\begin{aligned}v(t)&=\frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\&+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}h(s)\frac{ds}{s}.\end{aligned}\tag{II.3}$$

Proof: In view of Lemma 2.3, the coupled system (II.1) can be expressed by the following integral system

$$u(t)=a_0+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\phi(s)\frac{ds}{s},\tag{II.4}$$

$$v(t)=b_0+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}h(s)\frac{ds}{s},\tag{II.5}$$

$$\begin{aligned}u(1)&=\delta_1 v(T)\\&\Rightarrow\ a_0=\delta_1\left[b_0+\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right],\\ v(1)&=\delta_2 u(T)\\&\Rightarrow\ b_0=\delta_2\left[a_0+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right].\end{aligned}$$

where $a_0,b_0\in\mathbb{R}$. By using the boundary conditions $u(1)=\delta_1 v(T)$, and $v(1)=\delta_2 u(T)$, we obtain

$$\begin{aligned}u(1)&=\delta_1 v(T)\Rightarrow\\ a_0&=\delta_1\left[b_0+\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right],\end{aligned}\tag{II.6}$$

and

$$\begin{aligned}v(1)&=\delta_2 u(T)\Rightarrow\\ b_0&=\delta_2\left[a_0+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right].\end{aligned}\tag{II.7}$$

It follows from (II.6) and (II.7) that

$$\begin{aligned}a_0&=\frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\&\left.+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\&+\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s},\\ b_0&=\frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right].\end{aligned}$$

By substituting the values of a_0 and b_0 into (II.4) and (II.5), we conclude that the integral equations (II.2) and (II.3) are satisfied. The converse follows by direct computation. This completes the proof. ■

III. Main results

Let us present the following space

$$U=\{u(t):u(t)\in C([1,T],\mathbb{R})\}$$

equipped with the norm defined by

$$\|u\|=\sup\{|u(t)|,\ t\in[1,T]\}.$$

It is clear that $(U,\|\cdot\|)$ is a Banach space. Then the product space $(U\times V,\|(u,v)\|)$ is also a Banach space endowed with the norm defined by

$$\|(u,v)\|=\|u\|+\|v\|.$$

According to Lemma 2.7, we consider the operator $\mathcal{N}: U\times V\to U\times V$ defined by

$$\mathcal{N}(u,v)(t)=\begin{pmatrix}\mathcal{N}_1(u,v)(t)\\ \mathcal{N}_2(u,v)(t)\end{pmatrix},$$

where

$$\begin{aligned}&\mathcal{N}_1(u,v)(t)\\ =\ &\frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\ &\left.+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\ &+\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,u(s),v(s))\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}&\mathcal{N}_2(u,v)(t)\\ =\ &\frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\ &+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,u(s),v(s))\frac{ds}{s}.\end{aligned}$$

Sake for brevity, we set

$$W_1=\left[\frac{|\delta_1|\,|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\frac{(\log T)^\alpha}{\Gamma(\alpha+1)},\qquad\text{(III.1)}$$

$$W_2=\frac{|\delta_1|}{|1-\delta_1\delta_2|}[|\delta_1\delta_2|+|\delta_1|]\frac{(\log T)^\beta}{\Gamma(\beta+1)},\qquad\text{(III.2)}$$

$$W_3=\left[\frac{|\delta_2|}{|1-\delta_1\delta_2|}\right]\frac{(\log T)^\alpha}{\Gamma(\alpha+1)},\qquad\text{(III.3)}$$

$$W_4=\left[\frac{|\delta_2|\,|\delta_1|}{|1-\delta_1\delta_2|}+1\right]\frac{(\log T)^\beta}{\Gamma(\beta+1)}.\qquad\text{(III.4)}$$

Now, we prove the existence and uniqueness of solutions of the coupled system (I.1)-(I.2). To this aim, we shall apply the fixed point theorems by Banach and Schaefer. We first need the following hypotheses:

(H_1) $f_i:[1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are continuous functions and there exists $l_i>0$ $(i=1,2)$ such that

$$|f_1(t,u_1,u_2)-f_1(t,v_1,v_2)|\le l_1\sum_{i=1}^2|u_i-v_i|,$$

$$|f_2(t,u_1,u_2)-f_2(t,v_1,v_2)|\le l_2\sum_{i=1}^2|u_i-v_i|,$$

for all $t\in[1,T]$ and $u_i,v_i\in\mathbb{R}$ $i=1,2$.

(H_2) There exist real constants $k_i,\gamma_i\ge 0$ $(i=1,2)$ and $k_0,\gamma_0>0$ such that

$$\begin{aligned}|f_1(t,u_1,u_2)|&\le k_0+k_1|u_1|+k_2|u_2|,\\ |f_2(t,u_1,u_2)|&\le\gamma_0+\gamma_1|u_1|+\gamma_2|u_2|,\end{aligned}$$

for all $t\in[1,T]$ and $u_j\in\mathbb{R}$ $(j=1,2)$.

A. Existence and Uniqueness Theorems

First, we prove the uniqueness theorem based on the Banach fixed point theorem.

Theorem 3.1: Assume that (H_1) holds. If

$$\theta_1:=[(W_1+W_3)\,l_1+(W_2+W_4)\,l_2]<1,\qquad\text{(III.5)}$$

then the coupled system (I.1)-(I.2) has a unique solution, where W_i $(i=1,2,3,4)$ are given by equations (III.1), (III.2), (III.3), and (III.4).

Proof: Define $\sup_{t\in[1,T]}f_i(t,0,0)=P_i<\infty$ for all $(i=1,2)$, and $r>0$ such that $r\ge\frac{\theta_2}{1-\theta_1}$, where $\theta_1<1$ and

$$\theta_2:=(W_1+W_3)\,P_1+(W_2+W_4)\,P_2.$$

At first, we are going to show that $\mathcal{N}\mathcal{B}_r\subset\mathcal{B}_r$, where

$$\mathcal{B}_r=\{(u,v)\in U\times V:\|(u,v)\|<r\}.$$

Let $(u,v)\in\mathcal{B}_r$. Then for $t\in[1,T]$ we have

$$\begin{aligned}&|\mathcal{N}_1(u,v)(t)|\\ \le\ &\frac{\delta_1}{1-\delta_1\delta_2}\Big[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}\\ &\times|f_2(s,u(s),v(s))|\frac{ds}{s}+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\\ &\times|f_1(s,u(s),v(s))|\frac{ds}{s}\Big]+\delta_1\frac{1}{\Gamma(\beta)}\\ &\times\int_1^T(\log\frac{T}{s})^{\beta-1}|f_2(s,u(s),v(s))|\frac{ds}{s}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|f_1(s,u(s),v(s))|\frac{ds}{s}.\qquad\text{(III.6)}\end{aligned}$$

For each $(u,v)\in\mathcal{B}_r$ and $t\in[1,T]$, it follows from the hypothesis (H_1) that

$$\begin{aligned}&|f_1(t,u(t),v(t))|\\ \le\ &|f_1(t,u(t),v(t))-f_1(t,0,0)|+|f_1(t,0,0)|\\ \le\ &l_1(|u(t)|+|u(t)|)+P_1\\ \le\ &l_1r+P_1.\end{aligned}$$

As the same arguments, we get

$$|f_2(t,u(t),v(t))|\le l_2r+P_2.$$

Hence, the inequality (III.6) becomes

$$\begin{aligned}|\mathcal{N}_1(u,v)(t)|\ \le\ &\frac{|\delta_1|\,|\delta_2|\,(l_1r+P_1)}{|1-\delta_1\delta_2|}\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\\ &+\frac{|\delta_1|\,(l_2r+P_2)}{|1-\delta_1\delta_2|}[|\delta_1\delta_2|+|\delta_1|]\frac{(\log T)^\beta}{\Gamma(\beta+1)}\\ &+(l_1r+P_1)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\\ \le\ &(l_1W_1+l_2W_2)r+(P_1W_1+P_2W_2),\end{aligned}$$

which implies

$$\begin{aligned}\|\mathcal{N}_1(u,v)\|\ \le\ &(l_1W_1+l_2W_2)r\\ &+(P_1W_1+P_2W_2).\qquad\text{(III.7)}\end{aligned}$$

Similarly, we get

$$\begin{aligned}|\mathcal{N}_2(u,v)(t)| &\leq \frac{|\delta_1|\,|\delta_2|\,(l_2r+P_2)}{|1-\delta_1\delta_2|}\frac{(\log T)^{\beta}}{\Gamma(\beta+1)} \\ &\quad +\frac{|\delta_2|\,(l_1r+P_1)}{|1-\delta_1\delta_2|}\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad +(l_2r+P_2)\frac{(\log T)^{\beta}}{\Gamma(\beta+1)} \\ &\leq (l_2W_4+l_1W_3)r+(P_2W_4+P_1W_3).\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{N}_2(u,v)\| &\leq (l_2W_4+l_1W_3)r \\ &\quad +(P_2W_4+P_1W_3). \qquad \text{(III.8)}\end{aligned}$$

It follows from (III.7) and (III.8) that

$$\begin{aligned}\|\mathcal{N}(u,v)\| &\leq \|\mathcal{N}_1(u,v)\|+\|\mathcal{N}_2(u,v)\| \\ &\leq [(W_1+W_3)l_1+(W_2+W_4)l_2]\,r \\ &\quad [W_1+W_3]\,P_1+[W_2+W_4]\,P_2 \\ &\leq \theta_1 r+\theta_2=r.\end{aligned}$$

This proves that $\mathcal{N}\mathcal{B}_r\subset\mathcal{B}_r$. Next, we prove that $\mathcal{N}$ is contraction mapping. Indeed, for each $(u_1,v_1),(u_2,v_2)\in U\times V$ and for any $t\in[1,T]$, we have

$$\begin{aligned}&|\mathcal{N}_1(u_2,v_2)(t)-\mathcal{N}_1(u_1,v_1)(t)| \\ \leq\ &\frac{|\delta_1|\,|\delta_2|}{|1-\delta_1\delta_2|}\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\ &\times l_1\left(\|u_2-u_1\|+\|v_2-v_1\|\right) \\ &+\frac{|\delta_1|}{|1-\delta_1\delta_2|}[|\delta_1\delta_2|+|\delta_1|] \\ &\times\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}l_2\left(\|u_2-u_1\|+\|v_2-v_1\|\right) \\ &+\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}l_1\left(\|u_2-u_1\|+\|v_2-v_1\|\right) \\ \leq\ &\left[\frac{|\delta_1|\,|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\ &\times l_1\left(\|u_2-u_1\|+\|v_2-v_1\|\right) \\ &+\frac{|\delta_1|}{|1-\delta_1\delta_2|}[|\delta_1\delta_2|+|\delta_1|]\frac{(\log T)^{\beta}}{\Gamma(\beta+1)} \\ &\times l_2\left(\|u_2-u_1\|+\|v_2-v_1\|\right) \\ \leq\ &(W_1l_1+W_2l_2)\left(\|u_2-u_1\|+\|v_2-v_1\|\right),\end{aligned}$$

which implies

$$\begin{aligned}&\|\mathcal{N}_1(u_2,v_2)-\mathcal{N}_1(u_1,v_1)\| \\ \leq\ &(W_1l_1+W_2l_2) \\ &\times\left(\|u_2-u_1\|+\|v_2-v_1\|\right). \qquad \text{(III.9)}\end{aligned}$$

In a like manner, we get

$$\begin{aligned}&\|\mathcal{N}_2(u_2,v_2)-\mathcal{N}_2(u_1,v_1)\| \\ \leq\ &(W_3l_1+W_4l_2) \\ &\times\left(\|u_2-u_1\|+\|v_2-v_1\|\right). \qquad \text{(III.10)}\end{aligned}$$

From (III.9) and (III.10), we obtain

$$\begin{aligned}&\|\mathcal{N}(u_2,v_2)-\mathcal{N}(u_1,v_1)\| \\ \leq\ &[(W_1+W_3)\,l_1+(W_2+W_4)\,l_2] \\ &\times\left(\|u_2-u_1\|+\|v_2-v_1\|\right).\end{aligned}$$

Since $\theta_1<1$, the operator $\mathcal{N}$ is contraction mapping. So, we conclude that the coupled system (I.1)-(I.2) has a unique solution due to Banach's fixed point theorem. The proof is completed. ■

Next, we apply Schaefer's fixed point theorem to prove the existence theorem.

Theorem 3.2: Assume that (H_2) holds. If

$$[(W_1+W_3)\,k_1+(W_2+W_4)\,\gamma_1]<1,$$

and

$$[(W_1+W_3)\,k_2+(W_2+W_4)\,\gamma_2]<1,$$

then the coupled system (I.1)-(I.2) has at least one solution, where W_i $(i=1,2,3,4)$ are given by equations (III.1), (III.2), (III.3), and (III.4).

Proof: At first, we shall show that $\mathcal{N}:U\times V\to U\times V$ is a completely continuous mapping. Since f_1 and f_2 are continuous functions on $[1,T]$, it is obvious that the operator $\mathcal{N}$ is continuous. Following that, let $\Upsilon\subset U\times V$ be bounded. Then there exist $L_1(>0)$ and $L_2(>0)$ such that

$$|f_1(t,u(t),v(t))|<L_1\quad\forall(t,u,v)\in[1,T]\times\Upsilon,$$

and

$$|f_2(t,u(t),v(t))|<L_2\quad\forall(t,u,v)\in[1,T]\times\Upsilon.$$

Hence, for any $(u,v)\in\Upsilon$ we have

$$\begin{aligned}|\mathcal{N}_1(u,v)(t)| &\leq \left[\frac{|\delta_1|\,|\delta_2|}{|1-\delta_1\delta_2|}+1\right]\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}L_1 \\ &\quad+\frac{|\delta_1|}{|1-\delta_1\delta_2|}[|\delta_1\delta_2|+|\delta_1|] \\ &\quad\times\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}L_2 \\ &\leq W_1L_1+W_2L_2,\end{aligned}$$

which implies

$$\|\mathcal{N}_1(u,v)\|\leq W_1L_1+W_2L_2. \qquad \text{(III.11)}$$

Analogously, we get

$$\|\mathcal{N}_2(u,v)\|\leq W_3L_1+W_4L_2. \qquad \text{(III.12)}$$

It follows from (III.11) and (III.12) that

$$\|\mathcal{N}(u,v)\|\leq[(W_1+W_3)\,L_1+(W_2+W_4)\,L_2]:=\zeta.$$

This proves that $\mathcal{N}$ is uniformly bounded.

Next, we show that a bounded set Υ is mapped into an equicontinuous set of $U\times V$ by $\mathcal{N}$. Let $t_1,t_2\in[1,T]$ such that $t_1<t_2$. Then for any $(u,v)\in\Upsilon$, we have

$$\begin{aligned}&|\mathcal{N}_1(u(t_2),v(t_2))-\mathcal{N}_1(u(t_1),v(t_1))| \\ \leq\ &\frac{1}{\Gamma(\alpha)}\int_1^{t_2}(\log\frac{t_2}{s})^{\alpha-1}\,|f_1(s,u(s),v(s))|\,\frac{ds}{s} \\ &+\frac{1}{\Gamma(\alpha)}\int_1^{t_1}(\log\frac{t_1}{s})^{\alpha-1}\,|f_1(s,u(s),v(s))|\,\frac{ds}{s} \\ \leq\ &L_1\left\{\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{\alpha-1}-(\log\frac{t_1}{s})^{\alpha-1}\right]\frac{ds}{s}\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{\alpha-1}\frac{ds}{s}\right\} \\ \leq\ &\frac{2L_1}{\Gamma(\alpha+1)}\left(\log t_2-\log t_1\right)^{\alpha} \\ &+\frac{L_1}{\Gamma(\alpha+1)}\left(\log t_2\right)^{\alpha}-\left(\log t_1\right)^{\alpha}. \qquad \text{(III.13)}\end{aligned}$$

In a similar manner, we can easily get

$$\begin{aligned} & \left|\mathcal{N}_2(u(t_2),v(t_2))-\mathcal{N}_2(u(t_1),v(t_1)\right| \\ \leq\ & \frac{2L_2}{\Gamma(\beta+1)}\left(\log t_2-\log t_1\right)^{\beta} \\ & +\frac{L_2}{\Gamma(\beta+1)}\left(\log t_2\right)^{\beta}-\left(\log t_1\right)^{\beta}. \end{aligned} \tag{III.14}$$

Since the function $log(t)$ is uniformly continuous on $[1,T]$, the right-hand sides of the inequalities (III.13) and (III.14) tend to zero as $t_2\to t_1$. Therefore, the operator $\mathcal{N}(u,v)$ is equicontinuous. By collecting the above steps along with the Arzela-Ascoli theorem, we deduce that $\mathcal{N}:U\times V\to U\times V$ is completely continuous mapping.

In the end, we shall verify that the set

$$\Omega=\{(u,v)\in U\times\ V:(u,v)=\lambda\mathcal{N}(u,v),0\leq\lambda\leq 1\}$$

is bounded. Indeed, let $(u,v)\in\Omega$ such that $(u,v)=\lambda\mathcal{N}(u,v)$. Then for any $t\in[1,T]$, we have

$$u(t)=\lambda\mathcal{N}_1(u,v)(t),\ \text{and}\ \ v(t)=\lambda\mathcal{N}_2(u,v)(t).$$

Hence

$$\begin{aligned} |u(t)| &< |\mathcal{N}_1(u,v)(t)| \\ &\leq \frac{\delta_1}{1-\delta_1\delta_2}\Big[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1} \\ &\quad\times|f_2(s,u(s),v(s))|\frac{ds}{s}+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1} \\ &\quad\times|f_1(s,u(s),v(s))|\frac{ds}{s}\Big]+\delta_1\frac{1}{\Gamma(\beta)} \\ &\quad\times\int_1^T(\log\frac{T}{s})^{\beta-1}|f_2(s,u(s),v(s))|\frac{ds}{s} \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}|f_1(s,u(s),v(s))|\frac{ds}{s}. \end{aligned}$$

From the assumption (H_2), we obtain

$$\begin{aligned} |u(t)| &\leq W_1\left(k_0+k_1|u|+k_2|v|\right) \\ &\quad+W_2(\gamma_0+\gamma_1|u|+\gamma_2|v|) \\ &= W_1k_0+W_2\gamma_0+\left(W_1k_1+W_2\gamma_1\right)|u| \\ &\quad+(W_1k_2+W_2\gamma_2)|v|, \end{aligned}$$

By the same approach, we have

$$\begin{aligned} |v(t)| &\leq W_3\left(k_0+k_1|u|+k_2|v|\right) \\ &\quad+W_4(\gamma_0+\gamma_1|u|+\gamma_2|v|) \\ &\leq W_3k_0+W_4\gamma_0+\left(W_3k_1+W_4\gamma_1\right)|u| \\ &\quad+(W_3k_2+W_4\gamma_2)|v|, \end{aligned}$$

one can obtain

$$\begin{aligned} \|u\| &\leq W_1k_0+W_2\gamma_0+\left(W_1k_1+W_2\gamma_1\right)\|u\| \\ &\quad+(W_1k_2+W_2\gamma_2)\|v\|, \end{aligned}$$

and

$$\begin{aligned} \|v\| &\leq W_3k_0+W_4\gamma_0+\left(W_3k_1+W_4\gamma_1\right)\|u\| \\ &\quad+(W_3k_2+W_4\gamma_2)\|v\|. \end{aligned}$$

By a combination of the above norms, we can obtain

$$\begin{aligned} \|u\|+\|v\| &\leq (W_1+W_3)\,k_0+(W_2+W_4)\,\gamma_0 \\ &\quad+[(W_1+W_3)k_1+(W_2+W_4)\gamma_1]\,\|u\| \\ &\quad+[(W_1+W_3)k_2+(W_2+W_4)\gamma_2]\,\|v\|, \end{aligned}$$

which implies

$$\|u+v\|\leq\frac{(W_1+W_3)\,k_0+(W_2+W_4)\,\gamma_0}{W_0},$$

where

$$\begin{aligned} W_0 &= \min\{1-(W_1+W_3)k_1+(W_2+W_4)\gamma_1 \\ &\qquad,1-(W_1+W_3)k_2+(W_2+W_4)\gamma_2\}. \end{aligned}$$

Hence, the set Ω is bounded. As a consequence of Theorem (2.6), we conclude that $\mathcal{N}$ has at least one fixed point. This confirms exists at least one solution of the coupled system (I.1)-(I.2). The proof is completed. ■

B. Ulam-Hyers Stability

In this part, we mainly investigate the Ulam-Hyers (UH) stability and generalized Ulam-Hyers (GUH) stability of the coupled system (I.1)-(I.2).

Definition 3.3: Coupled system (I.1)-(I.2) is UH stable if there exists a real number $\lambda=(\lambda_{f_1},\lambda_{f_2})>0$ with the following property:

For some $\epsilon=(\epsilon_1,\epsilon_2)>0$, and each $(\widetilde{u},\widetilde{v})\in U\times V$, if

$$|D_1^{\alpha}\widetilde{u}(t)-f_1(t,\widetilde{u}(t),\widetilde{v}(t))|\leq\epsilon_1, \tag{III.15}$$

$$\left|D_1^{\beta}\widetilde{v}(t)-f_2(t,\widetilde{u}(t),\widetilde{v}(t))\right|\leq\epsilon_2, \tag{III.16}$$

then there exists $(u,v)\in U\times V$ satisfying the coupled system (I.1) with the following coupled boundary conditions:

$$\begin{cases} u(1)=\widetilde{u}(1), & v(T)=\widetilde{v}(T), \\ v(1)=\widetilde{v}(1), & u(T)=\widetilde{u}(T), \end{cases} \tag{III.17}$$

such that

$$|(\widetilde{u},\widetilde{v})\,(t)-(u,v)(t)|\leq\lambda\epsilon,\quad t\in[1,T].$$

Definition 3.4: Coupled system (I.1)-(I.2) is GUH stable if there exists $\varphi=\max(\varphi_{f_1},\varphi_{f_2})\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\varphi(0)=0$ such that for some $\epsilon=\max(\epsilon_1,\epsilon_2)>0$ and for each solution $(\widetilde{u},\widetilde{v})\in U\times V$ satisfying the inequalities (III.15)and (III.16) there exists a solution $(u,v)\in U\times V$ of the coupled system (I.1)-(III.17) with

$$|(\widetilde{u},\widetilde{v})\,(t)-(u,v)(t)|\leq\varphi(\epsilon),\quad t\in[1,T].$$

Remark 3.5: A function $(\widetilde{u},\widetilde{v})\in U\times V$ be the solution of the inequalities (III.15)-(III.16) if and only if there exists a function $(h_1,h_2)\in U\times V$ (where h_1 depends on solution $\widetilde{u}$ and h_2 depends on solution $\widetilde{v}$) such that

(i) $|h_1(t)|\leq\epsilon_1$ and $|h_2(t)|\leq\epsilon_2$ for all $t\in[1,T]$.

(ii) For all $t\in[1,T]$

$$\begin{cases} D_1^{\alpha}\widetilde{u}(t)=f_1(t,\widetilde{u}(t),\widetilde{v}(t))+h_1(t), \\ D_1^{\alpha}\widetilde{v}(t)=f_2(t,\widetilde{u}(t),\widetilde{v}(t))+h_2(t). \end{cases}$$

Lemma 3.6: Let $(\widetilde{u},\widetilde{v})\in U\times V$ is the solution of the inequalities (III.15)-(III.16). Then $(\widetilde{u},\widetilde{v})\in U\times V$ is the solution of the following integral inequalities:

$$\begin{aligned} & \left|\widetilde{u}(t)-Z_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right| \\ \leq\ & \Big[\left(\frac{\delta_1^2\delta_2}{1-\delta_1\delta_2}+\delta_1\right)\frac{(\log T)^{\beta}}{\Gamma(\beta+1)}+\delta_2\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big]\epsilon_1, \end{aligned}$$

and

$$\begin{aligned}&\left|\widetilde{v}(t)-Z_{\widetilde{v}}-\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right|\\ \leq\ &\left(\frac{\delta_2\delta_1}{1-\delta_1\delta_2}\frac{(\log T)^\beta}{\Gamma(\beta+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\epsilon_2,\end{aligned}$$

where

$$\begin{aligned}Z_{\widetilde{u}}\ :\ =&\ \frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\ &\left.+\delta_2\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right]\\ &+\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\end{aligned}$$

and

$$\begin{aligned}Z_{\widetilde{v}}\ :\ =&\ \frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\frac{1}{\Gamma(\beta)}\int_1^T(\log\frac{T}{s})^{\beta-1}h(s)\frac{ds}{s}\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi(s)\frac{ds}{s}\right].\end{aligned}$$

Proof: In view of Remark 3.5, we get

$$\begin{aligned}\widetilde{u}(t)\ =&\ Z_{\widetilde{u}}+\frac{\delta_1}{1-\delta_1\delta_2}\Big[\delta_1\delta_2\frac{1}{\Gamma(\beta)}\\ &\times\int_1^T(\log\frac{T}{s})^{\beta-1}h_1(s)\frac{ds}{s}+\delta_2\frac{1}{\Gamma(\alpha)}\\ &\int_1^T(\log\frac{T}{s})^{\alpha-1}h_1(s)\frac{ds}{s}\Big]+\delta_1\frac{1}{\Gamma(\beta)}\\ &\times\int_1^T(\log\frac{T}{s})^{\beta-1}h_1(s)\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\\ &\times\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}h_1(s)\frac{ds}{s}\end{aligned}\tag{III.18}$$

and

$$\begin{aligned}\widetilde{v}(t)\ =&\ Z_{\widetilde{v}}+\frac{\delta_2}{1-\delta_1\delta_2}\Big[\delta_1\frac{1}{\Gamma(\beta)}\\ &\times\int_1^T(\log\frac{T}{s})^{\beta-1}h_2(s)\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\\ &\times\int_1^T(\log\frac{T}{s})^{\alpha-1}h_2(s)\frac{ds}{s}\Big]+\frac{1}{\Gamma(\beta)}\\ &\times\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\\ &+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}h_2(s)\frac{ds}{s}\end{aligned}\tag{III.19}$$

By the inequalities (III.18), (III.19) and using Lemma 2.7, we obtain

$$\begin{aligned}&\left|\widetilde{u}(t)-Z_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right|\\ \leq\ &\Big[\left(\frac{\delta_1^2\delta_2}{1-\delta_1\delta_2}+\delta_1\right)\frac{(\log T)^\beta}{\Gamma(\beta+1)}+(\delta_2+1)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big]\epsilon_1,\end{aligned}$$

and

$$\begin{aligned}&\left|\widetilde{v}(t)-Z_{\widetilde{v}}-\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right|\\ \leq\ &\left[\left(\frac{\delta_2\delta_1}{1-\delta_1\delta_2}+1\right)\frac{(\log T)^\beta}{\Gamma(\beta+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\epsilon_2.\end{aligned}$$

■

Theorem 3.7: Under the assumptions of Theorem 3.1. If

$$(1-\mathcal{C}_\alpha)(1-\mathcal{C}_\beta)-\mathcal{C}_\alpha\mathcal{C}_\beta\neq0,$$

then the coupled system (I.1)-(I.2) will be UH and GUH stable in $U\times V$, where $\mathcal{C}_\alpha:=l_1\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}$, and $\mathcal{C}_\beta:=l_2\frac{(\log T)^\beta}{\Gamma(\beta+1)}$.

Proof: Let $(\widetilde{u},\widetilde{v})\in U\times V$ be the solution of the following coupled system

$$\begin{cases}D_1^\alpha\widetilde{u}(t)=f_1(t,\widetilde{u}(t),\widetilde{v}(t))+h_1(t),\\ D_1^\alpha\widetilde{v}(t)=f_2(t,\widetilde{u}(t),\widetilde{v}(t))+h_2(t),\\ \widetilde{u}(1)=\delta_1\widetilde{v}(T),\\ \widetilde{v}(1)=\delta_2\widetilde{u}(T),\end{cases}\tag{III.20}$$

and the function $(u,v)\in U\times V$ is a unique solution of the coupled system (I.1)-(I.2). Therefore, by using Lemma 2.7 we have

$$u(t)=Z_u+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,u(s),v(s))\frac{ds}{s}$$

and

$$v(t)=Z_v+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,u(s),v(s))\frac{ds}{s}.$$

From the boundary conditions (III.17), we get $Z_{\widetilde{u}}=Z_u$ and $Z_{\widetilde{v}}=Z_v$. Hence

$$u(t)=Z_{\widetilde{u}}+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,u(s),v(s))\frac{ds}{s}$$

and

$$v(t)=Z_{\widetilde{v}}+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,u(s),v(s))\frac{ds}{s}.$$

In view of Lemma 3.6, we have

$$\begin{aligned}&\left|\widetilde{u}(t)-Z_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right|\\ \leq\ &\mathcal{A}\epsilon_1,\end{aligned}\tag{III.21}$$

$$\begin{aligned}&\left|\widetilde{v}(t)-Z_{\widetilde{v}}-\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}f_2(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right|\\ \leq\ &\mathcal{B}\epsilon_2,\end{aligned}\tag{III.22}$$

where

$$\begin{aligned}\mathcal{A}\ :\ =&\left(\frac{\delta_1^2\delta_2}{1-\delta_1\delta_2}+\delta_1\right)\frac{(\log T)^\beta}{\Gamma(\beta+1)}\\ &+(\delta_2+1)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\end{aligned}$$

$$\mathcal{B}:=\left[\left(\frac{\delta_2\delta_1}{1-\delta_1\delta_2}+1\right)\frac{(\log T)^\beta}{\Gamma(\beta+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].$$

Thus by the assumption (H_1) and the inequalities (III.21), (III.22), we get

$$\begin{aligned}
&|\widetilde{u}(t)-u(t)| \\
= &\left|\widetilde{u}(t)-Z_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,u(s),v(s))\frac{ds}{s}\right| \\
\le &\left|\widetilde{u}(t)-Z_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}f_1(s,\widetilde{u}(s),\widetilde{v}(s))\frac{ds}{s}\right| \\
&+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1} \\
&\times|f_1(s,\widetilde{u}(s),\widetilde{v}(s))-f_1(s,u(s),v(s))|\,ds \\
\le &\mathcal{A}\epsilon_1+\frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1} \\
&\times l_1\left(|\widetilde{u}(s)-u(s)|+|\widetilde{v}(s)-v(s)|\right)ds,
\end{aligned}$$

which implies

$$(1-\mathcal{C}_\alpha)\left\|\widetilde{u}-u\right\|_U\le\mathcal{A}\epsilon_1+\mathcal{C}_\alpha\left\|\widetilde{v}-v\right\|_V. \tag{III.23}$$

Similarly, we obtain

$$(1-\mathcal{C}_\beta)\left\|\widetilde{v}-v\right\|_V\le\mathcal{B}\epsilon_2+\mathcal{C}_\beta\left\|\widetilde{u}-u\right\|_U. \tag{III.24}$$

The inequalities (III.23) and (III.24) can be written as

$$\begin{cases}(1-\mathcal{C}_\alpha)\left\|\widetilde{u}-u\right\|_U-\mathcal{C}_\alpha\left\|\widetilde{v}-v\right\|_V\le\mathcal{A}\epsilon_1,\\ (1-\mathcal{C}_\beta)\left\|\widetilde{v}-v\right\|_V-\mathcal{C}_\beta\left\|\widetilde{u}-u\right\|_U\le\mathcal{B}\epsilon_2.\end{cases} \tag{III.25}$$

Now, we will represent the relations in (III.25) as matrices as follows

$$\begin{aligned}
&\begin{pmatrix}(1-\mathcal{C}_\alpha) & -\mathcal{C}_\alpha\\ -\mathcal{C}_\beta & (1-\mathcal{C}_\beta)\end{pmatrix}\begin{pmatrix}\left\|\widetilde{u}-u\right\|_U\\ \left\|\widetilde{v}-v\right\|_V\end{pmatrix}\\
\le &\begin{pmatrix}\mathcal{A}\epsilon_1\\ \mathcal{B}\epsilon_2\end{pmatrix}.
\end{aligned}$$

After simple calculations of the above inequality, we can write

$$\begin{aligned}
\begin{pmatrix}\left\|\widetilde{u}-u\right\|_U\\ \left\|\widetilde{v}-v\right\|_V\end{pmatrix} \le &\begin{pmatrix}\frac{(1-\mathcal{C}_\alpha)}{\Delta} & \frac{\mathcal{C}_\alpha}{\Delta}\\ \frac{\mathcal{C}_\beta}{\Delta} & \frac{(1-\mathcal{C}_\beta)}{\Delta}\end{pmatrix}\\
&\times\begin{pmatrix}\mathcal{A}\epsilon_1\\ \mathcal{B}\epsilon_2\end{pmatrix}
\end{aligned}$$

where $\Delta=(1-\mathcal{C}_\alpha)(1-\mathcal{C}_\beta)-\mathcal{C}_\alpha\mathcal{C}_\beta\neq 0$. This leads to

$$\begin{aligned}
\left\|\widetilde{u}-u\right\|_U \le &\frac{(1-\mathcal{C}_\alpha)\mathcal{A}}{\Delta}\epsilon_1\\
&+\frac{\mathcal{C}_\alpha\mathcal{B}}{\Delta}\epsilon_2
\end{aligned}$$

and

$$\begin{aligned}
\left\|\widetilde{v}-v\right\|_V \le &\frac{\mathcal{C}_\beta\mathcal{A}}{\Delta}\epsilon_1\\
&+\frac{(1-\mathcal{C}_\beta)\mathcal{B}}{\Delta}\epsilon_2.
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
&\left\|\widetilde{u}-u\right\|_U+\left\|\widetilde{v}-v\right\|_V\\
\le &\left(\frac{(1-\mathcal{C}_\alpha)\mathcal{A}}{\Delta}+\frac{\mathcal{C}_\beta\mathcal{A}}{\Delta}\right)\epsilon_1\\
&+\left(\frac{\mathcal{C}_\alpha\mathcal{B}}{\Delta}+\frac{(1-\mathcal{C}_\beta)\mathcal{B}}{\Delta}\right)\epsilon_2.
\end{aligned}$$

For $\epsilon=\max\{\epsilon_1,\epsilon_2\}$ and

$$\begin{aligned}
\lambda = &\frac{(1-\mathcal{C}_\alpha)\mathcal{A}}{\Delta}+\frac{\mathcal{C}_\beta\mathcal{A}}{\Delta}\\
&+\frac{\mathcal{C}_\alpha\mathcal{B}}{\Delta}+\frac{(1-\mathcal{C}_\beta)\mathcal{B}}{\Delta}
\end{aligned}$$

we obtain

$$\|(\widetilde{u},\widetilde{v})-(u,v)\|_{U\times V}\le\lambda\epsilon. \tag{III.26}$$

This proves that the coupled system (I.1)-(I.2) is UH stable.

Moreover, the inequality (III.26) can be written as

$$\|(\widetilde{u},\widetilde{v})-(u,v)\|_{U\times V}\le\varphi(\epsilon),$$

where $\varphi(\epsilon)=\lambda\epsilon$ with $\varphi(0)=0$. This shows that the coupled system (I.1)-(I.2) is GUH stable. ■

C. Ulam-Hyers-Rassias Stability

Here, we discuss the two types of stability results namely, Ulam-Hyers-Rassias (UHR) stability, and generalized Ulam-Hyers-Rassias (GUHR) stability of the coupled system (I.1)-(I.2).

Definition 3.8: Coupled system (I.1)-(I.2) is UHR stable with respect to $\sigma\in C([1,T],\mathbb{R}^+)$ with $\sigma=\max(\sigma_1,\sigma_2)$ if there exists a $\lambda_\sigma=\max\left(\lambda_{\sigma_1},\lambda_{\sigma_2}\right)>0$ such that for each $\epsilon=\max\left(\epsilon_1,\epsilon_2\right)>0$, and for each solution $(\widetilde{u},\widetilde{v})\in U\times V$ of the inequalities

$$|D_1^\alpha\widetilde{u}(t)-f_1(t,\widetilde{u}(t),\widetilde{v}(t))|\le\epsilon_1\sigma_1(t),\quad t\in[1,T], \tag{III.27}$$

$$\left|D_1^\beta\widetilde{v}(t)-f_2(t,\widetilde{u}(t),\widetilde{v}(t))\right|\le\epsilon_2\sigma_2(t),\quad t\in[1,T], \tag{III.28}$$

there exists $(u,v)\in U\times V$ satisfying the coupled system (I.1) with

$$|(\widetilde{u},\widetilde{v})(t)-(u,v)(t)|\le\lambda_\sigma\epsilon\sigma(t),\quad t\in[1,T].$$

Definition 3.9: Coupled system (I.1)-(I.2) is UHR stable with respect to $\sigma\in C([1,T],\mathbb{R}^+)$ with $\sigma=\max(\sigma_1,\sigma_2)$ if there exists a $\lambda_\sigma=\max\left(\lambda_{\sigma_1},\lambda_{\sigma_2}\right)>0$ such that for each solution $(\widetilde{u},\widetilde{v})\in U\times V$ of the inequalities

$$|D_1^\alpha\widetilde{u}(t)-f_1(t,\widetilde{u}(t),\widetilde{v}(t))|\le\sigma_1(t),\quad t\in[1,T],$$

$$\left|D_1^\beta\widetilde{v}(t)-f_2(t,\widetilde{u}(t),\widetilde{v}(t))\right|\le\sigma_2(t),\quad t\in[1,T],$$

there exists $(u,v)\in U\times V$ satisfying the coupled system (I.1) with

$$|(\widetilde{u},\widetilde{v})(t)-(u,v)(t)|\le\lambda_\sigma\sigma(t),\quad t\in[1,T].$$

Remark 3.10: A function $(\widetilde{u},\widetilde{v})\in U\times V$ be the solution of the inequalities (III.27)-(III.28) if and only if there exists a function $(h_1,h_2),\sigma=\max(\sigma_1,\sigma_2)\in U\times V$ (where h_1 depends on solution $\widetilde{u}$ and h_2 depends on solution $\widetilde{v}$) such that

(i) $|h_1(t)|\le\epsilon_1\sigma_1(t)$ and $|h_2(t)|\le\epsilon_2\sigma_2(t)$ for all $t\in[1,T]$.

(ii) For all $t\in[1,T]$

$$\begin{cases}D_1^\alpha\widetilde{u}(t)=f_1(t,\widetilde{u}(t),\widetilde{v}(t))+h_1(t),\\ D_1^\alpha\widetilde{v}(t)=f_2(t,\widetilde{u}(t),\widetilde{v}(t))+h_2(t).\end{cases}$$

Theorem 3.11: Under the assumptions of Theorem 3.1. In addition, the following condition is satisfied:

(H_3) There exist $\sigma : [1,T] \to \mathbb{R}^+$ be a nondecreasing continuous function and $\lambda_\sigma > 0$ such that

$$I_1^\mu \sigma(t) \le \lambda_\sigma \sigma(t), \text{ where } \mu \in \{\alpha, \beta\}.$$

Then, the coupled system (I.1)-(I.2) is UHR stable and GUHR stable in $U \times V$.

Proof: Let $\epsilon = \max(\epsilon_1, \epsilon_2) > 0$ and $(\widetilde{u}, \widetilde{v}) \in U \times V$ be any solution of the coupled system (III.20). Taking $(u, v) \in U \times V$ as any solution of the coupled system (I.1)-(III.17). Therefore, by using Lemma 2.7 we have

$$u(t) = Z_{\widetilde{u}} + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} f_1(s, u(s), v(s)) \frac{ds}{s}$$

and

$$v(t) = Z_{\widetilde{v}} + \frac{1}{\Gamma(\beta)} \int_1^t (\log \frac{t}{s})^{\beta-1} f_2(s, u(s), v(s)) \frac{ds}{s}.$$

In view of Remark 3.10, we get same equations (III.18) and (III.19). It follows from integration of inequalities (III.27) and (III.28) with help of Remark (3.10) that

$$\begin{aligned} &\left|\widetilde{u}(t) - Z_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} f_1(s, \widetilde{u}(s), \widetilde{v}(s)) \frac{ds}{s}\right| \\ \le\; & \left(\frac{\delta_1^2\delta_2}{1-\delta_1\delta_2} + \delta_1 + \frac{\delta_1\delta_2}{1-\delta_1\delta_2} + 1\right) \epsilon_1 \lambda_{\sigma_1} \sigma_1(t) \\ =\; & \mathcal{A}^* \epsilon_1 \lambda_{\sigma_1} \sigma_1(t), \end{aligned} \tag{III.29}$$

and

$$\begin{aligned} &\left|\widetilde{v}(t) - Z_{\widetilde{v}} - \frac{1}{\Gamma(\beta)} \int_1^t (\log \frac{t}{s})^{\beta-1} f_2(s, \widetilde{u}(s), \widetilde{v}(s)) \frac{ds}{s}\right| \\ \le\; & \left(\frac{\delta_2\delta_1}{1-\delta_1\delta_2} + \frac{\delta_2}{1-\delta_1\delta_2} + 1\right) \epsilon_2 \lambda_{\sigma_2} \sigma_2(t) \\ =\; & \mathcal{B}^* \epsilon_2 \lambda_{\sigma_2} \sigma_2(t), \end{aligned} \tag{III.30}$$

where

$$\mathcal{A}^* = \frac{\delta_1^2\delta_2}{1-\delta_1\delta_2} + \delta_1 + \frac{\delta_1\delta_2}{1-\delta_1\delta_2} + 1,$$

and

$$\mathcal{B}^* = \frac{\delta_2\delta_1}{1-\delta_1\delta_2} + \frac{\delta_2}{1-\delta_1\delta_2} + 1.$$

Assumption (H_1) and inequalities (III.29), (III.30) give

$$\begin{aligned} &|\widetilde{u}(t) - u(t)| \\ =\; & \left|\widetilde{u}(t) - Z_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} f_1(s, u(s), v(s)) \frac{ds}{s}\right| \\ \le\; & \left|\widetilde{u}(t) - Z_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} f_1(s, \widetilde{u}(s), \widetilde{v}(s)) \frac{ds}{s}\right| \\ & + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \\ & \times |f_1(s, \widetilde{u}(s), \widetilde{v}(s)) - f_1(s, u(s), v(s))| \, ds \\ \le\; & \mathcal{A}^* \epsilon_1 \lambda_{\sigma_1} \sigma_1(t) + \frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \\ & \times l_1 \left(|\widetilde{u}(s) - u(s)| + |\widetilde{v}(s) - v(s)|\right) ds, \end{aligned}$$

which implies

$$(1 - \mathcal{C}_\alpha) \|\widetilde{u} - u\|_U \le \mathcal{A}^* \epsilon_1 \lambda_{\sigma_1} \sigma_1(t) + \mathcal{C}_\alpha \|\widetilde{v} - v\|_V . \tag{III.31}$$

Similarly, we obtain

$$(1 - \mathcal{C}_\beta) \|\widetilde{v} - v\|_V \le \mathcal{B}^* \epsilon_2 \lambda_{\sigma_2} \sigma_2(t) + \mathcal{C}_\beta \|\widetilde{u} - u\|_U . \tag{III.32}$$

After simple calculations of the inequalities (III.31) and (III.32), we can write

$$\begin{pmatrix} \|\widetilde{u} - u\|_U \\ \|\widetilde{v} - v\|_V \end{pmatrix} \le \begin{pmatrix} \frac{(1-\mathcal{C}_\alpha)}{\Delta} & \frac{\mathcal{C}_\alpha}{\Delta} \\ \frac{\mathcal{C}_\beta}{\Delta} & \frac{(1-\mathcal{C}_\beta)}{\Delta} \end{pmatrix} \times \begin{pmatrix} \mathcal{A}^* \epsilon_1 \lambda_{\sigma_1} \sigma_1(t) \\ \mathcal{B}^* \epsilon_2 \lambda_{\sigma_2} \sigma_2(t) \end{pmatrix}$$

where $\Delta = (1 - \mathcal{C}_\alpha)(1 - \mathcal{C}_\beta) - \mathcal{C}_\alpha \mathcal{C}_\beta \neq 0$. It follows that

$$\|\widetilde{u} - u\|_U \le \frac{(1-\mathcal{C}_\alpha)\mathcal{A}^*\lambda_{\sigma_1}}{\Delta} \epsilon_1 \sigma_1(t) + \frac{\mathcal{C}_\alpha \mathcal{B}^* \lambda_{\sigma_2}}{\Delta} \epsilon_2 \sigma_2(t)$$

and

$$\|\widetilde{v} - v\|_V \le \frac{\mathcal{C}_\beta \mathcal{A}^* \lambda_{\sigma_1}}{\Delta} \epsilon_1 \sigma_1(t) + \frac{(1-\mathcal{C}_\beta)\mathcal{B}^*\lambda_{\sigma_2}}{\Delta} \epsilon_2 \sigma_2(t).$$

For $\epsilon = \max(\epsilon_1, \epsilon_2)$ and $\sigma(t) = \max(\sigma_1, \sigma_2)$ with the above inequalities, we get

$$\begin{aligned} &\|\widetilde{u} - u\|_U + \|\widetilde{v} - v\|_V \\ \le\; & \left(\frac{(1-\mathcal{C}_\alpha)\mathcal{A}^*\lambda_{\sigma_1}}{\Delta} + \frac{\mathcal{C}_\beta \mathcal{A}^* \lambda_{\sigma_2}}{\Delta} \right. \\ & \left. + \frac{\mathcal{C}_\alpha \mathcal{B}^* \lambda_{\sigma_1}}{\Delta} + \frac{(1-\mathcal{C}_\beta)\mathcal{B}^*\lambda_{\sigma_2}}{\Delta}\right) \epsilon \sigma(t). \end{aligned}$$

For

$$\lambda_\sigma = \frac{(1-\mathcal{C}_\alpha)\mathcal{A}^*\lambda_{\sigma_1}}{\Delta} + \frac{\mathcal{C}_\beta \mathcal{A}^* \lambda_{\sigma_2}}{\Delta} + \frac{\mathcal{C}_\alpha \mathcal{B}^* \lambda_{\sigma_1}}{\Delta} + \frac{(1-\mathcal{C}_\beta)\mathcal{B}^*\lambda_{\sigma_2}}{\Delta},$$

we obtain

$$\|(\widetilde{u}, \widetilde{v}) - (u, v)\|_{U \times V} \le \lambda_\sigma \epsilon \sigma(t). \tag{III.33}$$

This proves that the coupled system (I.1)-(I.2) is UHR stable. Moreover, the inequality (III.33) can be written as

$$\|(\widetilde{u}, \widetilde{v}) - (u, v)\|_{U \times V} \le \varphi(\epsilon) \sigma(t).$$

where $\varphi(\epsilon) = \lambda_\sigma \epsilon$ with $\varphi(0) = 0$. This shows that the coupled system (I.1)-(I.2) is GUHR stable. ∎

IV. Examples

Here, we present two pertinent examples to illustrating the effectiveness of the acquired results.

Example 4.1: Let $\alpha = \beta = \frac{1}{2}$, $\delta_1 = \frac{4}{5}$, $\delta_2 = \frac{1}{2}$,

$$f_1(t, u, v) = \frac{1}{2(t+2)^2} \frac{|u(t)|}{1 + |u(t)|} + 1 + \frac{1}{16} \sin^2(v(t)) + \frac{1}{\sqrt{t^2+1}},$$

and

$$f_2(t, u, v) = \frac{1}{16\pi} \sin(2\pi u(t)) + \frac{|v(t)|}{8(1 + |v(t)|)} + \frac{1}{2}.$$

Consider the coupled system

$$\begin{cases} D_1^{\frac{1}{2}} u(t) = f_1(t, u, v), & t \in [1, e], \\ D_1^{\frac{1}{2}} v(t) = f_2(t, u, v), & t \in [1, e], \end{cases} \tag{IV.1}$$

with the coupled boundary conditions

$$\begin{array}{l} u(1) = \frac{4}{5}v(e), \\ v(1) = \frac{1}{2}u(e). \end{array} \tag{IV.2}$$

For $t \in [1,e]$ and $u, u^*, v, v^* \in \mathbb{R}^+$ we have

$$|f_1(t,u,u^*) - f_1(t,v,v^*)| \le \frac{1}{8}|u-v| + \frac{1}{8}|u^*-v^*|,$$

and

$$|f_2(t,u,u^*) - f_2(t,v,v^*)| \le \frac{1}{8}|u-v| + \frac{1}{8}|u^*-v^*|.$$

Hence the assumption (H_1) holds with $l_1 = l_2 = \frac{1}{8}$. Also we have $\sup_{t\in[1,e]} f_1(t,0,0) = 1 + \frac{1}{\sqrt{2}} = P_1 < \infty$ and $\sup_{t\in[1,e]} f_2(t,0,0) = \frac{1}{2} = P_2 < \infty$. Now, we shall check that condition (III.5) holds. Indeed, by some simple calculations we find that $W_1 = \frac{10}{3\sqrt{\pi}}$, $W_2 = \frac{52}{15\sqrt{\pi}}$, $W_3 = \frac{5}{3\sqrt{\pi}}$ and $W_4 = \frac{10}{3\sqrt{\pi}}$. So, with the given data, we see that $\theta_1 = \frac{59}{40\sqrt{\pi}} < 1$. Therefore, by Theorem 3.1, we conclude that the coupled system (IV.1)-(IV.2) has a unique solution on $[1,e]$. Moreover, for any $(\widetilde{u},\widetilde{v}) \in U \times V$ of the inequalities

$$\begin{aligned} &\left| D_1^{\alpha}\widetilde{u}(t) - \left(\frac{1}{2(t+2)^2}\frac{|\widetilde{u}(t)|}{1+|\widetilde{u}(t)|} + 1 \right.\right. \\ &\left.\left. + \frac{1}{16}\sin^2(\widetilde{v}(t)) + \frac{1}{\sqrt{t^2+1}} \right) \right| \\ \le\ & \epsilon_1,, \quad t \in [1,e], \end{aligned}$$

and

$$\begin{aligned} &\left| D_1^{\beta}\widetilde{v}(t) - \left(\frac{1}{16\pi}\sin(2\pi\widetilde{u}(t)) \right.\right. \\ &\left.\left. + \frac{|\widetilde{v}(t)|}{8(1+|\widetilde{v}(t)|)} + \frac{1}{2} \right) \right| \\ \le\ & \epsilon_2,, \quad t \in [1,e], \end{aligned}$$

then there exists a unique solution $(u,v) \in U \times V$ coupled system (IV.1)-(IV.2) such that

$$\|(\widetilde{u},\widetilde{v}) - (u,v)\|_{U\times V} \le \lambda\epsilon,$$

where $\lambda = 8.6447 > 0$ with $\mathcal{C}_\alpha = \mathcal{C}_\beta = \frac{1}{4\sqrt{\pi}}$, $\Delta = (1 - \mathcal{C}_\alpha)(1-\mathcal{C}_\beta) - \mathcal{C}_\alpha\mathcal{C}_\beta = 0.71791 \ne 0$,

$$\mathcal{A} := 3.1971,$$

and

$$\mathcal{B} := 3.009.$$

Since all the assumptions in Theorem 3.7 are satisfied, the coupled system (IV.1)-(IV.2) is UH and GUH stable. Moreover, let $\sigma(t) = \sqrt{\log t}$, $t \in [1,e]$ and $\mu \in \{\alpha,\beta\}$. Then $\sigma : [1,e] \to [0,+\infty)$ is continuous nondecreasing function such that

$$\begin{aligned} I_1^{\mu}\sigma(t) &= I_1^{\frac{1}{2}}(\log t)^{\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_1^t (\log\frac{t}{s})^{-\frac{1}{2}}(\log s)^{\frac{1}{2}}\frac{ds}{s} \\ &\le \frac{1}{\Gamma(\frac{1}{2})}\int_1^t (\log\frac{t}{s})^{-\frac{1}{2}}\frac{ds}{s} \\ &= \frac{2}{\sqrt{\pi}}(\log t)^{\frac{1}{2}} \\ &= \lambda_\sigma\sigma(t), \end{aligned}$$

This proves σ satisfies condition (H_3) with $\lambda_\sigma = \frac{2}{\sqrt{\pi}}$. Thus, Theorem (3.11) shows that the coupled system (IV.1)-(IV.2) is RUH and GRUH stable.

Example 4.2: Let $\alpha = \beta = \frac{2}{3}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{3}$,

$$\begin{aligned} f_1(t,u(t),v(t)) &= \frac{1}{(t+4)^3}\frac{|u(t)|}{1+|u(t)|} \\ &\quad + \frac{1}{30}\frac{|v(t)|}{1+|v(t)|} + \frac{1}{\sqrt{t+8}}, \end{aligned}$$

and

$$\begin{aligned} f_2(t,u(t),v(t)) &= \frac{1}{16}\sin u(t) \\ &\quad + \frac{|v(t)|}{50(1+|v(t)|)} + \frac{1}{5\sqrt{t^2+15}}. \end{aligned}$$

Consider the coupled system

$$\begin{cases} D_1^{\frac{2}{3}}u(t) = f_1(t,u(t),v(t)), & t \in [1,e], \\ D_1^{\frac{2}{3}}v(t) = f_2(t,u(t),v(t)), & t \in [1,e], \end{cases} \tag{IV.3}$$

with the coupled boundary conditions

$$\begin{array}{l} u(1) = \frac{1}{4}v(e), \\ v(1) = \frac{1}{3}u(e). \end{array} \tag{IV.4}$$

For $t \in [1,e]$ and $u, v \in \mathbb{R}^+$, we have

$$|f_1(t,u,v)| \le \frac{1}{125}|u| + \frac{1}{30}|v| + \frac{1}{3}$$

and

$$|f_2(t,u,v)| \le \frac{1}{16}|u| + \frac{1}{50}|v| + \frac{1}{20}.$$

Clearly, the assumption (H_2) holds with $k_0 = \frac{1}{3}$, $k_1 = \frac{1}{125}$, $k_2 = \frac{1}{30}$, $\gamma_0 = \frac{1}{20}$, $\gamma_1 = \frac{1}{16}$, and $\gamma_2 = \frac{1}{50}$. By some simple computations we find that $W_1 = \frac{12}{11\Gamma(\frac{5}{3})}$, $W_2 = \frac{1}{11\Gamma(\frac{5}{3})}$, $W_3 = \frac{4}{11\Gamma(\frac{5}{3})}$, and $W_4 = \frac{12}{11\Gamma(\frac{5}{3})}$. Hence

$$[(W_1+W_3)k_1 + (W_2+W_4)\gamma_1] \approx 0.09 < 1,$$

$$[(W_1+W_3)k_2 + (W_2+W_4)\gamma_2] \approx 0.08 < 1,$$

Also

$$W_0 \approx \min\{1.07, 0.9\} \ne 0.$$

The conditions of Theorem 3.2 are satisfied. Hence there exists at least one solution of the coupled system (IV.3)-(IV.4) on $[1,e]$.

V. Conclusion

This paper mainly investigated some existence and uniqueness results of boundary value problems for a coupled system of FDEs with coupled boundary conditions involving Hadamard fractional derivatives. Moreover, the Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability results of the considered system are discussed. Our analysis based on the reduction of FDEs to integral equations and applying some fixed point theorems which are quite effective. We trust the reported results here will have a positive impact on the development of further applications in engineering and applied sciences.

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