A Method for Group Decision Making with Multiplicative Consistent Interval-valued Intuitionistic Fuzzy Preference Relation

Ziyu Yang, Liyuan Zhang and Chunlei Liang

Abstract—**Interval-valued intuitionistic fuzzy preference relation (IVIFPR) is an appropriate tool for describing the group decision making (GDM) problems with complex and uncertain information because of its inclusiveness and flexibility. Based on the newly defined multiplicative consistency, this paper investigates an algorithm for the GDM method with IVIFPRs. Firstly, a novel multiplicative consistency concept is proposed, which is proved to satisfy an important property: robustness. A conversion formula is devised to accomplish the multiplicative consistent IVIFPRs by utilizing the normalized interval-valued intuitionistic fuzzy (IVIF) weights. Subsequently, a consistency measure and inconsistent repairing process are put forward to ensure that every individual IVIFPR is of acceptable multiplicative consistency. Afterward, in the context of minimizing the deviations between the given IVIFPRs and their corresponding consistent IVIFPRs, two fractional programming models are constructed to generate the normalized individual IVIF weights and collective ones, where the experts are considered as individuals and a group, respectively. Finally, an example is cited and comparative analyses with previous approaches are conducted to demonstrate the applicability and validity of the proposed method.**

Index Terms—**Group decision making, multiplicative consistency, interval-valued intuitionistic fuzzy preference relation.**

I. Introduction

GROUP decision making (GDM) [1], [2], [3], with the purpose of inviting decision makers (DMs) to estimate alternatives then prioritize the optimal one, is utilized in diverse areas of operations research. Preference relations are comprehensively applied to the GDM process to express the preference information[4] of DMs over the alternative set [5], such as fuzzy preference relation (FPR) [6], multiplicative preference relation [7], interval-valued fuzzy preference relation [8] and triangular fuzzy preference relation [9]. However, only the membership degree of one alternative to another can be represented by these preference relations, while the inherent uncertainty and hesitation are often ignored. To deal with this issue, Szmidt and Kacprzyk [10] proposed intuitionistic fuzzy preference relation (IFPR), which used degrees of membership, non-membership and hesitation to represent the preferences of DMs for alternatives. Hinduja and Pandey [11] explored an approach to determine the priorities from crisp values in IFPRs. Due to the diversity of values and goals of group DMs and the complexity in acquired information, it might be hard for DMs to give expression of preferences for a certain alternative or attribute by exact numbers. Thus, Xu and Chen [12] put forward the conception of interval-valued intuitionistic fuzzy preference relation (IVIFPR), in which the degrees of membership, non-membership and hesitation were presented by interval-valued intuitionistic fuzzy values (IVIFVs) [13], [14]. When describing uncertain preferences, IVIFPRs [15], [16], [17] are often more effective, practical and comprehensive.

In the context of GDM environment, exploring the methods with IVIFPRs has been focused by the worldwide scholars. Yang et al. [18] yielded the priorities from the IVIFPR by building interval-valued optimal priority optimization model, and established the corresponding algorithm flow of IVIF analytic network process. Zhou et al. [19] solved GDM with IVIFPRs through combining the fuzzy cooperative game method with the continuous IVIF ordered weighted averaging operator. Mohammadi and Makui [20] integrated Multi-attribute GDM with evidential reasoning methodology, then based on IVIFPRs, they proposed a new approach for supporting such decision situation. To rank alternatives, Wu and Chiclana [21] defined the IVIF continuous OWA (IVIF-COWA) operator, and gave an original score function for IVIFPRs. Different from Wu and Chiclana [21], the ranking order of alternatives defined by Wang et al. [22] was generated according to the proposed order relation of IVIFVs which were obtained by defining the possibility degree and divergence degree.

Recently, consistency analysis is an area of research value in GDM with IVIFPRs. Due to the complexity of GDM problems, DMs often cannot give a completely consistent judgment, and then cannot use priority weights to obtain scientific results. Therefore, it is of vital importance to pay attention to the defining, checking and repairing processes of the consistency. The current research is mainly divided into additive consistency [23], [24] and multiplicative consistency [25], [26], [27], we concentrate on the latter here. Based on the consistency, Li et al. [28] introduced two algorithms for GDM problems to deduce the optimal choice. Xu [29] defined IFPR, consistent IFPR and acceptable IFPR, and analyzed their properties. According to the definition of multiplicative consistency in [30], Xu and Liao [31] investigated a method to modify the inconsistent IFPRs. However, Liao and Xu [32] indicated the defects of the multiplicative consistency concept in [31], and proposed a new conception of multiplicative consistent IFPR. Subsequently, the multiplicative consistency in [32] is modified via a programming

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model by Xu et al. [33]. Based on multiplicative consistency, Li et al. [34] devised two techniques for multi-criteria GDM with IFPRs. Wan et al. [35] defined the consistency of IVIFPR. Liao et al. [36] described different kinds of multiplicative consistent IVIFPRs, and explored a multiplicative consistency improving procedure. Wan et al. [37] pointed out that when an IVIFPR degenerated into an IFPR, the definition of Liao et al. [36] would be unreasonable. Then a new acceptable multiplicative consistency was introduced, and a new algorithm was designed to repair inconsistent IVIFPRs. However, Wan et al. [26] emphasize that the parameters of iterative algorithm are hard to determine during the consistency repairing process in [37]. Furthermore, the consistency definition given by Wan et al. [37] might be unreliable because it did not satisfy robustness [38].

Based on the limitations mentioned above, we initiate a novel multiplicative consistency concept and put forward an approach for dealing with GDM with IVIFPRs. The major innovations are listed as follows:

1) A new multiplicative consistency definition of IVIFPRs is introduced which is confirmed to overcome the limitations in [36] and [37], i.e., the satisfaction of robustness can be proved, furthermore, when the IVIFPRs degenerate to IFPRs or even FPRs, this new definition is still valid.

2) A new normalized IVIF weight concept is put forward and a conversion formula is developed to transform the priority weights into consistent IVIFPRs. Subsequently, considering the minimized deviations of the given IVIFPRs and the corresponding consistent ones, two new programming models are constructed to derive normalized IVIF weights from aspects of individual expert and an expert group.

3) The consistency measure is calculated between IVIFPRs given by experts and the derived multiplicative consistent IVIFPRs to choose the inconsistent ones. Inspired by Liao et al.'s repairing process of IFPR [39], an inconsistency repairing process is presented to transform IVIFPRs into acceptable multiplicative consistent IVIFPRs.

4) Considering experts as individuals and a group, respectively, two fractional programming models are built to retrive the normalized individual and collective IVIF weights.

The rest parts is designed below: Section 2 briefly reflects on the relevant conceptions of IFPRs and IVIFPRs and the ranking method of interval-valued intuitionistic fuzzy numbers (IVIFNs). In section 3, we describe the definitions of multiplicative consistency characterized in [36] and [37] and analyzes their limitations. Later, a multiplicative consistency of IVIFPRs is newly introduced. Then we obtain the normalized IVIF weights based on this innovative definition. Afterwards, a transformation formula is introduced, by which the corresponding multiplicative consistent IVIFPRs are composed. With respect to Section 4, we propose consistency checking and inconsistency repairing procedure for an IVIFPR. In Section 5, considering all experts as a group, we establish a programming model to gain the collective IVIF weights, and introduce a specific algorithm process to solve GDM problems. Section 6 uses a mathematical example to expound the practicability and highlights the advantages of this method by comparing it with methods in Wan [37] and Liao [36]. The paper ends with the conclusion in Section 7.

II. Preliminaries

In order to facilitate our introduction, let us first do some reviews about the related concepts.

Definition 1: ([29]) An IFPR on the discrete set $X = \{x_1, x_2, \cdots, x_n\}$ is denoted by a preference matrix $R = (r_{ij})_{n\times n}$, where $r_{ij} = \langle (x_i, x_j), \mu(x_i, x_j), \nu(x_i, x_j)\rangle$ $(i, j = 1, 2, \cdots, n)$. Let $r_{ij} = (\mu_{ij}, \nu_{ij})$, where μ_{ij} indicates the degree to which x_i is preferred to x_j, ν_{ij} represents the degree to which x_i is not preferred to x_j. Furthermore, μ_{ij} and ν_{ij} satisfy the following conditions:

$\mu_{ij}, \nu_{ij} \in [0, 1],\ 0 \le \mu_{ij} + \nu_{ij} \le 1,\ \mu_{ij} = \nu_{ji},\ \nu_{ij} = \mu_{ji},\ \mu_{ii} = \nu_{ii} = 0.5$ for all $i, j = 1, 2, \cdots, n$.

Definition 2: ([32]) An IFPR $R = (r_{ij})_{n\times n}$ with $r_{ij} = (\mu_{ij}, \nu_{ij})$ on the set X is called multiplicative consistent if it satisfies the following transiivity:

$$\mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = \nu_{ij} \cdot \nu_{jk} \cdot \nu_{ki}, \tag{1}$$

for all $i, j, k = 1, 2, \cdots, n$.

Definition 3: ([12]) An IVIFPR on the set X is denoted by a preference matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times n} \subset X \times X$, where $\tilde{r}_{ij} = \langle (x_i, x_j), \tilde{\mu}(x_i, x_j), \tilde{\nu}(x_i, x_j)\rangle$ $(i, j = 1, 2, \cdots, n)$. For convenience, let $\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij})$, where $\tilde{\mu}_{ij} = [\underline{\mu}_{ij}, \overline{\mu}_{ij}]$ represents the degree range to which x_i is preferred to x_j, $\tilde{\nu}_{ij} = [\underline{\nu}_{ij}, \overline{\nu}_{ij}]$ indicates the degree range to which the object x_i is not preferred to x_j. Moreover, $\tilde{\mu}_{ij}$ and $\tilde{\nu}_{ij}$ fulfill the conditions:

$\tilde{\mu}_{ij} = [\underline{\mu}_{ij}, \overline{\mu}_{ij}] \subseteq [0, 1]$, $\tilde{\nu}_{ij} = [\underline{\nu}_{ij}, \overline{\nu}_{ij}] \subseteq [0, 1]$, $0 \le \overline{\mu}_{ij} + \overline{\nu}_{ij} \le 1$, $\tilde{\mu}_{ij} = \tilde{\nu}_{ji}$, $\tilde{\nu}_{ij} = \tilde{\mu}_{ji}$, $\tilde{\mu}_{ii} = \tilde{\nu}_{ii} = [0.5, 0.5]$, for all $i, j = 1, 2, \cdots, n$.

To rank IVIFNs, the score function and accuracy function given by Xu[40] are shown as follows.

Definition 4: ([40]) Let $\tilde{\beta} = ([a, b], [c, d])$ be an IVIFN, then

$$s(\tilde{\beta}) = \frac{1}{2}(a - c + b - d) \tag{2}$$

and

$$h(\tilde{\beta}) = \frac{1}{2}(a + c + b + d) \tag{3}$$

are called the score function and accuracy function of $\tilde{\beta}$, respectively.

Let $\tilde{\beta}_1$ and $\tilde{\beta}_2$ be any two IVIFNs. Based on above two functions, the order relations between IVIFNs are demonstrated as follows [40].

If $s(\tilde{\beta}_1) > s(\tilde{\beta}_2)$, then $\tilde{\beta}_1 > \tilde{\beta}_2$.
If $s(\tilde{\beta}_1) = s(\tilde{\beta}_2)$, and
  if $h(\tilde{\beta}_1) > h(\tilde{\beta}_2)$, then $\tilde{\beta}_1 > \tilde{\beta}_2$;
  if $h(\tilde{\beta}_1) = h(\tilde{\beta}_2)$, then $\tilde{\beta}_1 = \tilde{\beta}_2$.

III. Multiplicative Consistency of IVIFPRs

A. A New Multiplicative Consistency Definition of IVIFPRs

In the evaluation procedure through decision making, it may be challenging for a DM to give the crisp numbers of membership degrees about alternatives. This is owning to the inadequate information and imprecise evaluation of the preference degree between alternatives. In this situation, the pairwise comparison judgments can be appropriately presented by IVIFNs, then an IVIFPR can be constituted.

Similar to IFPR, due to the lack of consistency in IVIFPRs, irrational conclusions can be drawn. Liao et al.[36] and

Wan et al.[37] characterized the multiplicative consistency of IVIFPRs, respectively.

Definition 5: ([36]) The IVIFPR $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ is multiplicative consistent if

$$\begin{aligned}
&\underline{\mu}_{ij} = \\
&\begin{cases} 0, & \left(\underline{\mu}_{ik}, \underline{\mu}_{kj}\right) \in \{(0,1),(1,0)\} \\ \frac{\underline{\mu}_{ik}\underline{\mu}_{kj}}{\underline{\mu}_{ik}\underline{\mu}_{kj}+(1-\underline{\mu}_{ik})(1-\underline{\mu}_{kj})}, & otherwise \end{cases}, \\
&\overline{\mu}_{ij} = \\
&\begin{cases} 0, & \left(\overline{\mu}_{ik}, \overline{\mu}_{kj}\right) \in \{(0,1),(1,0)\} \\ \frac{\overline{\mu}_{ik}\overline{\mu}_{kj}}{\overline{\mu}_{ik}\overline{\mu}_{kj}+(1-\overline{\mu}_{ik})(1-\overline{\mu}_{kj})}, & otherwise \end{cases}, \\
&\underline{\nu}_{ij} = \\
&\begin{cases} 0, & \left(\underline{\nu}_{ik}, \underline{\nu}_{kj}\right) \in \{(0,1),(1,0)\} \\ \frac{\underline{\nu}_{ik}\underline{\nu}_{kj}}{\underline{\nu}_{ik}\underline{\nu}_{kj}+(1-\underline{\nu}_{ik})(1-\underline{\nu}_{kj})}, & otherwise \end{cases}, \\
&\overline{\nu}_{ij} = \\
&\begin{cases} 0, & \left(\overline{\nu}_{ik}, \overline{\nu}_{kj}\right) \in \{(0,1),(1,0)\} \\ \frac{\overline{\nu}_{ik}\overline{\nu}_{kj}}{\overline{\nu}_{ik}\overline{\nu}_{kj}+(1-\overline{\nu}_{ik})(1-\overline{\nu}_{kj})}, & otherwise \end{cases},
\end{aligned} \tag{4}$$

for all $i < k < j$.

Definition 6: ([37]) An IVIFPR $\tilde{R} = \left(\left(\tilde{\mu}_{ij}, \tilde{\nu}_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}_{ij} = \left[\underline{\mu}_{ij}, \overline{\mu}_{ij}\right]$ and $\tilde{\nu}_{ij} = \left[\underline{\nu}_{ij}, \overline{\nu}_{ij}\right]$ is multiplicative consistent if

$$\begin{aligned}
\underline{\mu}_{ij} \cdot \underline{\mu}_{jk} \cdot \overline{\mu}_{ki} &= \overline{\nu}_{ij} \cdot \overline{\nu}_{jk} \cdot \underline{\nu}_{ki}, \\
\overline{\mu}_{ij} \cdot \overline{\mu}_{jk} \cdot \underline{\mu}_{ki} &= \underline{\nu}_{ij} \cdot \underline{\nu}_{jk} \cdot \overline{\nu}_{ki},
\end{aligned} \tag{5}$$

for all $i, j, k = 1, 2, \cdots, n$.

Remark 1: ([38]) Definition 5 indicates that Eq. (4) is a development of the multiplicative consistency conception in Tanino [41]. Note that Tanino's multiplicative consistency conception has the most important property: robustness, while Definition 5 does not satisfy it. Definition 6 has the same drawback as Definition 5, which means that consistency defined by Wan et al. [37] is also lack of robustness. That is to say, contradictory consistency conclusions might be accessed by different comparison orders.

For more details, one can see the Example 2 and Example 3 in Meng et al. [38]

Since the above two descriptions in [36] and [37] do not meet the robust condition [41], a novel multiplicative consistency is depicted below.

Definition 7: An IVIFPR $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n} = \left(\left(\tilde{\mu}_{ij}, \tilde{\nu}_{ij}\right)\right)_{n\times n}$ with $\tilde{\mu}_{ij} = \left[\underline{\mu}_{ij}, \overline{\mu}_{ij}\right]$ and $\tilde{\nu}_{ij} = \left[\underline{\nu}_{ij}, \overline{\nu}_{ij}\right]$ is called multiplicative consistent if the following multiplicative transitivity satisfies:

$$\underline{\mu}_{ij} \cdot \overline{\mu}_{ij} \cdot \underline{\mu}_{jk} \cdot \overline{\mu}_{jk} \cdot \underline{\mu}_{ki} \cdot \overline{\mu}_{ki} = \underline{\nu}_{ij} \cdot \overline{\nu}_{ij} \cdot \underline{\nu}_{jk} \cdot \overline{\nu}_{jk} \cdot \underline{\nu}_{ki} \cdot \overline{\nu}_{ki}, \tag{6}$$

for all $i, j, k = 1, 2, \cdots, n$.

In particular, if $\underline{\mu}_{ij} = \overline{\mu}_{ij} = \mu_{ij}$ and $\underline{\nu}_{ij} = \overline{\nu}_{ij} = \nu_{ij}$, $\tilde{R}$ will reduce to an IFPR $R = \left(r_{ij}\right)_{n\times n}$ with $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$. Then the multiplicative transitivity reduces to Eq. (1). Furthermore, if $\mu_{ij} + \nu_{ij} = 1$, R reduces to a FPR.

Then we will check the robustness of Definition 7.

Let $\tilde{R}^{\iota} = \left(\tilde{r}^{\iota}_{ij}\right)_{n\times n} = \left(\left(\tilde{\mu}^{\iota}_{ij}, \tilde{\nu}^{\iota}_{ij}\right)\right)_{n\times n}$, where $\tilde{\mu}^{\iota}_{ij} = \left[\underline{\mu}^{\iota}_{ij}, \overline{\mu}^{\iota}_{ij}\right] = \left[\underline{\mu}_{\iota(i)\iota(j)}, \overline{\mu}_{\iota(i)\iota(j)}\right], \tilde{\nu}^{\iota}_{ij} = \left[\underline{\nu}^{\iota}_{ij}, \overline{\nu}^{\iota}_{ij}\right] = \left[\underline{\nu}_{\iota(i)\iota(j)}, \overline{\nu}_{\iota(i)\iota(j)}\right]$, and ι is a permutation of $\{1, 2, \cdots, n\}$. Set $\iota(i) = i', \iota(j) = j'$, then $\tilde{\mu}^{\iota}_{ij} = \left[\underline{\mu}_{i'j'}, \overline{\mu}_{i'j'}\right]$, $\tilde{\nu}^{\iota}_{ij} = \left[\underline{\nu}_{i'j'}, \overline{\nu}_{i'j'}\right]$.

Theorem 8: The IVIFPR $\tilde{R}$ is multiplicative consistent if and only if there exists a multiplicative consistent $\tilde{R}^{\iota}$ for any permutation ι.

Proof: Necessity. Presume that $\tilde{R}$ is multiplicative consistent, according to Definition 7, we have $\underline{\mu}_{ij} \cdot \overline{\mu}_{ij} \cdot \underline{\mu}_{jk} \cdot \overline{\mu}_{jk} \cdot \underline{\mu}_{ki} \cdot \overline{\mu}_{ki} = \underline{\nu}_{ij} \cdot \overline{\nu}_{ij} \cdot \underline{\nu}_{jk} \cdot \overline{\nu}_{jk} \cdot \underline{\nu}_{ki} \cdot \overline{\nu}_{ki}$, for all $i, j, k = 1, 2, \cdots, n$. Therefore, $\underline{\mu}^{\iota}_{ij} \cdot \overline{\mu}^{\iota}_{ij} \cdot \underline{\mu}^{\iota}_{jk} \cdot \overline{\mu}^{\iota}_{jk} \cdot \underline{\mu}^{\iota}_{ki} \cdot \overline{\mu}^{\iota}_{ki} = \underline{\mu}_{i'j'} \cdot \overline{\mu}_{i'j'} \cdot \underline{\mu}_{j'k'} \cdot \overline{\mu}_{j'k'} \cdot \underline{\mu}_{k'i'} \cdot \overline{\mu}_{k'i'} = \underline{\nu}_{i'j'} \cdot \overline{\nu}_{i'j'} \cdot \underline{\nu}_{j'k'} \cdot \overline{\nu}_{j'k'} \cdot \underline{\nu}_{k'i'} \cdot \overline{\nu}_{k'i'} = \underline{\nu}^{\iota}_{ij} \cdot \overline{\nu}^{\iota}_{ij} \cdot \underline{\nu}^{\iota}_{jk} \cdot \overline{\nu}^{\iota}_{jk} \cdot \underline{\nu}^{\iota}_{ki} \cdot \overline{\nu}^{\iota}_{ki}$, for all $i, j, k = 1, 2, \cdots, n$. Through Definition 7, we can get $\tilde{R}^{\iota}$ is multiplicative consistent.

Sufficiency. A multiplicative consistent $\tilde{R}^{\iota}$ is equal to $\underline{\mu}^{\iota}_{ij} \cdot \overline{\mu}^{\iota}_{ij} \cdot \underline{\mu}^{\iota}_{jk} \cdot \overline{\mu}^{\iota}_{jk} \cdot \underline{\mu}^{\iota}_{ki} \cdot \overline{\mu}^{\iota}_{ki} = \underline{\nu}^{\iota}_{ij} \cdot \overline{\nu}^{\iota}_{ij} \cdot \underline{\nu}^{\iota}_{jk} \cdot \overline{\nu}^{\iota}_{jk} \cdot \underline{\nu}^{\iota}_{ki} \cdot \overline{\nu}^{\iota}_{ki}$, i.e., $\underline{\mu}_{i'j'} \cdot \overline{\mu}_{i'j'} \cdot \underline{\mu}_{j'k'} \cdot \overline{\mu}_{j'k'} \cdot \underline{\mu}_{k'i'} \cdot \overline{\mu}_{k'i'} = \underline{\nu}_{i'j'} \cdot \overline{\nu}_{i'j'} \cdot \underline{\nu}_{j'k'} \cdot \overline{\nu}_{j'k'} \cdot \underline{\nu}_{k'i'} \cdot \overline{\nu}_{k'i'}$. In accordance with the Definition 7, $\tilde{R}$ is multiplicative consistent. ■

B. Deriving the IVIF Weights Based on the Multiplicative Consistency

During the decision making with IVIFPR, it is of vital importance to obtain the priority weights. In this part, we will concentrate on this theme and come up with an approach for generating the priority weight vector of IVIFPRs under the condition of multiplicative consistency.

Let $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, \cdots, \tilde{\varpi}_n)^T$ be an IVIF priority weight vector of the IVIFPR $\tilde{R}$, where $\tilde{\varpi}_i = \left(\tilde{\varpi}^{\mu}_i, \tilde{\varpi}^{\nu}_i\right) = \left(\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right], \left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right]\right)$ is an IVIFN, which satisfies $\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right] \subseteq [0,1]$, $\left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right] \subseteq [0,1]$ and $\overline{\varpi}^{\mu}_i + \overline{\varpi}^{\nu}_i \leq 1$ for $i = 1, 2, \cdots, n$.

We first introduce the definition of normalized IVIF priority weights as:

Definition 9: An IVIF priority weight vector $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, \cdots, \tilde{\varpi}_n)^T$ with $\tilde{\varpi}_i = \left(\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right], \left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right]\right)$, $\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right] \subseteq [0,1]$, $\left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right] \subseteq [0,1]$ and $\overline{\varpi}^{\mu}_i + \overline{\varpi}^{\nu}_i \leq 1$ is said to be normalized if it satisfies

$$\sum_{j=1, j\neq i}^{n} \overline{\varpi}^{\mu}_j \leq \underline{\varpi}^{\nu}_i, \quad \underline{\varpi}^{\mu}_i + n - 2 \geq \sum_{j=1, j\neq i}^{n} \overline{\varpi}^{\nu}_j \tag{7}$$

for $i = 1, 2, \cdots, n$.

Influenced by the multiplicative consistent IFPR defined by [32] and Eq. (6), we confirm the multiplicative consistent IVIFPR.

Assume that

$$\begin{aligned}
\tilde{r}_{ij} &= \left(\tilde{\mu}_{ij}, \tilde{\nu}_{ij}\right) \\
&= \begin{cases} ([0.5, 0.5], [0.5, 0.5]), & if\, i = j \\ \left(\left[\frac{\underline{\varpi}^{\mu}_i}{2-\overline{\varpi}^{\nu}_i-\overline{\varpi}^{\nu}_j}, \frac{\overline{\varpi}^{\mu}_i}{2-\underline{\varpi}^{\nu}_i-\underline{\varpi}^{\nu}_j}\right], \right. & \\ \left. \left[\frac{\underline{\varpi}^{\mu}_j}{2-\overline{\varpi}^{\nu}_i-\overline{\varpi}^{\nu}_j}, \frac{\overline{\varpi}^{\mu}_j}{2-\underline{\varpi}^{\nu}_i-\underline{\varpi}^{\nu}_j}\right] \right) & if\, i \neq j \end{cases}
\end{aligned} \tag{8}$$

where $\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right] \subseteq [0,1]$, $\left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right] \subseteq [0,1]$, $\overline{\varpi}^{\mu}_i + \overline{\varpi}^{\nu}_i \leq 1$, $\sum_{j=1, j\neq i}^{n} \overline{\varpi}^{\mu}_j \leq \underline{\varpi}^{\nu}_i$, and $\underline{\varpi}^{\mu}_i + n - 2 \geq \sum_{j=1, j\neq i}^{n} \overline{\varpi}^{\nu}_j$ $(i = 1, 2, \cdots, n)$. Then we can obtain:

Theorem 10: $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ is an IVIFPR if all $\tilde{r}_{ij}(i, j = 1, 2, \cdots, n)$ are represented as in Eq. (8).

Proof: Obviously, $\tilde{\mu}_{ij} = \tilde{\nu}_{ji}$ for $i, j = 1, 2, \cdots, n$. Since $\left[\underline{\varpi}^{\mu}_i, \overline{\varpi}^{\mu}_i\right] \subseteq [0,1]$, $\left[\underline{\varpi}^{\nu}_i, \overline{\varpi}^{\nu}_i\right] \subseteq [0,1]$, and $\overline{\varpi}^{\mu}_i + \overline{\varpi}^{\nu}_i \leq 1$, it follows that $\frac{\overline{\varpi}^{\mu}_i}{2-\overline{\varpi}^{\nu}_i-\overline{\varpi}^{\nu}_j} + \frac{\overline{\varpi}^{\mu}_j}{2-\overline{\varpi}^{\nu}_i-\overline{\varpi}^{\nu}_j} \leq \frac{\overline{\varpi}^{\mu}_i}{\overline{\varpi}^{\nu}_i+\overline{\varpi}^{\nu}_j} + \frac{\overline{\varpi}^{\mu}_j}{\overline{\varpi}^{\nu}_i+\overline{\varpi}^{\nu}_j} = 1.$

According to Definition 3, $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}$ is an IVIFPR, then the proof is completed. ∎

Theorem 11: The IVIFPR $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}$ is multiplicative consistent, where the elements $\tilde{r}_{ij}(i,j=1,2,\cdots,n)$ are identified as in Eq. (8).

Proof: Since Eq. (6) is equal to

$$\underline{\mu}_{ij}\cdot\overline{\mu}_{ij}\cdot\underline{\mu}_{jk}\cdot\overline{\mu}_{jk}\cdot\underline{\mu}_{ki}\cdot\overline{\mu}_{ki}=\underline{\nu}_{ij}\cdot\overline{\nu}_{ij}\cdot\underline{\nu}_{jk}\cdot\overline{\nu}_{jk}\cdot\underline{\nu}_{ki}\cdot\overline{\nu}_{ki},$$

where $i<j<k$, then from Eq. (8), we have

$$\begin{aligned}&\underline{\mu}_{ij}\cdot\overline{\mu}_{ij}\cdot\underline{\mu}_{jk}\cdot\overline{\mu}_{jk}\cdot\underline{\mu}_{ki}\cdot\overline{\mu}_{ki}\\ =&\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}\cdot\frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}\cdot\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\\ &\cdot\frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\cdot\frac{\underline{\varpi}_k^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\cdot\frac{\overline{\varpi}_k^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}},\end{aligned}$$

$$\begin{aligned}&\underline{\nu}_{ij}\cdot\overline{\nu}_{ij}\cdot\underline{\nu}_{jk}\cdot\overline{\nu}_{jk}\cdot\underline{\nu}_{ki}\cdot\overline{\nu}_{ki}\\ =&\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}\cdot\frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}\cdot\frac{\underline{\varpi}_k^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\\ &\cdot\frac{\overline{\varpi}_k^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\cdot\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\cdot\frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}.\end{aligned}$$

It is obvious that $\underline{\mu}_{ij}\cdot\overline{\mu}_{ij}\cdot\underline{\mu}_{jk}\cdot\overline{\mu}_{jk}\cdot\underline{\mu}_{ki}\cdot\overline{\mu}_{ki}=\underline{\nu}_{ij}\cdot\overline{\nu}_{ij}\cdot\underline{\nu}_{jk}\cdot\overline{\nu}_{jk}\cdot\underline{\nu}_{ki}\cdot\overline{\nu}_{ki}$, which satisfies Eq. (6).

Especially, when $i=j$, which equals $\underline{\mu}_{ij}=\overline{\mu}_{ij}=0.5$, $\underline{\nu}_{ij}=\overline{\nu}_{ij}=0.5$, then

$$\begin{aligned}&\underline{\mu}_{ij}\cdot\overline{\mu}_{ij}\cdot\underline{\mu}_{jk}\cdot\overline{\mu}_{jk}\cdot\underline{\mu}_{ki}\cdot\overline{\mu}_{ki}\\ =&\,0.25\cdot\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\cdot\frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\\ &\cdot\frac{\underline{\varpi}_k^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\cdot\frac{\overline{\varpi}_k^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\\ =&\,0.25\cdot\frac{\underline{\varpi}_k^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\cdot\frac{\overline{\varpi}_k^{\mu}}{2-\overline{\varpi}_j^{\nu}-\overline{\varpi}_k^{\nu}}\\ &\cdot\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\cdot\frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_k^{\nu}-\overline{\varpi}_i^{\nu}}\\ =&\,\underline{\nu}_{ij}\cdot\overline{\nu}_{ij}\cdot\underline{\nu}_{jk}\cdot\overline{\nu}_{jk}\cdot\underline{\nu}_{ki}\cdot\overline{\nu}_{ki},\end{aligned}$$

which also satisfies Eq. (6). Similarly, Eq. (6) still holds when $j=k$ or $k=i$.

Consequently, the IVIFPR $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}$ defined as in Eq. (8) fulfills the multiplicative consistency in Definition 7, then the proof is completed. ∎

Note that if $\underline{w}_i^{\mu}=\overline{\varpi}_i^{\mu}=\varpi_i^{\mu}$, $\underline{\varpi}_i^{\nu}=\overline{\varpi}_i^{\nu}=\varpi_i^{\nu}$, i.e., all IVIF weights are degraded to intuitionistic fuzzy weights, then $\tilde{R}$ reduces to an IFPR R, and $\tilde{r}_{ij}$ reduces to $r_{ij}=(\mu_{ij},\nu_{ij})$, where $\mu_{ij}=\frac{\varpi_i^{\mu}}{2-\varpi_i^{\nu}-\varpi_j^{\nu}}$, $\nu_{ij}=\frac{\varpi_j^{\mu}}{2-\varpi_i^{\nu}-\varpi_j^{\nu}}$. Moreover, the multiplicative transitivity reduces to Eq. (1).

Corollary 1: $\tilde{R}^{*}=\left(\tilde{r}_{ij}^{*}\right)_{n\times n}$ is called a multiplicative consistent IVIFPR if there is a normalized IVIF priority weight vector $\tilde{\varpi}=(\tilde{\varpi}_1,\tilde{\varpi}_2,\cdots,\tilde{\varpi}_n)^T$, which satisfies

$$\begin{aligned}\tilde{r}_{ij}^{*}&=\left(\tilde{\mu}_{ij}^{*},\tilde{\nu}_{ij}^{*}\right)\\ &=\begin{cases}([0.5,0.5],[0.5,0.5]), & if\ i=j\\ \left(\left[\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}},\frac{\overline{\varpi}_i^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}\right],\right.\\ \left.\left[\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}},\frac{\overline{\varpi}_j^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}\right]\right) & if\ i\neq j\end{cases}\end{aligned}\tag{9}$$

where $\left[\underline{\varpi}_i^{\mu},\overline{\varpi}_i^{\mu}\right]\subseteq[0,1]$, $\left[\underline{\varpi}_i^{\nu},\overline{\varpi}_i^{\nu}\right]\subseteq[0,1]$, $\overline{\varpi}_i^{\mu}+\overline{\varpi}_i^{\nu}\leq 1$, $\sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\mu}\leq\underline{\varpi}_i^{\nu}$, and $\underline{\varpi}_i^{\mu}+n-2\geq\sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\nu}$ $(i=1,2,\cdots,n)$, then $\tilde{R}^{*}$ is called to be multiplicative consistent.

In order to access reasonable results, the IVIFPRs given by experts are supposed to be multiplicative consistent then denoted as in Corollary 1. However, it may be too difficult for experts to establish such an IVIFPR in actual decision making, especially when there are too many alternatives. Therefore, the smallest deviation between an IVIFPR and its corresponding consistent IVIFPR is requested. Inspired by Corollary 1, a novel method is exploited to deduce the normalized priority weights for an IVIFPR. Let $\tilde{R}^{(h)}=\left(\tilde{r}_{ij}^{(h)}\right)_{n\times n}$ be an IVIFPR given by expert $E_h(h=1,2,\cdots,s)$, the deviation variables are indicated as follows:

$$\underline{\vartheta}_{ij}^{(h)}=\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\mu}_{ij}^{(h)},\tag{10}$$

$$\overline{\vartheta}_{ij}^{(h)}=\frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\overline{\mu}_{ij}^{(h)},\tag{11}$$

$$\underline{\zeta}_{ij}^{(h)}=\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\nu}_{ij}^{(h)},\tag{12}$$

$$\overline{\zeta}_{ij}^{(h)}=\frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\overline{\nu}_{ij}^{(h)},\tag{13}$$

where $i,j=1,2,\cdots,n;i\neq j,h=1,2,\cdots,s$.

For the propose of producing more exact results, the deviations should be as small as possible. Then we obtain the following objective function for the kth expert:

$$Min\ Z=\sum_{i=1}^{n}\sum_{j=1}^{n}(|\underline{\vartheta}_{ij}^{(h)}|+|\overline{\vartheta}_{ij}^{(h)}|+|\underline{\xi}_{ij}^{(h)}|+|\overline{\zeta}_{ij}^{(h)}|),\tag{14}$$

where $k=1,2,\cdots,s$.

Considering $\underline{\mu}_{ij}=\underline{\nu}_{ji}$, $\overline{\mu}_{ij}=\overline{\nu}_{ji}$, $\underline{\nu}_{ij}=\underline{\mu}_{ji}$, $\overline{\nu}_{ij}=\overline{\mu}_{ji}$, and $\underline{\vartheta}_{ij}^{(h)}=\underline{\zeta}_{ji}^{(h)}$, $\overline{\vartheta}_{ij}^{(h)}=\overline{\zeta}_{ji}^{(h)}$, Eq. (14) is equivalent to:

$$Min\ Z=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\underline{\vartheta}_{ij}^{(h)}|+|\overline{\vartheta}_{ij}^{(h)}|+|\underline{\zeta}_{ij}^{(h)}|+|\overline{\zeta}_{ij}^{(h)}|),\tag{15}$$

where $h=1,2,\cdots,s$.

Let $\underline{\vartheta}_{ij}^{(h)+}=\frac{|\underline{\vartheta}_{ij}^{(h)}|+\underline{\vartheta}_{ij}^{(h)}}{2}$, $\underline{\vartheta}_{ij}^{(h)-}=\frac{|\underline{\vartheta}_{ij}^{(h)}|-\underline{\vartheta}_{ij}^{(h)}}{2}$, $\overline{\vartheta}_{ij}^{(h)+}=\frac{|\overline{\vartheta}_{ij}^{(h)}|+\overline{\vartheta}_{ij}^{(h)}}{2}$, $\overline{\vartheta}_{ij}^{(h)-}=\frac{|\overline{\vartheta}_{ij}^{(h)}|-\overline{\vartheta}_{ij}^{(h)}}{2}$, $\underline{\zeta}_{ij}^{(h)+}=\frac{|\underline{\zeta}_{ij}^{(h)}|+\underline{\zeta}_{ij}^{(h)}}{2}$, $\underline{\zeta}_{ij}^{(h)-}=\frac{|\underline{\zeta}_{ij}^{(h)}|-\underline{\zeta}_{ij}^{(h)}}{2}$, $\overline{\zeta}_{ij}^{(h)+}=\frac{|\overline{\zeta}_{ij}^{(h)}|+\overline{\zeta}_{ij}^{(h)}}{2}$, $\overline{\zeta}_{ij}^{(h)-}=\frac{|\overline{\zeta}_{ij}^{(h)}|-\overline{\zeta}_{ij}^{(h)}}{2}$, then $\underline{\vartheta}_{ij}^{(h)}=\underline{\vartheta}_{ij}^{(h)+}-\underline{\vartheta}_{ij}^{(h)-}$, $|\underline{\vartheta}_{ij}^{(h)}|=\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}$, $\overline{\vartheta}_{ij}^{(h)}=\overline{\vartheta}_{ij}^{(h)+}-\overline{\vartheta}_{ij}^{(h)-}$, $|\overline{\vartheta}_{ij}^{(h)}|=\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-}$, $\underline{\zeta}_{ij}^{(h)}=\underline{\zeta}_{ij}^{(h)+}-\underline{\zeta}_{ij}^{(h)-}$, $|\underline{\zeta}_{ij}^{(h)}|=\underline{\zeta}_{ij}^{(h)+}+\underline{\zeta}_{ij}^{(h)-}$, $\overline{\zeta}_{ij}^{(h)}=\overline{\zeta}_{ij}^{(h)+}-\overline{\zeta}_{ij}^{(h)-}$, $|\overline{\zeta}_{ij}^{(h)}|=\overline{\zeta}_{ij}^{(h)+}+\overline{\zeta}_{ij}^{(h)-}$, where $\underline{\vartheta}_{ij}^{(h)+}\geq 0$, $\underline{\vartheta}_{ij}^{(h)-}\geq 0$, $\overline{\vartheta}_{ij}^{(h)+}\geq 0$, $\overline{\vartheta}_{ij}^{(k)-}\geq 0$, $\underline{\zeta}_{ij}^{(h)+}\geq 0$, $\underline{\zeta}_{ij}^{(h)-}\geq 0$, $\overline{\zeta}_{ij}^{(h)+}\geq 0$, $\overline{\zeta}_{ij}^{(h)-}\geq 0$, $\underline{\vartheta}_{ij}^{(h)+}\cdot\underline{\vartheta}_{ij}^{(h)-}=0$, $\overline{\vartheta}_{ij}^{(h)+}\cdot\overline{\vartheta}_{ij}^{(h)-}=0$, $\underline{\zeta}_{ij}^{(h)+}\cdot\underline{\zeta}_{ij}^{(h)-}=0$, and $\overline{\zeta}_{ij}^{(h)+}\cdot\overline{\zeta}_{ij}^{(h)-}=0$. Thus, the fractional Model 1 composed for the hth expert is established.

We solve Model 1 by using LINGO. The optimal solution f^{*} and the normalized IVIF priority weighs are soon generated. If $f^{*}=0$, the IVIFPR $\tilde{R}^{(h)}$ is multiplicative consistent,

Model 1 $$Min\ f=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}+\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-}+\underline{\zeta}_{ij}^{(h)+}+\underline{\zeta}_{ij}^{(h)-}+\overline{\zeta}_{ij}^{(h)+}+\overline{\zeta}_{ij}^{(h)-})$$

$$s.t.\begin{cases}\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\mu}_{ij}^{(h)}-\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}=0, \quad i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\overline{\mu}_{ij}^{(h)}-\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-}=0, \quad i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\nu}_{ij}^{(h)}-\underline{\zeta}_{ij}^{(h)+}+\underline{\zeta}_{ij}^{(h)-}=0, \quad i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\overline{\nu}_{ij}^{(h)}-\overline{\zeta}_{ij}^{(h)+}+\overline{\zeta}_{ij}^{(h)-}=0, \quad i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \left[\underline{\varpi}_i^{\mu},\overline{\varpi}_i^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_i^{\nu},\overline{\varpi}_i^{\nu}\right]\subseteq[0,1],\ \overline{\varpi}_i^{\mu}+\overline{\varpi}_i^{\nu}\leq 1, \quad i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\mu}\leq\underline{\varpi}_i^{\nu},\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\nu}\leq\underline{\varpi}_i^{\mu}+n-2, \quad i=1,2,\cdots,n\\ \underline{\vartheta}_{ij}^{(h)+},\ \underline{\vartheta}_{ij}^{(h)-},\ \overline{\vartheta}_{ij}^{(h)+},\ \overline{\vartheta}_{ij}^{(h)-},\ \underline{\zeta}_{ij}^{(h)+},\ \underline{\zeta}_{ij}^{(h)-},\ \overline{\zeta}_{ij}^{(h)+},\ \overline{\zeta}_{ij}^{(h)-}\geq 0, \quad i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \underline{\vartheta}_{ij}^{(h)+}\cdot\underline{\vartheta}_{ij}^{(h)-}=0,\ \overline{\vartheta}_{ij}^{(h)+}\cdot\overline{\vartheta}_{ij}^{(h)-}=0,\ \underline{\zeta}_{ij}^{(h)+}\cdot\underline{\zeta}_{ij}^{(h)-}=0,\ \overline{\zeta}_{ij}^{(h)+}\cdot\overline{\zeta}_{ij}^{(h)-}=0,\ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\end{cases}$$

additionally, the yielded normalized IVIF priority weights are rational.

IV. Consistency Checking and Inconsistency Repairing Process for IVIFPRs

Under the consideration of various problems that may be encountered in decision making and the limitations of experts' knowledge, completely multiplicative consistent IVIFPRs may be extremely hard to be established. Inspired from the acceptable multiplicative consistent IFPR [42], we define the acceptable multiplicative consistent IVIFPR.

Definition 12: $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}=\left(\left(\tilde{\mu}_{ij},\tilde{\nu}_{ij}\right)\right)_{n\times n}$ $(i,j=1,2,\cdots,n)$ is called an acceptable multiplicative consistent IVIFPR if

$$d\big(\tilde{R},\tilde{R}^{*}\big)\leq 1-\alpha. \tag{16}$$

α is the consistency threshold, $d\big(\tilde{R},\tilde{R}^{*}\big)$ represents the distance measure between $\tilde{R}$ and its corresponding multiplicative consistent IVIFPR $\tilde{R}^{*}$, which can be counted by

$$d\big(\tilde{R},\tilde{R}^{*}\big)=\frac{1}{2n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\underline{\mu}_{ij}-\underline{\mu}_{ij}^{*}| +|\overline{\mu}_{ij}-\overline{\mu}_{ij}^{*}|+|\underline{\nu}_{ij}-\underline{\nu}_{ij}^{*}|+|\overline{\nu}_{ij}-\overline{\nu}_{ij}^{*}|). \tag{17}$$

In the course of Eq. (17), the consistency index is introduced below.

Definition 13: Let $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}$ be an IVIFPR, where $\tilde{r}_{ij}=\left(\tilde{\mu}_{ij},\tilde{\nu}_{ij}\right)=\left(\left[\underline{\mu}_{ij},\overline{\mu}_{ij}\right],\left[\underline{\nu}_{ij},\overline{\nu}_{ij}\right]\right), i,j=1,2,\cdots,n.$ The consistency measure $C_{\tilde{R}}$ of $\tilde{R}$ satisfies

$$C_{\tilde{R}}=1-\frac{1}{2n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\underline{\mu}_{ij}-\underline{\mu}_{ij}^{*}| +|\overline{\mu}_{ij}-\overline{\mu}_{ij}^{*}|+|\underline{\nu}_{ij}-\underline{\nu}_{ij}^{*}|+|\overline{\nu}_{ij}-\overline{\nu}_{ij}^{*}|). \tag{18}$$

Note that $\tilde{R}^{*}$ is the corresponding multiplicative consistent IVIFPR of $\tilde{R}$ generated by Eq. (9).

Hence, according to Eq. (9) and Eq. (18), the consistency measure can be computed for the IVIFPR $\tilde{R}$.

Taking the little possibility for developing a perfectly consistent IVIFPR into account, the experts should also give a threshold α for the consistency degree, which is used to judge whether the IVIFPRs are acceptable consistent. If yes, proceed to the subsequent steps, otherwise repair IVIFPRs until they are of acceptable consistency. The repairing process for IVIFPRs is planned as follows.

Let t be the iteration times and ε be the step size, where $0\leq t\varepsilon\leq 1$. The inconsistency IVIFPR $\tilde{R}^{t}=\left(\tilde{r}_{ij}^{t}\right)_{n\times n}$ with $C_{\tilde{R}^{t}}<\alpha$ can be transformed into $\tilde{R}^{t+1}=\left(\tilde{r}_{ij}^{t+1}\right)_{n\times n}$ by the application of the iterative formulas below:

$$\underline{\mu}_{ij}^{t+1}=(\underline{\mu}_{ij}^{t})^{1-t\varepsilon}\cdot(\underline{\mu}_{ij}^{t*})^{t\varepsilon}, \tag{19}$$

$$\overline{\mu}_{ij}^{t+1}=(\overline{\mu}_{ij}^{t})^{1-t\varepsilon}\cdot(\overline{\mu}_{ij}^{t*})^{t\varepsilon}, \tag{20}$$

$$\underline{\nu}_{ij}^{t+1}=(\underline{\nu}_{ij}^{t})^{1-t\varepsilon}\cdot(\underline{\nu}_{ij}^{t*})^{t\varepsilon}, \tag{21}$$

$$\overline{\nu}_{ij}^{t+1}=(\overline{\nu}_{ij}^{t})^{1-t\varepsilon}\cdot(\overline{\nu}_{ij}^{t*})^{t\varepsilon}, \tag{22}$$

where $\tilde{R}^{t*}=\left(\tilde{r}_{ij}^{t*}\right)_{n\times n}=\left(\tilde{\mu}_{ij}^{t*},\tilde{\nu}_{ij}^{t*}\right)_{n\times n}$ being the corresponding multiplicative consistent IVIFPR of $\tilde{R}^{t}$ generated by Eq. (9), $i,j=1,2,\cdots,n.$

Theorem 14: The iteration process using Eqs. (19) - (22) is convergent.

Proof: We complete the iteration when the repaired IVIFPR $\tilde{R}^{t}$ meets acceptable consistency, i.e., $C_{\tilde{R}^{t}}\geq\alpha$. Let $\gamma=t\varepsilon$, then $\gamma\in[0,1]$ under the condition that $0\leq t\varepsilon\leq 1$. Suppose M to be the maximum iteration number and step size $\varepsilon=1/M$, according to Eqs. (19) - (22), after $t=M$ iterations of calculation, we can yield $\underline{\mu}_{ij}^{t+1}=\underline{\mu}_{ij}^{t*}$, $\overline{\mu}_{ij}^{t+1}=\overline{\mu}_{ij}^{t*}$, $\underline{\nu}_{ij}^{t+1}=\underline{\nu}_{ij}^{t*}$ and $\overline{\nu}_{ij}^{t+1}=\overline{\nu}_{ij}^{t*}$, i.e., $\tilde{R}^{t+1}=\tilde{R}^{t*}$. Since $\tilde{R}^{t*}$ is multiplicative consistent, $\tilde{R}^{t+1}$ should have the same property. Therefore, the iteration process is convergent. ■

V. Fractional Programming Model and the Procedure for Interval-valued Intuitionistic Fuzzy GDM

A. Fractional Programming Model for Interval-valued Intuitionistic Fuzzy GDM

Suppose in a GDM problem with IVIFPRs, $P=\{P_1,P_2,\cdots,P_n\}$ represents a set of alternatives, $E=\{E_1,E_2,\cdots,E_s\}$ represents the experts set and $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_s)^T$ is the weight vector of the experts, with the conditions that $\lambda_h\geq 0$ and $\sum_{h=1}^{s}\lambda_h=1$. After pairwise comparisons of alternatives, every expert can accomplish an IVIFPR $\tilde{R}^{(h)}=\left(\tilde{r}_{ij}^{(h)}\right)_{n\times n}(h=1,2,\cdots,s)$.

Considering the original IVIFPRs may not be perfectly consistent, the overall IVIF priority weights $\tilde{\varpi}_1,\tilde{\varpi}_2,\cdots,\tilde{\varpi}_n$

are used to establish a multiplicative consistent IVIFPR with Eq. (9). Inspired by Model 1, a fractional programming model will be constructed to identify the normalized collective IVIF priority weights $\tilde{\varpi}_1, \tilde{\varpi}_2, \cdots, \tilde{\varpi}_n$ with treating experts as a group. That is to say, the minimum deviations should exist between each initial IVIFPR $\tilde{R}^{(h)}$ given by the expert $E_h(h = 1, 2, \cdots, s)$ and the corresponding multiplicative consistent IVIFPR $\tilde{R}$. Thus, the overall deviation between $\tilde{R}^{(h)}$ and $\tilde{R}$ can be denoted as follows:

$$Min\ Z=\sum_{h=1}^{s}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_h(|\underline{\vartheta}_{ij}^{(h)}|+|\overline{\vartheta}_{ij}^{(h)}|+|\underline{\zeta}_{ij}^{(h)}|+|\overline{\zeta}_{ij}^{(h)}|). \quad (23)$$

Similarly, Eq. (23) is identical to

$$Min\ Z=\sum_{h=1}^{s}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_h(\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}+\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-} + \underline{\zeta}_{ij}^{(h)+} + \underline{\zeta}_{ij}^{(h)-} + \overline{\zeta}_{ij}^{(h)+} + \overline{\zeta}_{ij}^{(h)-}), \quad (24)$$

for $i, j = 1, 2, \cdots, n; i < j; h = 1, 2, \cdots, s$. Then, Model 2 can be established.

For the equation $\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}} - \underline{\mu}_{ij}^{(h)} - \underline{\vartheta}_{ij}^{(h)+} + \underline{\vartheta}_{ij}^{(h)-} = 0,$, let the both sides multiply λ_h, it yields

$$\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}\lambda_h - \lambda_h\underline{\mu}_{ij}^{(h)} - \lambda_h\underline{\vartheta}_{ij}^{(h)+} + \lambda_h\underline{\vartheta}_{ij}^{(h)-} = 0. \quad (25)$$

Since $\sum_{h=1}^{s}\lambda_h = 1$, summing all the s equations, we can obtain

$$\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}} -\sum_{h=1}^{s}\lambda_h\underline{\mu}_{ij}^{(h)} -\sum_{h=1}^{s}\lambda_h\underline{\vartheta}_{ij}^{(h)+} +\sum_{h=1}^{s}\lambda_h\underline{\vartheta}_{ij}^{(h)-} = 0.$$

Similarly,

$$\frac{\overline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}} -\sum_{h=1}^{s}\lambda_h\overline{\mu}_{ij}^{(h)} -\sum_{h=1}^{s}\lambda_h\overline{\vartheta}_{ij}^{(h)+} +\sum_{h=1}^{s}\lambda_h\overline{\vartheta}_{ij}^{(h)-} = 0,$$

$$\frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}} -\sum_{h=1}^{s}\lambda_h\underline{\nu}_{ij}^{(h)} -\sum_{h=1}^{s}\lambda_h\underline{\zeta}_{ij}^{(h)+} +\sum_{h=1}^{s}\lambda_h\underline{\zeta}_{ij}^{(h)-} = 0,$$

$$\frac{\overline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}} -\sum_{h=1}^{s}\lambda_h\overline{\nu}_{ij}^{(h)} -\sum_{h=1}^{s}\lambda_h\overline{\zeta}_{ij}^{(h)+} +\sum_{h=1}^{s}\lambda_h\overline{\zeta}_{ij}^{(h)-} = 0.$$

Let $\underline{\vartheta}_{ij}^{-} = \sum_{h=1}^{s}\lambda_h\underline{\vartheta}_{ij}^{(h)-}$, $\underline{\vartheta}_{ij}^{+} = \sum_{h=1}^{s}\lambda_h\underline{\vartheta}_{ij}^{(h)+}$, $\overline{\vartheta}_{ij}^{-} = \sum_{h=1}^{s}\lambda_h\overline{\vartheta}_{ij}^{(h)-}$, $\overline{\vartheta}_{ij}^{+} = \sum_{h=1}^{s}\lambda_h\overline{\vartheta}_{ij}^{(h)+}$, $\underline{\zeta}_{ij}^{-} = \sum_{h=1}^{s}\lambda_h\underline{\zeta}_{ij}^{(h)-}$, $\underline{\zeta}_{ij}^{+} = \sum_{h=1}^{s}\lambda_h\underline{\zeta}_{ij}^{(h)+}$, $\overline{\zeta}_{ij}^{-} = \sum_{h=1}^{s}\lambda_h\overline{\zeta}_{ij}^{(h)-}$, and $\overline{\zeta}_{ij}^{+} = \sum_{h=1}^{s}\lambda_h\overline{\zeta}_{ij}^{(h)+}$. Then, Model 2 can be rewritten as Model 3.

Utilizing LINGO, we solve the model and yield the overall IVIF priority weights $\tilde{\varpi}_i = \left(\left[\underline{\varpi}_i^{\mu}, \overline{\varpi}_i^{\mu}\right], \left[\underline{\varpi}_i^{\nu}, \overline{\varpi}_i^{\nu}\right]\right)$ $(i = 1, 2, \cdots, n)$. The ranking order can be further derived.

B. Procedure For GDM with IVIFPRs

Based on all the above analyses, a step by step procedure for GDM with IVIFPRs is described as follows.

Algorithm:

Step 1: Invite an expert group $E = \{E_1, E_2, \cdots, E_h, \cdots, E_s\}$ to give their preferences over alternatives $P_i(i = 1, 2, \cdots, n)$. The weighting vector of experts is $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)^T$ with $\lambda_h > 0$, $(h = 1, 2, \cdots, s)$, and $\sum_{h=1}^{s}\lambda_h = 1$. The consistency threshold α is determined in advance by experts and DMs. Let $\tilde{R}^{(t)(h)} = \tilde{R}^{(h)}$, and set t=1.

Step 2: According to the programming Model 1, compute the normalized IVIF priority vector $\tilde{\varpi}^{(t)(h)} = (\tilde{\varpi}_1^{(t)(h)}, \tilde{\varpi}_2^{(t)(h)}, \cdots, \tilde{\varpi}_n^{(t)(h)})^T$ for the IVIFPR $\tilde{R}^{(t)(h)}$, then build the corresponding multiplicative consistent IVIFPR $\tilde{R}^{(t)(h)^*} = \left(\tilde{r}_{ij}^{(t)(h)^*}\right)_{n\times n}$ by Eq. (9).

Step 3: Calculate the consistency degree $C_{\tilde{R}^{(t)(h)}}$ via Eq. (18), and then judge the acceptable consistency of each IVIFPR $\tilde{R}^{(t)(h)}$ by comparing with the consistency threshold α. If $C_{\tilde{R}^{(t)(h)}} \geq \alpha$, then $\tilde{R}^{(t)(h)}$ is acceptable consistent, go to Step 5; otherwise, go to the next step.

Step 4: Repair the inconsistent IVIFPR. Let the parameter $\varepsilon \in [0, 1]$ and use the following formulas to iterate $\tilde{R}^{(t)(h)}$ to $\tilde{R}^{(t+1)(h)}$ with $\tilde{R}^{(t+1)(h)} = \left(\tilde{r}_{ij}^{(t+1)(h)}\right)_{n\times n} = \left(\left(\tilde{\mu}_{ij}^{(t+1)(h)}, \tilde{\nu}_{ij}^{(t+1)(h)}\right)\right)_{n\times n}$, where

$$\underline{\mu}_{ij}^{(t+1)(h)} = (\underline{\mu}_{ij}^{(t)(h)})^{1-t\varepsilon} \cdot (\underline{\mu}_{ij}^{(t)(h)*})^{t\varepsilon}, \quad (26)$$

$$\overline{\mu}_{ij}^{(t+1)(h)} = (\overline{\mu}_{ij}^{(t)(h)})^{1-t\varepsilon} \cdot (\overline{\mu}_{ij}^{(t)(h)*})^{t\varepsilon}, \quad (27)$$

$$\underline{\nu}_{ij}^{(t+1)(h)} = (\underline{\nu}_{ij}^{(t)(h)})^{1-t\varepsilon} \cdot (\underline{\nu}_{ij}^{(t)(h)*})^{t\varepsilon}, \quad (28)$$

$$\overline{\nu}_{ij}^{(t+1)(h)} = (\overline{\nu}_{ij}^{(t)(h)})^{1-t\varepsilon} \cdot (\overline{\nu}_{ij}^{(t)(h)*})^{t\varepsilon}, \quad (29)$$

for $i, j = 1, 2, \cdots, n$. Let $t = t + 1$. Go to step 2.

Step 5: Establish a fractional programming model by Model 3. According to LINGO, the model can be solved and the normalized collective IVIF priority weight vector $\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, \cdots, \tilde{\varpi}_n)^T$ is retrived.

Step 6: Using Eq. (2) and Eq. (3) in Definition 4 to generate the rank of alternatives, then find the best alternative P^*. End.

VI. Numerical Example

In this section, the algorithm is applied to a quoted example from [37]. Then we compare our approach with two previous methods in [37] and [36] to illustrate the advantages of our method.

A. A Practical Example of Virtual Enterprise Partner Selection

To facilitate health reimbursement management, AHEAD Information Technology Co., LTD (AHEAD for short) plans to establish a new-type rural cooperative medical care management information system. The system is made up of a software system and hardware devices with integrated chips. Since software systems can be developed by itself, a partner is in need to produce the hardware device.Four partners $\{P_1, P_2, P_3, P_4\}$ remain for further evaluation. Three experts $\{E_1, E_2, E_3\}$, whose weighting vector are established as $\lambda = (1/3, 1/3, 1/3)^T$, are invited to estimate these four partners. After comparing the partners in pairs, experts give their IVIFPRs $\tilde{R}^{(1)}$, $\tilde{R}^{(2)}$ and $\tilde{R}^{(3)}$ as follows.

Next, this example will be solved with the application of our approach.

Step 1: Take the consistency threshold $\alpha = 0.9$.

Model 2 $$Min\ f=\sum_{h=1}^{s}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_h(\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}+\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-}+\underline{\zeta}_{ij}^{(h)+}+\underline{\zeta}_{ij}^{(h)-}+\overline{\zeta}_{ij}^{(h)+}+\overline{\zeta}_{ij}^{(h)-})$$

$$s.t.\begin{cases}\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\mu}_{ij}^{(h)}-\underline{\vartheta}_{ij}^{(h)+}+\underline{\vartheta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\overline{\varpi}_i^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}-\overline{\mu}_{ij}^{(h)}-\overline{\vartheta}_{ij}^{(h)+}+\overline{\vartheta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\underline{\nu}_{ij}^{(h)}-\underline{\zeta}_{ij}^{(h)+}+\underline{\zeta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \frac{\overline{\varpi}_j^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}-\overline{\nu}_{ij}^{(h)}-\overline{\zeta}_{ij}^{(h)+}+\overline{\zeta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \left[\underline{\varpi}_i^{\mu},\overline{\varpi}_i^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_i^{\nu},\overline{\varpi}_i^{\nu}\right]\subseteq[0,1],\ \overline{\varpi}_i^{\mu}+\overline{\varpi}_i^{\nu}\le 1,\ \ i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\mu}\le\underline{\varpi}_i^{\nu},\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\nu}\le\underline{\varpi}_i^{\mu}+n-2,\ \ i=1,2,\cdots,n\\ \underline{\vartheta}_{ij}^{(h)+},\ \underline{\vartheta}_{ij}^{(h)-},\ \overline{\vartheta}_{ij}^{(h)+},\ \overline{\vartheta}_{ij}^{(h)-}\ge 0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \underline{\zeta}_{ij}^{(h)+},\ \underline{\zeta}_{ij}^{(h)-},\ \overline{\zeta}_{ij}^{(h)+},\ \overline{\zeta}_{ij}^{(h)-}\ge 0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \underline{\vartheta}_{ij}^{(h)+}\cdot\underline{\vartheta}_{ij}^{(h)-}=0,\ \overline{\vartheta}_{ij}^{(h)+}\cdot\overline{\vartheta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\\ \underline{\zeta}_{ij}^{(h)+}\cdot\underline{\zeta}_{ij}^{(h)-}=0,\ \overline{\zeta}_{ij}^{(h)+}\cdot\overline{\zeta}_{ij}^{(h)-}=0,\ \ i,j=1,2,\cdots,n;i<j;\ h=1,2,\cdots,s\end{cases}$$

Model 3 $$Min\ f=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\underline{\vartheta}_{ij}^{+}+\underline{\vartheta}_{ij}^{-}+\overline{\vartheta}_{ij}^{+}+\overline{\vartheta}_{ij}^{-}+\underline{\zeta}_{ij}^{+}+\underline{\zeta}_{ij}^{-}+\overline{\zeta}_{ij}^{+}+\overline{\zeta}_{ij}^{-})$$

$$s.t.\begin{cases}\frac{\underline{\varpi}_i^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\sum_{h=1}^{s}\lambda_h\underline{\mu}_{ij}-\underline{\vartheta}_{ij}^{+}+\underline{\vartheta}_{ij}^{-}=0,\ \ i,j=1,2,\cdots,n;i<j\\ \frac{\overline{\varpi}_i^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}-\sum_{h=1}^{s}\lambda_h\overline{\mu}_{ij}-\overline{\vartheta}_{ij}^{+}+\overline{\vartheta}_{ij}^{-}=0,\ \ i,j=1,2,\cdots,n;i<j\\ \frac{\underline{\varpi}_j^{\mu}}{2-\overline{\varpi}_i^{\nu}-\overline{\varpi}_j^{\nu}}-\sum_{h=1}^{s}\lambda_h\underline{\nu}_{ij}-\underline{\zeta}_{ij}^{+}+\underline{\zeta}_{ij}^{-}=0,\ \ i,j=1,2,\cdots,n;i<j\\ \frac{\overline{\varpi}_j^{\mu}}{2-\underline{\varpi}_i^{\nu}-\underline{\varpi}_j^{\nu}}-\sum_{h=1}^{s}\lambda_h\overline{\nu}_{ij}-\overline{\zeta}_{ij}^{+}+\overline{\zeta}_{ij}^{-}=0,\ \ i,j=1,2,\cdots,n;i<j\\ \left[\underline{\varpi}_i^{\mu},\overline{\varpi}_i^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_i^{\nu},\overline{\varpi}_i^{\nu}\right]\subseteq[0,1],\ \overline{\varpi}_i^{\mu}+\overline{\varpi}_i^{\nu}\le 1,\ \ i=1,2,\cdots,n\\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\mu}\le\underline{\varpi}_i^{\nu},\ \sum_{j=1,j\neq i}^{n}\overline{\varpi}_j^{\nu}\le\underline{\varpi}_i^{\mu}+n-2,\ \ i=1,2,\cdots,n\\ \underline{\vartheta}_{ij}^{+},\ \underline{\vartheta}_{ij}^{-},\ \overline{\vartheta}_{ij}^{+},\ \overline{\vartheta}_{ij}^{-},\ \underline{\zeta}_{ij}^{+},\ \underline{\zeta}_{ij}^{-},\ \overline{\zeta}_{ij}^{+},\ \overline{\zeta}_{ij}^{-}\ge 0,\ \ i,j=1,2,\cdots,n;i<j\end{cases}$$

$$\tilde{R}^{(1)}=\begin{pmatrix}([0.5000,0.5000],[0.5000,0.5000]) & ([0.1400,0.3000],[0.6500,0.7000]) & ([0.6200,0.6500],[0.1200,0.1500]) & ([0.8300,0.9000],[0.0400,0.1000])\\ ([0.6500,0.7000],[0.1400,0.3000]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.7300,0.7800],[0.0400,0.0500]) & ([0.9000,0.9500],[0.0200,0.0220])\\ ([0.1200,0.1500],[0.6200,0.6500]) & ([0.0400,0.0500],[0.7300,0.7800]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.5500,0.6000],[0.1000,0.2000])\\ ([0.0400,0.1000],[0.8300,0.9000]) & ([0.0200,0.0220],[0.9000,0.9500]) & ([0.1000,0.2000],[0.5500,0.6000]) & ([0.5000,0.5000],[0.5000,0.5000])\end{pmatrix}$$

$$\tilde{R}^{(2)}=\begin{pmatrix}([0.5000,0.5000],[0.5000,0.5000]) & ([0.6500,0.7000],[0.0500,0.3300]) & ([0.6200,0.6500],[0.0360,0.3500]) & ([0.7800,0.8200],[0.0060,0.1400])\\ ([0.0500,0.3300],[0.6500,0.7000]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.5000,0.5500],[0.3000,0.4000]) & ([0.7500,0.8000],[0.0800,0.1800])\\ ([0.0360,0.3500],[0.6200,0.6500]) & ([0.3000,0.4000],[0.5000,0.5500]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.7000,0.7500],[0.1000,0.2000])\\ ([0.0060,0.1400],[0.7800,0.8200]) & ([0.0800,0.1800],[0.7500,0.8000]) & ([0.1000,0.2000],[0.7000,0.7500]) & ([0.5000,0.5000],[0.5000,0.5000])\end{pmatrix}$$

$$\tilde{R}^{(3)}=\begin{pmatrix}([0.5000,0.5000],[0.5000,0.5000]) & ([0.4500,0.5000],[0.1000,0.2000]) & ([0.5000,0.6000],[0.1100,0.1700]) & ([0.8000,0.8500],[0.1000,0.1200])\\ ([0.1000,0.2000],[0.4500,0.5000]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.7500,0.8000],[0.1500,0.2000]) & ([0.6500,0.7000],[0.1000,0.1500])\\ ([0.1100,0.1700],[0.5000,0.6000]) & ([0.1500,0.2000],[0.7500,0.8000]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.8000,0.8500],[0.0500,0.1000])\\ ([0.1000,0.1200],[0.8000,0.8500]) & ([0.1000,0.1500],[0.6500,0.7000]) & ([0.0500,0.1000],[0.8000,0.8500]) & ([0.5000,0.5000],[0.5000,0.5000])\end{pmatrix}$$

Step 2: According to Model 1, three fractional programming models are constructed for IVIFPRs above (we only show the first model as Model 4 here).

Use LINGO to solve the models and acquire the underlying normalized IVIF priority weight vectors for the individual IVIFPRs as follows:

$$\tilde{\varpi}^{(1)}=\big(([0.2441,0.2646],[0.7281,0.7281]),([0.5840,0.5912],[0.3734,0.3734]),([0.0793,0.0865],[0.8781,0.8781]),([0.0144,0.0223],[0.9423,0.9777])\big)^T.$$

$$\tilde{\varpi}^{(2)}=\big(([0.4722,0.4722],[0.4962,0.4962]),([0.2375,0.2534],[0.7162,0.7361]),([0.1847,0.1900],[0.7880,0.7889]),([0.0253,0.0528],[0.9211,0.9472])\big)^T.$$

$$\tilde{\varpi}^{(3)}=\big(([0.4933,0.4991],[0.3920,0.3920]),([0.2755,0.2813],[0.6098,0.6098]),([0.0932,0.0990],[0.8154,0.8952]),([0.0058,0.0116],[0.9500,0.9884])\big)^T.$$

Correspondingly, the multiplicative consistent IVIFPRs $\tilde{R}^{*}$ can be generated as $\tilde{R}^{(1)*}$, $\tilde{R}^{(2)*}$ and $\tilde{R}^{(3)*}$ by Eq. (9).

Step 3: Via Eq. (18), the consistency degree of these three IVIFPRs are calculated as $C_{\tilde{R}^{(1)}}=0.9715$, $C_{\tilde{R}^{(2)}}=0.9503$, $C_{\tilde{R}^{(3)}}=0.8727$. Note that the consistency threshold α = 0.9, IVIFPRs $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$ are both acceptable consistent. Since $C_{\tilde{R}^{(3)}}=0.8727<0.9$, then we repair the third IVIFPR $\tilde{R}^{(3)}$ because it is inconsistent.

Step 4: Let $t=1$ and $\tilde{R}^{(1)(3)}=\tilde{R}^{(3)}$; then, according to Eqs. (26)-(29) with the parameter $\varepsilon=0.2$, we can obtain $\tilde{R}^{(2)(3)}$ as follows.

Model 4

$$
\begin{aligned}
Min\ f = (&\underline{\vartheta}_{12}^{(1)+} + \underline{\vartheta}_{12}^{(1)-} + \overline{\vartheta}_{12}^{(1)+} + \overline{\vartheta}_{12}^{(1)-} + \underline{\zeta}_{12}^{(1)+} + \underline{\zeta}_{12}^{(1)-} + \overline{\zeta}_{12}^{(1)+} + \overline{\zeta}_{12}^{(1)-} + \underline{\vartheta}_{13}^{(1)+} + \underline{\vartheta}_{13}^{(1)-} + \overline{\vartheta}_{13}^{(1)+} + \overline{\vartheta}_{13}^{(1)-} \\
&+ \underline{\zeta}_{13}^{(1)+} + \underline{\zeta}_{13}^{(1)-} + \overline{\zeta}_{13}^{(1)+} + \overline{\zeta}_{13}^{(1)-} + \underline{\vartheta}_{14}^{(1)+} + \underline{\vartheta}_{14}^{(1)-} + \overline{\vartheta}_{14}^{(1)+} + \overline{\vartheta}_{14}^{(1)-} + \underline{\zeta}_{14}^{(1)+} + \underline{\zeta}_{14}^{(1)-} + \overline{\zeta}_{14}^{(1)+} + \overline{\zeta}_{14}^{(1)-} \\
&+ \underline{\vartheta}_{23}^{(1)+} + \underline{\vartheta}_{23}^{(1)-} + \overline{\vartheta}_{23}^{(1)+} + \overline{\vartheta}_{23}^{(1)-} + \underline{\zeta}_{23}^{(1)+} + \underline{\zeta}_{23}^{(1)-} + \overline{\zeta}_{23}^{(1)+} + \overline{\zeta}_{23}^{(1)-} + \underline{\vartheta}_{24}^{(1)+} + \underline{\vartheta}_{24}^{(1)-} + \overline{\vartheta}_{24}^{(1)+} + \overline{\vartheta}_{24}^{(1)-} \\
&+ \underline{\zeta}_{24}^{(1)+} + \underline{\zeta}_{24}^{(1)-} + \overline{\zeta}_{24}^{(1)+} + \overline{\zeta}_{24}^{(1)-} + \underline{\vartheta}_{34}^{(1)+} + \underline{\vartheta}_{34}^{(1)-} + \overline{\vartheta}_{34}^{(1)+} + \overline{\vartheta}_{34}^{(1)-} + \underline{\zeta}_{34}^{(1)+} + \underline{\zeta}_{34}^{(1)-} + \overline{\zeta}_{34}^{(1)+} + \overline{\zeta}_{34}^{(1)-})
\end{aligned}
$$

$$
s.t.\begin{cases}
\frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_2^{\nu}} - 0.14 - \underline{\vartheta}_{12}^{(1)+} + \underline{\vartheta}_{12}^{(1)-} = 0,\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_3^{\nu}} - 0.62 - \underline{\vartheta}_{13}^{(1)+} + \underline{\vartheta}_{13}^{(1)-} = 0,\\
\frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_4^{\nu}} - 0.83 - \underline{\vartheta}_{14}^{(1)+} + \underline{\vartheta}_{14}^{(1)-} = 0,\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_3^{\nu}} - 0.73 - \underline{\vartheta}_{23}^{(1)+} + \underline{\vartheta}_{23}^{(1)-} = 0,\\
\frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_4^{\nu}} - 0.9 - \underline{\vartheta}_{24}^{(1)+} + \underline{\vartheta}_{24}^{(1)-} = 0,\ \frac{\underline{\varpi}_3^{\mu}}{2-\overline{\varpi}_3^{\nu}-\overline{\varpi}_4^{\nu}} - 0.55 - \underline{\vartheta}_{34}^{(1)+} + \underline{\vartheta}_{34}^{(1)-} = 0,\\
\frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_2^{\nu}} - 0.3 - \overline{\vartheta}_{12}^{(1)+} + \overline{\vartheta}_{12}^{(1)-} = 0,\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_3^{\nu}} - 0.65 - \overline{\vartheta}_{13}^{(1)+} + \overline{\vartheta}_{13}^{(1)-} = 0,\\
\frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_4^{\nu}} - 0.9 - \overline{\vartheta}_{14}^{(1)+} + \overline{\vartheta}_{14}^{(1)-} = 0,\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_3^{\nu}} - 0.78 - \overline{\vartheta}_{23}^{(1)+} + \overline{\vartheta}_{23}^{(1)-} = 0,\\
\frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_4^{\nu}} - 0.95 - \overline{\vartheta}_{24}^{(1)+} + \overline{\vartheta}_{24}^{(1)-} = 0,\ \frac{\overline{\varpi}_3^{\mu}}{2-\underline{\varpi}_3^{\nu}-\underline{\varpi}_4^{\nu}} - 0.6 - \overline{\vartheta}_{34}^{(1)+} + \overline{\vartheta}_{34}^{(1)-} = 0,\\
\frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_2^{\nu}} - 0.65 - \underline{\zeta}_{12}^{(1)+} + \underline{\zeta}_{12}^{(1)-} = 0,\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_3^{\nu}} - 0.12 - \underline{\zeta}_{13}^{(1)+} + \underline{\zeta}_{13}^{(1)-} = 0,\\
\frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_4^{\nu}} - 0.04 - \underline{\zeta}_{14}^{(1)+} + \underline{\zeta}_{14}^{(1)-} = 0,\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_3^{\nu}} - 0.04 - \underline{\zeta}_{23}^{(1)+} + \underline{\zeta}_{23}^{(1)-} = 0,\\
\frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_4^{\nu}} - 0.02 - \underline{\zeta}_{24}^{(1)+} + \underline{\zeta}_{24}^{(1)-} = 0,\ \frac{\underline{\varpi}_3^{\mu}}{2-\overline{\varpi}_3^{\nu}-\overline{\varpi}_4^{\nu}} - 0.1 - \underline{\zeta}_{34}^{(1)+} + \underline{\zeta}_{34}^{(1)-} = 0,\\
\frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_2^{\nu}} - 0.7 - \overline{\zeta}_{12}^{(1)+} + \overline{\zeta}_{12}^{(1)-} = 0,\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_3^{\nu}} - 0.15 - \overline{\zeta}_{13}^{(1)+} + \overline{\zeta}_{13}^{(1)-} = 0,\\
\frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_4^{\nu}} - 0.1 - \overline{\zeta}_{14}^{(1)+} + \overline{\zeta}_{14}^{(1)-} = 0,\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_3^{\nu}} - 0.05 - \overline{\zeta}_{23}^{(1)+} + \overline{\zeta}_{23}^{(1)-} = 0,\\
\frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_4^{\nu}} - 0.022 - \overline{\zeta}_{24}^{(1)+} + \overline{\zeta}_{24}^{(1)-} = 0,\ \frac{\overline{\varpi}_3^{\mu}}{2-\underline{\varpi}_3^{\nu}-\underline{\varpi}_4^{\nu}} - 0.2 - \overline{\zeta}_{34}^{(1)+} + \overline{\zeta}_{34}^{(1)-} = 0,\\
\left[\underline{\varpi}_1^{\mu}, \overline{\varpi}_1^{\mu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_2^{\mu}, \overline{\varpi}_2^{\mu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_3^{\mu}, \overline{\varpi}_3^{\mu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_4^{\mu}, \overline{\varpi}_4^{\mu}\right] \subseteq [0,1],\\
\left[\underline{\varpi}_1^{\nu}, \overline{\varpi}_1^{\nu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_2^{\nu}, \overline{\varpi}_2^{\nu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_3^{\nu}, \overline{\varpi}_3^{\nu}\right] \subseteq [0,1],\ \left[\underline{\varpi}_4^{\nu}, \overline{\varpi}_4^{\nu}\right] \subseteq [0,1],\\
\overline{\varpi}_1^{\mu} + \overline{\varpi}_1^{\nu} \le 1,\ \overline{\varpi}_2^{\mu} + \overline{\varpi}_2^{\nu} \le 1,\ \overline{\varpi}_3^{\mu} + \overline{\varpi}_3^{\nu} \le 1,\ \overline{\varpi}_4^{\mu} + \overline{\varpi}_4^{\nu} \le 1,\\
\overline{\varpi}_1^{\mu} + \overline{\varpi}_2^{\mu} + \overline{\varpi}_3^{\mu} \le \underline{\varpi}_4^{\nu},\ \overline{\varpi}_1^{\mu} + \overline{\varpi}_2^{\mu} + \overline{\varpi}_4^{\mu} \le \underline{\varpi}_3^{\nu}, \overline{\varpi}_1^{\mu} + \overline{\varpi}_3^{\mu} + \overline{\varpi}_4^{\mu} \le \underline{\varpi}_2^{\nu},\ \overline{\varpi}_2^{\mu} + \overline{\varpi}_3^{\mu} + \overline{\varpi}_4^{\mu} \le \underline{\varpi}_1^{\nu},\\
\underline{\varpi}_4^{\mu} + 2 \ge \overline{\varpi}_1^{\nu} + \overline{\varpi}_2^{\nu} + \overline{\varpi}_3^{\nu},\ \underline{\varpi}_3^{\mu} + 2 \ge \overline{\varpi}_1^{\nu} + \overline{\varpi}_2^{\nu} + \overline{\varpi}_4^{\nu},\\
\underline{\varpi}_2^{\mu} + 2 \ge \overline{\varpi}_1^{\nu} + \overline{\varpi}_3^{\nu} + \overline{\varpi}_4^{\nu},\ \underline{\varpi}_1^{\mu} + 2 \ge \overline{\varpi}_2^{\nu} + \overline{\varpi}_3^{\nu} + \overline{\varpi}_4^{\nu},\\
\underline{\vartheta}_{12}^{(1)+} \ge 0,\ \underline{\vartheta}_{12}^{(1)-} \ge 0,\ \overline{\vartheta}_{12}^{(1)+} \ge 0,\ \overline{\vartheta}_{12}^{(1)-} \ge 0,\ \underline{\zeta}_{12}^{(1)+} \ge 0,\ \underline{\zeta}_{12}^{(1)-} \ge 0,\ \overline{\zeta}_{12}^{(1)+} \ge 0,\ \overline{\xi}_{12}^{(1)-} \ge 0,\\
\underline{\vartheta}_{13}^{(1)+} \ge 0,\ \underline{\vartheta}_{13}^{(1)-} \ge 0,\ \overline{\vartheta}_{13}^{(1)+} \ge 0,\ \overline{\vartheta}_{13}^{(1)-} \ge 0,\ \underline{\zeta}_{13}^{(1)+} \ge 0,\ \underline{\zeta}_{13}^{(1)-} \ge 0,\ \overline{\zeta}_{13}^{(1)+} \ge 0,\ \overline{\zeta}_{13}^{(1)-} \ge 0,\\
\underline{\vartheta}_{14}^{(1)+} \ge 0,\ \underline{\vartheta}_{14}^{(1)-} \ge 0,\ \overline{\vartheta}_{14}^{(1)+} \ge 0,\ \overline{\vartheta}_{14}^{(1)-} \ge 0,\ \underline{\zeta}_{14}^{(1)+} \ge 0,\ \underline{\zeta}_{14}^{(1)-} \ge 0,\ \overline{\zeta}_{14}^{(1)+} \ge 0,\ \overline{\zeta}_{14}^{(1)-} \ge 0,\\
\underline{\vartheta}_{23}^{(1)+} \ge 0,\ \underline{\vartheta}_{23}^{(1)-} \ge 0,\ \overline{\vartheta}_{23}^{(1)+} \ge 0,\ \overline{\vartheta}_{23}^{(1)-} \ge 0,\ \underline{\zeta}_{23}^{(1)+} \ge 0,\ \underline{\zeta}_{23}^{(1)-} \ge 0,\ \overline{\zeta}_{23}^{(1)+} \ge 0,\ \overline{\zeta}_{23}^{(1)-} \ge 0,\\
\underline{\vartheta}_{24}^{(1)+} \ge 0,\ \underline{\vartheta}_{24}^{(1)-} \ge 0,\ \overline{\vartheta}_{24}^{(1)+} \ge 0,\ \overline{\vartheta}_{24}^{(1)-} \ge 0,\ \underline{\zeta}_{24}^{(1)+} \ge 0,\ \underline{\zeta}_{24}^{(1)-} \ge 0,\ \overline{\zeta}_{24}^{(1)+} \ge 0,\ \overline{\zeta}_{24}^{(1)-} \ge 0,\\
\underline{\vartheta}_{34}^{(1)+} \ge 0,\ \underline{\vartheta}_{34}^{(1)-} \ge 0,\ \overline{\vartheta}_{34}^{(1)+} \ge 0,\ \overline{\vartheta}_{34}^{(1)-} \ge 0,\ \underline{\zeta}_{34}^{(1)+} \ge 0,\ \underline{\zeta}_{34}^{(1)-} \ge 0,\ \overline{\zeta}_{34}^{(1)+} \ge 0,\ \overline{\zeta}_{34}^{(1)-} \ge 0,\\
\underline{\vartheta}_{12}^{(1)+} \cdot \underline{\vartheta}_{12}^{(1)-} = 0,\ \overline{\vartheta}_{12}^{(1)+} \cdot \overline{\vartheta}_{12}^{(1)-} = 0,\ \underline{\zeta}_{12}^{(1)+} \cdot \underline{\zeta}_{12}^{(1)-} = 0,\ \overline{\zeta}_{12}^{(1)+} \cdot \overline{\zeta}_{12}^{(1)-} = 0,\\
\underline{\vartheta}_{13}^{(1)+} \cdot \underline{\vartheta}_{13}^{(1)-} = 0,\ \overline{\vartheta}_{13}^{(1)+} \cdot \overline{\vartheta}_{13}^{(1)-} = 0,\ \underline{\zeta}_{13}^{(1)+} \cdot \underline{\zeta}_{13}^{(1)-} = 0,\ \overline{\zeta}_{13}^{(1)+} \cdot \overline{\zeta}_{13}^{(1)-} = 0,\\
\underline{\vartheta}_{14}^{(1)+} \cdot \underline{\vartheta}_{14}^{(1)-} = 0,\ \overline{\vartheta}_{14}^{(1)+} \cdot \overline{\vartheta}_{14}^{(1)-} = 0,\ \underline{\zeta}_{14}^{(1)+} \cdot \underline{\zeta}_{14}^{(1)-} = 0,\ \overline{\zeta}_{14}^{(1)+} \cdot \overline{\zeta}_{14}^{(1)-} = 0,\\
\underline{\vartheta}_{23}^{(1)+} \cdot \underline{\vartheta}_{23}^{(1)-} = 0,\ \overline{\vartheta}_{23}^{(1)+} \cdot \overline{\vartheta}_{23}^{(1)-} = 0,\ \underline{\zeta}_{23}^{(1)+} \cdot \underline{\zeta}_{23}^{(1)-} = 0,\ \overline{\zeta}_{23}^{(1)+} \cdot \overline{\zeta}_{23}^{(1)-} = 0,\\
\underline{\vartheta}_{24}^{(1)+} \cdot \underline{\vartheta}_{24}^{(1)-} = 0,\ \overline{\vartheta}_{24}^{(1)+} \cdot \overline{\vartheta}_{24}^{(1)-} = 0,\ \underline{\zeta}_{24}^{(1)+} \cdot \underline{\zeta}_{24}^{(1)-} = 0,\ \overline{\zeta}_{24}^{(1)+} \cdot \overline{\zeta}_{24}^{(1)-} = 0,\\
\underline{\vartheta}_{34}^{(1)+} \cdot \underline{\vartheta}_{34}^{(1)-} = 0,\ \overline{\vartheta}_{34}^{(1)+} \cdot \overline{\vartheta}_{34}^{(1)-} = 0,\ \underline{\zeta}_{34}^{(1)+} \cdot \underline{\zeta}_{34}^{(1)-} = 0,\ \overline{\zeta}_{34}^{(1)+} \cdot \overline{\zeta}_{34}^{(1)-} = 0.
\end{cases}
$$

$$
\tilde{R}^{(1)*} = \begin{pmatrix}
([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.2717, 0.2946], [0.6500, 0.6580]) & ([0.6200, 0.6721], [0.2014, 0.2197]) & ([0.8300, 0.8997], [0.0490, 0.0758]) \\
([0.6500, 0.6580], [0.2717, 0.2946]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.7802, 0.7898], [0.1060, 0.1156]) & ([0.9000, 0.9111], [0.0222, 0.0343]) \\
([0.2014, 0.2197], [0.6200, 0.6721]) & ([0.1060, 0.1156], [0.7802, 0.7898]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.5500, 0.6000], [0.1000, 0.1545]) \\
([0.0490, 0.0758], [0.8300, 0.8997]) & ([0.0222, 0.0343], [0.9000, 0.9111]) & ([0.1000, 0.1545], [0.5500, 0.6000]) & ([0.5000, 0.5000], [0.5000, 0.5000])
\end{pmatrix}
$$

$$
\tilde{R}^{(2)*} = \begin{pmatrix}
([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.6150, 0.6150], [0.3094, 0.3300]) & ([0.6604, 0.6604], [0.2584, 0.2658]) & ([0.8483, 0.8483], [0.0455, 0.0949]) \\
([0.3094, 0.3300], [0.6150, 0.6150]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.5000, 0.5333], [0.3889, 0.4000]) & ([0.8000, 0.9019], [0.0799, 0.1667]) \\
([0.2584, 0.2658], [0.6604, 0.6604]) & ([0.3889, 0.4000], [0.5000, 0.5333]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.7000, 0.7200], [0.0959, 0.2001]) \\
([0.0455, 0.0949], [0.8483, 0.8483]) & ([0.0799, 0.1667], [0.8000, 0.9019]) & ([0.0959, 0.2001], [0.7000, 0.7200]) & ([0.5000, 0.5000], [0.5000, 0.5000])
\end{pmatrix}
$$

$$
\tilde{R}^{(3)*} = \begin{pmatrix}
([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.4942, 0.5000], [0.2760, 0.2818]) & ([0.6920, 0.7002], [0.1307, 0.1389]) & ([0.0094, 0.0187], [0.0455, 0.0949]) \\
([0.2760, 0.2818], [0.4942, 0.5000]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.5565, 0.5682], [0.1883, 0.2000]) & ([0.6855, 0.7000], [0.0144, 0.0289]) \\
([0.1307, 0.1389], [0.6920, 0.7002]) & ([0.1883, 0.2000], [0.5565, 0.5682]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.8001, 0.8499], [0.0498, 0.0996]) \\
([0.0094, 0.0187], [0.0455, 0.0949]) & ([0.0144, 0.0289], [0.6855, 0.7000]) & ([0.0498, 0.0996], [0.8001, 0.8499]) & ([0.5000, 0.5000], [0.5000, 0.5000])
\end{pmatrix}
$$

$$\tilde{R}^{(2)(3)} = \begin{pmatrix} ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.4585, 0.5000], [0.1225, 0.2142]) & ([0.5336, 0.6188], [0.1139, 0.1633]) & ([0.3289, 0.3962], [0.0623, 0.0827]) \\ ([0.1225, 0.2142], [0.4585, 0.5000]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.7065, 0.7471], [0.1570, 0.2000]) & ([0.6569, 0.7000], [0.0679, 0.1079]) \\ ([0.1139, 0.1633], [0.5336, 0.6188]) & ([0.1570, 0.2000], [0.7065, 0.7471]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.8000, 0.8500], [0.0500, 0.0999]) \\ ([0.0623, 0.0827], [0.3289, 0.3962]) & ([0.0679, 0.1079], [0.6569, 0.7000]) & ([0.0500, 0.0999], [0.8000, 0.8500]) & ([0.5000, 0.5000], [0.5000, 0.5000]) \end{pmatrix}$$

In virtue of Model 4, the underlying IVIF weights of $\tilde{R}^{(2)(3)}$ can be yielded as:

$$\tilde{\varpi}^{(2)(3)} = \big([0.3985,0.3985],[0.4700,0.4700]\big),\big([0.3202,0.3386],[0.6260,0.6608]\big),\big([0.1052,0.1052],[0.8618,0.8860]\big),\big([0.0168, 0.0262], [0.8423, 0.8518]\big)\big)^T.$$

Therefore, the corresponding multiplicative consistent IVIFPR $\tilde{R}^{(2)(3)^*}$ for $\tilde{R}^{(2)(3)}$ can be shown.

The consistency degree $C_{\tilde{R}^{(2)(3)}} = 0.9087 > \alpha$ with Eq. (18), so $\tilde{R}^{(2)(3)}$ is acceptably consistent. Till now, all the IVIFPRs are acceptably consistent.

Step 5: Using Model 5, we can derive the normalized collective IVIF priority weight vector

$$\tilde{\varpi} = (\tilde{\varpi}_1, \tilde{\varpi}_2, \tilde{\varpi}_3, \tilde{\varpi}_4)^T = \big([0.3806,0.3830], [0.5335,0.5335]\big),\big([0.3746, 0.3792], [0.5521, 0.5521]\big),\big([0.1172, 0.1257],[0.8591, 0.8677]\big),\big([0.0143,0.0286], [0.9546,0.9609]\big)\big)^T.$$

Step 6: Using Eq. (2), we have $s(\tilde{\varpi}_1) = -0.1517$, $s(\tilde{\varpi}_2) = -0.1752$, $s(\tilde{\varpi}_3) = -0.7419$, $s(\tilde{\varpi}_4) = -0.9363$. Since $s(\tilde{\varpi}_1) > s(\tilde{\varpi}_2) > s(\tilde{\varpi}_3) > s(\tilde{\varpi}_4)$, the optimal alternative is A_1.

B. Comparative Analyses

To demostrate the advantages of our approach, this subsection conducts comparisons with Wan et al.'s method [37] and Liao et al.'s method [36].

1) Comparison with Wan et al.'s method: In this part, the Example is solved by Wan's method [37].

Step 1: Check and repair the consistency of individual IVIFPRs.

Wan et al. [37] take the consistency threshold α = 0.2. According to Eqs. (4) and (5) in [37], the consistency indices can be obtained: $\text{MCI}(\tilde{R}_1^L) = 0.1738$, $\text{MCI}(\tilde{R}_1^U) = 0.4507$, $\text{MCI}(\tilde{R}_2^L) = 0.2181$, $\text{MCI}(\tilde{R}_2^U) = 0.1638$, $\text{MCI}(\tilde{R}_3^L) = 1.1575$, and $\text{MCI}(\tilde{R}_3^U) = 1.9883$. According to Theorem 5 in [37], all IVIFPRs $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3$ are of unacceptable consistency. Then we can get the repaired IVIFPRs $\bar{\tilde{R}}_1$, $\bar{\tilde{R}}_2$ and $\bar{\tilde{R}}_3$ by Algorithm II and Eq. (33) in [37]. And the new consistency indices can be calculated: $\text{MCI}(\bar{\tilde{R}}_1^U) = 0.1694$, $\text{MCI}(\bar{\tilde{R}}_2^L) = 0.1727$, $\text{MCI}(\bar{\tilde{R}}_2^U) = 0.1984$, $\text{MCI}(\bar{\tilde{R}}_3^L) = 0.1646$ and $\text{MCI}(\bar{\tilde{R}}_3^U) = 0.1638$. Compared with the threshold α, they are all acceptable consistent, thus, by Theorem 5 in [37], all the repaired IVIFPRs meet the acceptable consistent condition.

Step 2: Determine the experts' weight vector. Here we set the weights of experts as $\lambda = (1/3, 1/3, 1/3)^T$.

Step 3: The collective IVIFPR can be obtained via Eq. (34) in [37] with experts' weight vector λ. According to the solution of the model Eq. (53) in [37], the optimal solution can be derived. With Eq. (45) in [37], the IVIF priority weight vector is derived and shown in Table 1.

Step 4: According to Eq. (2), the ranking results of alternatives can be derived. As Table 1 shows, in accordance with Wan et al.'s method [37], we can generate the ranking result $P_1 > P_2 > P_3 > P_4$, which means the optimal alternative is P_1.

2) Comparison with Liao et al.'s method: In this part, the Example is solved by Liao's method [36].

Step 1: Let $\tilde{R}_k^{(0)} = \tilde{R}_k$ $(k = 1, 2, 3)$. According to Algorithm 2 in [36], consistent IVIFPRs $\bar{\tilde{R}}_1^{(0)}$, $\bar{\tilde{R}}_2^{(0)}$ and $\bar{\tilde{R}}_3^{(0)}$ can be constructed. Then use Eq. (31) in [36] to integrate individual IVIFPRs into the collective $\bar{\tilde{R}}^{(0)}$. By Eq. (34) in [36], the distances between $\bar{\tilde{R}}_k^{(0)}$ and $\bar{\tilde{R}}^{(0)}$ are calculated as: $d(\bar{\tilde{R}}_1^{(0)}, \bar{\tilde{R}}^{(0)}) = 0.2007$, $d(\bar{\tilde{R}}_2^{(0)}, \bar{\tilde{R}}^{(0)}) = 0.1377$, $d(\bar{\tilde{R}}_3^{(0)}, \bar{\tilde{R}}^{(0)}) = 0.0985$. Take the consistency threshold γ^* = 0.2. Owing to $d(\bar{\tilde{R}}_1^{(0)}, \bar{\tilde{R}}^{(0)}) > \gamma^*$, $\bar{\tilde{R}}_1^{(0)}$ is unacceptable consistent, so we need to repair it.

Step 2: Let the parameter $\eta = 0.2$. With Eqs. (35)-(38) in [36], the repaired IVIFPR $\bar{\tilde{R}}_1^{(1)}$ can be generated. Then we derive the new collective IVIFPR $\bar{\tilde{R}}^{(1)}$. Thus, the distances between $\bar{\tilde{R}}_k^{(1)}$ and $\bar{\tilde{R}}^{(1)}$ are obtained as: $d(\bar{\tilde{R}}_1^{(1)}, \bar{\tilde{R}}^{(1)}) = 0.1813$, $d(\bar{\tilde{R}}_2^{(1)}, \bar{\tilde{R}}^{(1)}) = 0.0920$, $d(\bar{\tilde{R}}_3^{(1)}, \bar{\tilde{R}}^{(1)}) = 0.1313$. Since all the distances $d(\bar{\tilde{R}}_k^{(1)}, \bar{\tilde{R}}^{(1)}) < \gamma^* = 0.2$, the repair process can be finished.

Step 3: By Eq. (39) in [36], the priority weights and the ranking orders are acquired and viewed in Table 1. Furthermore, the same result $P_1 > P_2 > P_3 > P_4$ can be found according to Eq. (2) and then the optimal alternative P_1 can be selected.

Compared with above two previously approaches given in [37] and [36], our approach has the following advantages:

1) A more reasonable multiplicative consistency definition of IVIFPRs has been characterized. When an IVIFPR reduces to an IFPR, the multiplicative consistency definitions in the proposed method and Liao's method respectively degenerate into the definitions of an IFPR in Xu [30] and Liao [32]. Since Liao and Xu [32] has indicated that their consistency definition is more rational than that presented in [30], our definition is more reliable. Furthermore, the definitions of multiplicative consistency described in [36] and [37] do not satisfy robustness. In this situation, the consistency of an IVIFPR described in this paper is more reasonable.

2) It is necessary to emphasize that intervals in our paper are normalized, while in [36] and [37] are not. Since the weights derived by a symmetric IVIF weighted averaging (SIVIFWA) operator in Liao et al.'s method [36] are not normalized, which may lead to imprecise results. Furthermore, as is defined in [37], an IVIFPR can deduce two matching IFPRs by Eqs. (22) and (23) in [37]. However, the IVIF weights are not equal to the weights of corresponding IFPRs. Thus, Wan et al.'s method is lack of rationality. We build up an optimization model which can overcome this issue adequately. With paying attention to the normalizing process of the IVIF weights, our method is more convincing.

VII. Conclusion

This paper presents a novel approach for GDM problems with IVIFPRs. An innovative definition of the multiplicative consistency for IVIFPRs has been proposed. One distinctive

$$\tilde{R}^{(2)(3)^*}=\begin{pmatrix} ([0.5000,0.5000],[0.5000,0.5000]) & ([0.4585,0.4585],[0.3684,0.3895]) & ([0.6188,0.6188],[0.1633,0.1633]) & ([0.5876,0.5876],[0.0248,0.0386]) \\ ([0.3684,0.3895],[0.4585,0.4585]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.7065,0.7471],[0.2321,0.2321]) & ([0.6569,0.6946],[0.0345,0.0538]) \\ ([0.1633,0.1633],[0.6188,0.6188]) & ([0.2321,0.2321],[0.7065,0.7471]) & ([0.5000,0.5000],[0.5000,0.5000]) & ([0.4010,0.4010],[0.0641,0.0999]) \\ ([0.0248,0.0386],[0.5876,0.5876]) & ([0.0345,0.0538],[0.6569,0.6946]) & ([0.0641,0.0999],[0.4010,0.4010]) & ([0.5000,0.5000],[0.5000,0.5000]) \end{pmatrix}$$

Model 5

$$\begin{aligned} Min\ f=(&\underline{\vartheta}_{12}^{+}+\underline{\vartheta}_{12}^{-}+\overline{\vartheta}_{12}^{+}+\overline{\vartheta}_{12}^{-}+\underline{\zeta}_{12}^{+}+\underline{\zeta}_{12}^{-}+\overline{\zeta}_{12}^{+}+\overline{\zeta}_{12}^{-}+\underline{\vartheta}_{13}^{+}+\underline{\vartheta}_{13}^{-}+\overline{\vartheta}_{13}^{+}+\overline{\vartheta}_{13}^{-}+\underline{\zeta}_{13}^{+}+\underline{\zeta}_{13}^{-}+\overline{\zeta}_{13}^{+}+\overline{\zeta}_{13}^{-}\\ &+\underline{\vartheta}_{14}^{+}+\underline{\vartheta}_{14}^{-}+\overline{\vartheta}_{14}^{+}+\overline{\vartheta}_{14}^{-}+\underline{\zeta}_{14}^{+}+\underline{\zeta}_{14}^{-}+\overline{\zeta}_{14}^{+}+\overline{\zeta}_{14}^{-}+\underline{\vartheta}_{23}^{+}+\underline{\vartheta}_{23}^{-}+\overline{\vartheta}_{23}^{+}+\overline{\vartheta}_{23}^{-}+\underline{\zeta}_{23}^{+}+\underline{\zeta}_{23}^{-}+\overline{\zeta}_{23}^{+}+\overline{\zeta}_{23}^{-}\\ &+\underline{\vartheta}_{24}^{+}+\underline{\vartheta}_{24}^{-}+\overline{\vartheta}_{24}^{+}+\overline{\vartheta}_{24}^{-}+\underline{\zeta}_{24}^{+}+\underline{\zeta}_{24}^{-}+\overline{\zeta}_{24}^{+}+\overline{\zeta}_{24}^{-}+\underline{\vartheta}_{34}^{+}+\underline{\vartheta}_{34}^{-}+\overline{\varepsilon}_{34}^{+}+\overline{\vartheta}_{34}^{-}+\underline{\zeta}_{34}^{+}+\underline{\zeta}_{34}^{-}+\overline{\zeta}_{34}^{+}+\overline{\zeta}_{34}^{-})\end{aligned}$$

$$s.t.\begin{cases} \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_2^{\nu}}-\frac{1}{3}(0.14+0.65+0.4585)-\underline{\vartheta}_{12}^{+}+\underline{\vartheta}_{12}^{-}=0,\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_3^{\nu}}-\frac{1}{3}(0.62+0.62+0.5336)-\underline{\vartheta}_{13}^{+}+\underline{\vartheta}_{13}^{-}=0,\\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.83+0.78+0.3289)-\underline{\vartheta}_{14}^{+}+\underline{\vartheta}_{14}^{-}=0,\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_3^{\nu}}-\frac{1}{3}(0.73+0.5+0.7065)-\underline{\vartheta}_{23}^{+}+\underline{\vartheta}_{23}^{-}=0,\\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.9+0.75+0.6569)-\underline{\vartheta}_{24}^{+}+\underline{\vartheta}_{24}^{-}=0,\ \frac{\underline{\varpi}_3^{\mu}}{2-\overline{\varpi}_3^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.55+0.7+0.8)-\underline{\vartheta}_{34}^{+}+\underline{\vartheta}_{34}^{-}=0,\\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_2^{\nu}}-\frac{1}{3}(0.3+0.67+0.5)-\overline{\vartheta}_{12}^{+}+\overline{\vartheta}_{12}^{-}=0,\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_3^{\nu}}-\frac{1}{3}(0.65+0.65+0.6188)-\overline{\vartheta}_{13}^{+}+\overline{\vartheta}_{13}^{-}=0,\\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.14+0.65+0.4585)-\overline{\vartheta}_{14}^{+}+\overline{\vartheta}_{14}^{-}=0,\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_3^{\nu}}-\frac{1}{3}(0.14+0.65+0.4585)-\overline{\vartheta}_{23}^{+}+\overline{\vartheta}_{23}^{-}=0,\\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.95+0.8+0.7)-\overline{\vartheta}_{24}^{+}+\overline{\vartheta}_{24}^{-}=0,\ \frac{\overline{\varpi}_3^{\mu}}{2-\underline{\varpi}_3^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.6+0.75+0.85)-\overline{\vartheta}_{34}^{+}+\overline{\vartheta}_{34}^{-}=0,\\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_2^{\nu}}-\frac{1}{3}(0.65+0.05+0.1225)-\underline{\zeta}_{12}^{+}+\underline{\zeta}_{12}^{-}=0,\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_3^{\nu}}-\frac{1}{3}(0.12+0.036+0.1139)-\underline{\zeta}_{13}^{+}+\underline{\zeta}_{13}^{-}=0,\\ \frac{\underline{\varpi}_1^{\mu}}{2-\overline{\varpi}_1^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.04+0.006+0.0623)-\underline{\zeta}_{14}^{+}+\underline{\zeta}_{14}^{-}=0,\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_3^{\nu}}-\frac{1}{3}(0.04+0.3+0.157)-\underline{\zeta}_{23}^{+}+\underline{\zeta}_{23}^{-}=0,\\ \frac{\underline{\varpi}_2^{\mu}}{2-\overline{\varpi}_2^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.02+0.08+0.679)-\underline{\zeta}_{24}^{+}+\underline{\zeta}_{24}^{-}=0,\ \frac{\underline{\varpi}_3^{\mu}}{2-\overline{\varpi}_3^{\nu}-\overline{\varpi}_4^{\nu}}-\frac{1}{3}(0.1+0.1+0.05)-\underline{\zeta}_{34}^{+}+\underline{\zeta}_{34}^{-}=0,\\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_2^{\nu}}-\frac{1}{3}(0.7+0.33+0.2142)-\overline{\zeta}_{12}^{+}+\overline{\zeta}_{12}^{-}=0,\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_3^{\nu}}-\frac{1}{3}(0.15+0.35+0.1633)-\overline{\zeta}_{13}^{+}+\overline{\zeta}_{13}^{-}=0,\\ \frac{\overline{\varpi}_1^{\mu}}{2-\underline{\varpi}_1^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.1+0.14+0.0827)-\overline{\zeta}_{14}^{+}+\overline{\zeta}_{14}^{-}=0,\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_3^{\nu}}-\frac{1}{3}(0.05+0.4+0.2)-\overline{\zeta}_{23}^{+}+\overline{\zeta}_{23}^{-}=0,\\ \frac{\overline{\varpi}_2^{\mu}}{2-\underline{\varpi}_2^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.022+0.18+0.1079)-\overline{\zeta}_{24}^{+}+\overline{\zeta}_{24}^{-}=0,\ \frac{\overline{\varpi}_3^{\mu}}{2-\underline{\varpi}_3^{\nu}-\underline{\varpi}_4^{\nu}}-\frac{1}{3}(0.2+0.2+0.0999)-\overline{\zeta}_{34}^{+}+\overline{\zeta}_{34}^{-}=0,\\ \left[\underline{\varpi}_1^{\mu},\overline{\varpi}_1^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_2^{\mu},\overline{\varpi}_2^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_3^{\mu},\overline{\varpi}_3^{\mu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_4^{\mu},\overline{\varpi}_4^{\mu}\right]\subseteq[0,1],\\ \left[\underline{\varpi}_1^{\nu},\overline{\varpi}_1^{\nu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_2^{\nu},\overline{\varpi}_2^{\nu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_3^{\nu},\overline{\varpi}_3^{\nu}\right]\subseteq[0,1],\ \left[\underline{\varpi}_4^{\nu},\overline{\varpi}_4^{\nu}\right]\subseteq[0,1],\\ \overline{\varpi}_1^{\mu}+\overline{\varpi}_1^{\nu}\leq 1,\ \overline{\varpi}_2^{\mu}+\overline{\varpi}_2^{\nu}\leq 1,\ \overline{\varpi}_3^{\mu}+\overline{\varpi}_3^{\nu}\leq 1,\ \overline{\varpi}_4^{\mu}+\overline{\varpi}_4^{\nu}\leq 1,\\ \overline{\varpi}_1^{\mu}+\overline{\varpi}_2^{\mu}+\overline{\varpi}_3^{\mu}\leq\underline{\varpi}_4^{\nu},\ \overline{\varpi}_1^{\mu}+\overline{\varpi}_2^{\mu}+\overline{\varpi}_4^{\mu}\leq\underline{\varpi}_3^{\nu},\overline{\varpi}_1^{\mu}+\overline{\varpi}_3^{\mu}+\overline{\varpi}_4^{\mu}\leq\underline{\varpi}_2^{\nu},\ \overline{\varpi}_2^{\mu}+\overline{\varpi}_3^{\mu}+\overline{\varpi}_4^{\mu}\leq\underline{\varpi}_1^{\nu},\\ \underline{\varpi}_4^{\mu}+2\geq\overline{\varpi}_1^{\nu}+\overline{\varpi}_2^{\nu}+\overline{\varpi}_3^{\nu},\ \underline{\varpi}_3^{\mu}+2\geq\overline{\varpi}_1^{\nu}+\overline{\varpi}_2^{\nu}+\overline{\varpi}_4^{\nu},\\ \underline{\varpi}_2^{\mu}+2\geq\overline{\varpi}_1^{\nu}+\overline{\varpi}_3^{\nu}+\overline{\varpi}_4^{\nu},\ \underline{\varpi}_1^{\mu}+2\geq\overline{\varpi}_2^{\nu}+\overline{\varpi}_3^{\nu}+\overline{\varpi}_4^{\nu},\\ \underline{\vartheta}_{12}^{+}\geq 0,\ \underline{\vartheta}_{12}^{-}\geq 0,\ \overline{\vartheta}_{12}^{+}\geq 0,\ \overline{\vartheta}_{12}^{-}\geq 0,\ \underline{\zeta}_{12}^{+}\geq 0,\ \underline{\zeta}_{12}^{-}\geq 0,\ \overline{\zeta}_{12}^{+}\geq 0,\ \overline{\zeta}_{12}^{-}\geq 0,\\ \underline{\vartheta}_{13}^{+}\geq 0,\ \underline{\vartheta}_{13}^{-}\geq 0,\ \overline{\vartheta}_{13}^{+}\geq 0,\ \overline{\vartheta}_{13}^{-}\geq 0,\ \underline{\zeta}_{13}^{+}\geq 0,\ \underline{\zeta}_{13}^{-}\geq 0,\ \overline{\zeta}_{13}^{+}\geq 0,\ \overline{\zeta}_{13}^{-}\geq 0,\\ \underline{\vartheta}_{14}^{+}\geq 0,\ \underline{\vartheta}_{14}^{-}\geq 0,\ \overline{\vartheta}_{14}^{+}\geq 0,\ \overline{\vartheta}_{14}^{-}\geq 0,\ \underline{\zeta}_{14}^{+}\geq 0,\ \underline{\zeta}_{14}^{-}\geq 0,\ \overline{\zeta}_{14}^{+}\geq 0,\ \overline{\zeta}_{14}^{-}\geq 0,\\ \underline{\vartheta}_{23}^{+}\geq 0,\ \underline{\vartheta}_{23}^{-}\geq 0,\ \overline{\vartheta}_{23}^{+}\geq 0,\ \overline{\vartheta}_{23}^{-}\geq 0,\ \underline{\zeta}_{23}^{+}\geq 0,\ \underline{\zeta}_{23}^{-}\geq 0,\ \overline{\zeta}_{23}^{+}\geq 0,\ \overline{\zeta}_{23}^{-}\geq 0,\\ \underline{\vartheta}_{24}^{+}\geq 0,\ \underline{\vartheta}_{24}^{-}\geq 0,\ \overline{\vartheta}_{24}^{+}\geq 0,\ \overline{\vartheta}_{24}^{-}\geq 0,\ \underline{\zeta}_{24}^{+}\geq 0,\ \underline{\zeta}_{24}^{-}\geq 0,\ \overline{\zeta}_{24}^{+}\geq 0,\ \overline{\zeta}_{24}^{-}\geq 0,\\ \underline{\vartheta}_{34}^{+}\geq 0,\ \underline{\vartheta}_{34}^{-}\geq 0,\ \overline{\vartheta}_{34}^{+}\geq 0,\ \overline{\vartheta}_{34}^{-}\geq 0,\ \underline{\zeta}_{34}^{+}\geq 0,\ \underline{\zeta}_{34}^{-}\geq 0,\ \overline{\zeta}_{34}^{+}\geq 0,\ \overline{\zeta}_{34}^{-}\geq 0. \end{cases}$$

TABLE I

COMPARISONS OF PRIORITY WEIGHT VECTORS AND THE RANKING RESULTS OF ALTERNATIVES FOR DIFFERENT METHODS.

Methods	Priority weight vector $(\tilde{\varpi}_1,\tilde{\varpi}_2,\tilde{\varpi}_3,\tilde{\varpi}_4)^T$	Ranking orders
The proposed method	$\big(([0.3806,0.3830],[0.5335,0.5335]),([0.3746,0.3792],[0.5521,0.5521]),$ $([0.1172,0.1257],[0.8591,0.8677]),([0.0143,0.0286],[0.9546,0.9609])\big)^T$	$P_1>P_2>P_3>P_4$
Wan et al.'s method	$\big(([0.3015,0.4307],[0.3871,0.4673]),([0.3265,0.3629],[0.4476,0.5445]),$ $([0.0567,0.0846],[0.8738,0.9061]),([0.0039,0.0199],[0.9801,0.9801])\big)^T$	$P_1>P_2>P_3>P_4$
Liao et al.'s method	$\big(([0.5595,0.6600],[0.0950,0.2454]),([0.5349,0.6504],[0.1935,0.2817]),$ $([0.2860,0.4073],[0.4084,0.5120]),([0.0652,0.1531],[0.7012,0.7702])\big)^T$	$P_1>P_2>P_3>P_4$

characteristic is that it satisfies robustness, which means that during the IVIFPR establishing procedure, the multiplicative consistent conclusion remains stable whether the comparison order of objects changes or not. By virtue of the newly defined normalized IVIF weights, a conversion formula has been developed to transform the normalized weights into multiplicative consistent IVIFPRs. Consistency measure and inconsistent repairing process have been introduced to guarantee that all IVIFPRs derived conform to the acceptable consistency. Furthermore, the normalized individual and collective IVIF weights have been obtained from two different programming models corresponding to individual and group experts respectively. In this way, a step-by-step algorithm has been formed. The practical validation of the proposed algorithm are analyzed subsequently and the advantages of this method have been demonstrated by comparing with other approaches.

We have given consistency checking and inconsistency repairing process of interval-valued intuitionistic fuzzy GDM. However, further studies need to be fulfilled. In GDM problems with IVIFPRs, as an equally important part, the consensus reaching procedure is also an open question. Furthermore, in the proposed method, the weight vector of experts is determined in advance, how to achieve the objectiveness of expert weights is also a crucial issue worth analysing in the future.

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