Derivative Free Iterative Scheme for Monotone Nonlinear Ill-posed Hammerstein-Type Equations

Shobha M Erappa and Santhosh George

Abstract—**An iterative scheme which is free of derivative is employed to approximately solve nonlinear ill-posed Hammerstein type operator equations $TG(x) = y$, where G is a non-linear monotone operator and T is a bounded linear operator defined on Hilbert spaces X, Y, Z. The convergence analysis adapted in the paper includes weaker Lipschitz condition and adaptive choice of Perverzev and Schock(2005) is employed to choose the regularization parameter α. Furthermore, order optimal error bounds are obtained and the method is validated by a numerical example.**

Index Terms—**Derivative free Iterative method, Newton type method, Non-linear Ill-posed problems, Lipschitz condition, Hammerstein Operators, Adaptive Choice, Tikhonov regularization**

I. Introduction

Consider a nonlinear Hammerstein integral operator

$$(Ax)(t) := \int_0^1 k(s,t)f(s,x(s))ds$$

where

$$k(s,t) \in L^2([0,1]\times[0,1]), \ \ x \in L^2[0,1]$$

and $t \in [0,1]$. The above integral operator A admits a representation of the form $A = TG$ where

$$T : L^2[0,1] \to L^2[0,1]$$

is a linear integral operator with kernel $k(t,s)$: defined as

$$Tx(t) = \int_0^1 k(t,s)x(s)ds$$

and

$$G : D(G) \subseteq L^2[0,1] \to L^2[0,1]$$

is a nonlinear superposition operator (cf. [13]) defined as $Gx(s) = f(s,x(s))$.

The non-linear integral equation arises in a variety of application in various fields such as geophysics, electricity and magnestism, radiation, fluid mechanics, reactor theory, etc. and equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory.

In this paper, our study focuses on regularization of such non-linear ill-posed Hammerstein type operator([6]- [10]) equation of the form

$$(TG)x = y. \tag{1}$$

Shobha M Erappa is an Assistant Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal India-576104, e-mail id: shobha.me@gmail.com.

Santhosh George is a Professor in the Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, India-575025, e-mail id: sgeorge@nitk.ac.in

Here $G : D(G) \subseteq X \to X$, is a nonlinear operator, $T : X \to Y$ is a bounded linear operator and X and Y are Hilbert spaces with corresponding inner product $\langle .,. \rangle$ and norm $\|.\|$ respectively. We study the case where G is a monotone operator (i.e., $\langle G(x) - G(y), x - y\rangle \geq 0, \quad \forall x, y \in D(G)$) ([15], [18]) and $G'(x_0)^{-1}$ does not exist. Hence, G is ill-posed and thus along with non-closedness of the range of the operator T, we see that now (1) is ill-posed.

It is further assumed that y^δ is the available data with

$$\|y - y^\delta\| \leq \delta$$

and hence we approximate

$$(TG)x = y^\delta \tag{2}$$

instead of (1). Observe that (2) can be approximated by solving

$$Tz = y^\delta \tag{3}$$

for z and then solving the non-linear problem

$$G(x) = z. \tag{4}$$

We also assume that the solution $\hat{x}$ of (1) satisfies x_0-minimum norm solution i.e.,

$$\begin{aligned}\|G(\hat{x}) - G(x_0)\| := \min\{&\|G(x) - G(x_0)\| : \\ &TG(x) = y, x \in D(G)\}.\end{aligned} \tag{5}$$

Various methods have been proposed for approximating the solution of (2) which involves Frechet derivative of F. The most common method for solving the linear operator equation (3) is Tikhonov regularization ([5]-[11]). In particular, we consider

$$z_\alpha^\delta = (T^*T + \alpha I)^{-1}T^*(y^\delta - TG(x_0)) + F(x_0) \tag{6}$$

to approximate (3).

Newton method is usually applied to solve nonlinear equation (4). In the literature, ([1], [2], [3], [4], [12], [14], [17]) we see that a series of modification to Newton's scheme, which is studied and analyzed to improve the local convergence. In all these methods, one has to compute the inverse involving Fr$\acute{e}$chet derivative of G at each iterate x_k or at initial guess x_0. The high computational efficiency of Newtons formula, would still fail at some stages of evaluation, if the derivative of the functions vanishes or is too small. These limitations of the existing method led us to define a new iterative sequence for (3) which do not involve the Fr$\acute{e}$chet derivative and is given in Section 3.

Note that the regularization parameter α is chosen as per the adaptive scheme studied by Pereverzev and Schock ([16]) for the linear ill-posed operator equations and the same parameter α is used for solving the non-linear operator equation (4).

The paper is structured as follows. Preparatory results and adaptive choice strategy is given in section 2 and section 3 comprises the proposed derivative free iterative method. Section 4 deals with the algorithm for implementing the proposed method. Finally, in section 5, the method is elaborated with a numerical example where the performance of the proposed method is better when compared to that of the method in [9], [10].

II. Preliminaries

Let $B_r(x_0)$ and $\overline{B_r(x_0)}$ denotes the open and closed ball of radius r with centre at x_0. The following assumption is required for error estimation.

Assumption 2.1: There exists a continuous, strictly monotonically increasing function

$$\varphi : (0, a] \to (0, \infty)$$

with $a \geq \|T^2\|$ satisfying;

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \lambda \in (0, a]$$
  and
- there exists $v \in X, \|v\| \leq 1$ such that
  $$G(\hat{x}) - G(x_0) = \varphi(T^*T)v.$$

It can be seen that (see (4.3) in [8]) under Assumption 2.1,

$$\|G(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{7}$$

A. A priori choice of the parameter

For the choice, $\alpha := \alpha_\delta$ the estimate

$$\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$$

in (7) is of optimal order if it satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let

$$\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|T\|^2.$$

Then we have

$$\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$$

and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)).$$

So the relation (7) leads to

$$\|G(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta).$$

B. Adaptive scheme for choice of the parameter

Pereverzev and Shock[16], introduced a parameter choice strategy called adaptive parameter choice strategy, modified suitably to choose the regularization parameter α in our method.

Let

$$D_M = \{\alpha_i = \alpha_0\mu^{2i}, i = 0, 1, 2, ..., M\}, \mu > 1,$$

$$k := \max\{i : \alpha_i \in D_M^+\} \tag{8}$$

and

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\}, \tag{9}$$

where, $D_M^+ = \{\alpha_i \in D_M : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$.

THEOREM 2.2: (cf. [8], Theorem 4.3) Let l be as in (9), k be as in (8) and $z_{\alpha_k}^\delta$ be as in (6) with $\alpha = \alpha_k$. Then $l \leq k$ and

$$\|G(\hat{x}) - z_{\alpha_k}^\delta\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

III. Iterative Method and Convergence Analysis

Let $\delta \in (0, d]$ and $\alpha \in (\delta, a]$, for some positive constants a, d with $d < a$, and $\|G'(x)\| \leq \beta_0$, for all $x \in D(G)$. Let

$$x_{n+1,\alpha_k}^\delta = x_{n,\alpha_k}^\delta - \beta[G(x_{n,\alpha_k}^\delta) - z_{\alpha_k}^\delta + \frac{\alpha_k}{c}(x_{n,\alpha_k}^\delta - x_0)] \tag{10}$$

where α_k is as in Theorem 2.2, $x_{0,\alpha_k}^\delta := x_0$ is the initial guess and $\beta := \frac{c}{\beta_0 c + a}$, with $c \leq \alpha_k$. First we prove that x_{n,α_k}^δ converges to the zero x_{c,α_k}^δ of

$$G(x) + \frac{\alpha_k}{c}(x - x_0) = z_{\alpha_k}^\delta \tag{11}$$

and then we prove that x_{c,α_k}^δ is an approximation for $\hat{x}$.

The following parameters and notation are necessary for the analysis. Let

$$0 < \beta < \min\{1, \frac{1}{c}\}$$

and

$$\|\hat{x} - x_0\| := \rho < r$$

where $r = min\{\frac{1}{\beta}, \beta_0\rho + \frac{d}{\sqrt{\alpha_0}}\}$.
Define

$$\gamma_\rho := \beta[\beta_0\rho + \frac{d}{\sqrt{\alpha_0}}]$$

and

$$q = 1 - \frac{\beta\alpha_k}{c}.$$

LEMMA 3.1: Let x_{n,α_k}^δ and $z_{\alpha_k}^\delta$ be as in (10) and (6) respectively, then $x_{1,\alpha_k}^\delta \in B_{\gamma_\rho}(x_0)$.
Proof. From (10), we have that

$$\begin{aligned} \|x_{1,\alpha_k}^\delta - x_0\| &= \|\beta(G(x_0) - z_{\alpha_k}^\delta)\| \\ &\leq \beta\|G(x_0) - z_{\alpha_k} + z_{\alpha_k} - z_{\alpha_k}^\delta\| \\ &\leq \beta[\|G(x_0) - G(\hat{x})\| + \|z_{\alpha_k} - z_{\alpha_k}^\delta\|] \\ &\leq \beta[\beta_0\rho + \frac{\delta}{\sqrt{\alpha_k}}] = \gamma_\rho. \end{aligned}$$

THEOREM 3.2: Let x_{n,α_k}^δ be as in (10) and Lemma 3.1 holds. Then $(x_{n,\alpha_k}^\delta) \in B_r(x_0)$ is a Cauchy sequence converging to $x_{c,\alpha_k}^\delta \in \overline{B_r(x_0)}$. Further

$$G(x_{c,\alpha_k}^\delta) + \frac{\alpha_k}{c}(x_{c,\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$$

and

$$\|x_{n,\alpha_k}^\delta - x_{c,\alpha_k}^\delta\| \leq Cq^n$$

where $C = \frac{\gamma_\rho}{1-q}$.
Proof. Note that

$$\begin{aligned} x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta &= x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta - \beta[G(x_{n,\alpha_k}^\delta) \\ &\quad -G(x_{n-1,\alpha_k}^\delta) + \frac{\alpha_k}{c}(x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta)] \\ &= (x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta) - \beta[A_n + \\ &\quad \frac{\alpha_k}{c}I](x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta) \end{aligned}$$

where $A_n = \int_0^1 G'(x_n + t(x_n - x_{n-1}))dt$.

Then,

$$\begin{aligned} \|x^\delta_{n+1,\alpha_k} - x^\delta_{n,\alpha_k}\| &\leq \|I - \beta(A_n + \frac{\alpha_k}{c}I)\| \\ &\quad \times \|(x^\delta_{n,\alpha_k} - x^\delta_{n-1,\alpha_k})\|. \end{aligned}$$

Since the operator A_n is positive self adjoint operator with $\|A_n\| \leq \beta_0$, we have

$$\begin{aligned} \|I - \beta(A_n + \frac{\alpha_k}{c}I)\| &= \sup_{\|x\|=1} |\langle[(I - \frac{\beta\alpha_k}{c})I - \beta A_n]x, x\rangle| \\ &\leq 1 - \frac{\beta\alpha_k}{c} \end{aligned}$$

Therefore,

$$\|x^\delta_{n+1,\alpha_k} - x^\delta_{n,\alpha_k}\| \leq q\|x^\delta_{n,\alpha_k} - x^\delta_{n-1,\alpha_k}\|. \quad (12)$$

Next we show that $x^\delta_{n,\alpha_k} \in \overline{B_r(x_0)}$, for all $n \geq 0$. By Lemma 3.1, we have $\|x^\delta_{1,\alpha_k} - x_0\| \leq r$.

Further,

$$\begin{aligned} \|x^\delta_{n+1,\alpha_k} - x_0\| &\leq \|x^\delta_{n+1,\alpha_k} - x^\delta_{n,\alpha_k}\| + \|x^\delta_{n,\alpha_k} - x^\delta_{n-1,\alpha_k}\| \\ &\quad + \cdots + \|x^\delta_{1,\alpha_k} - x_0\| \\ &\leq q^n\|x^\delta_{1,\alpha_k} - x_0\| + q^{n-1}\|x^\delta_{1,\alpha_k} - x_0\| \\ &\quad + \cdots + \|x^\delta_{1,\alpha_k} - x_0\| \\ &\leq (q^n + q^{n-1} + \cdots \\ &\quad \cdots + q^2 + q + 1)\|x^\delta_{1\alpha_k} - x_0\| \\ &\leq \frac{1-q^n}{1-q}\gamma_\rho \\ &\leq \frac{\gamma_\rho}{1-q} \leq r \end{aligned}$$

i.e., $x^\delta_{n+1,\alpha_k} \in \overline{B_r(x_0)}$.

Observe that by (12),

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \sum_{i=0}^{i=m-1} \|x^\delta_{n+i+1,\alpha_k} - x^\delta_{n+i,\alpha_k}\| \\ &\leq \sum_{i=0}^{i=m-1} q\|x^\delta_{n+i,\alpha_k} - x^\delta_{n-1+i,\alpha_k}\| \\ &\leq (q^n + q^{n+1} + q^{n+2} + \\ &\quad ... + q^{n+m})\|x^\delta_{1,\alpha_k} - x_0\| \\ &\leq q^n(1 + q + q^2 + ... + q^m)\gamma_\rho \\ &\leq q^n[\frac{1-q^{m+1}}{1-q}]\gamma_\rho \\ &\leq Cq^n \end{aligned}$$

Thus $x^\delta_{n,\alpha_k} \in B_r(x_0)$ is a Cauchy sequence in and hence it converges, say to $x^\delta_{c,\alpha_k} \in \overline{B_r(x_0)}$ and

$$\begin{aligned} \beta\|G(x^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(x^\delta_{n,\alpha_k} - x_0)\| &= \|x^\delta_{n+1,\alpha_k} - x^\delta_{n,\alpha_k}\| \\ &\leq q^n\gamma_\rho \quad (13) \end{aligned}$$

As $n \to \infty$ in (13) we see that

$$G(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}. \quad (14)$$

Hence the proof of the Theorem.

The assumption below leads us to prove our desired results.

Assumption 3.3: (cf.[18]) Let $x_0 \in X$ be fixed. There exists a constant k_0 such that for every $v \in B_r(x_0) \subseteq D(G)$ and $w \in X$, there exists an element $\Phi(x_0, u, v) \in X$ satisfying $[G'(x_0) - G'(v)]w = G'(x_0)\Phi(x_0, v, w), \|\Phi(x_0, v, w)\| \leq k_0\|w\|\|x_0 - v\|$.

Assumption 3.4: There exists a continuous, strictly monotonically increasing function

$$\varphi_1 : (0, b] \to (0, \infty)$$

with $b \geq \|G'(x_0)\|$ satisfying;

- $\lim_{\lambda\to 0}\varphi_1(\lambda) = 0$,
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \qquad \forall\lambda \in (0, b]$$
- there exists $w \in X$ with $\|w\| \leq 1$ (cf. [15]) such that
$$x_0 - \hat{x} = \varphi_1(G'(x_0))w.$$
- For each $y \in B_r(x_0)$ there exists a bounded linear operator $S(y, x_0)$ (see [17]) such that
$$G'(y) = G'(x_0)S(y, x_0)$$
with $\|S(y, x_0)\| \leq k_1$.

Assume that

$$k_1 < \frac{1 - k_0 r}{1 - \beta c}$$

and for $\alpha > 0$, consider $\varphi_1(\alpha) \leq \varphi(\alpha)$ for the sake of simplicity.

THEOREM 3.5: Suppose x^δ_{c,α_k} is the zero of (11) and Assumptions 3.3 and 3.4 hold. Then

$$\|\hat{x} - x^\delta_{c,\alpha_k}\| = O(\psi^{-1}(\delta)).$$

Proof. Let $M_1 = \int_0^1 G'(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}))dt$. Then

$$G(x^\delta_{c,\alpha_k}) - G(\hat{x}) = M_1(x^\delta_{c,\alpha_k} - \hat{x})$$

and hence by (14),

$$c\beta[G(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}] + \alpha_k(x^\delta_{c,\alpha_k} - x_0) = 0,$$

so

$$\begin{aligned} c\beta M_1(x^\delta_{c,\alpha_k} - \hat{x}) + \alpha\beta(x^\delta_{c,\alpha_k} - \hat{x}) &= c\beta(z^\delta_{\alpha_k} - G(\hat{x})) \\ &\quad + \alpha_k\beta(x_0 - \hat{x}), \end{aligned}$$

$G'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) + \beta c(M_1 + \alpha_k\beta I)(x^\delta_{c,\alpha_k} - \hat{x}) = c\beta(z^\delta_{\alpha_k} - G(\hat{x})) + \alpha_k\beta(x_0 - \hat{x}) + G'(x_0)(x^\delta_{c,\alpha_k} - \hat{x})$.

Note that

$$\begin{aligned} \|x^\delta_{c,\alpha_k} - \hat{x}\| &\leq \|\alpha_k\beta(G'(x_0) + \alpha_k\beta I)^{-1}(x_0 - \hat{x})\| + \\ &\quad \|(G'(x_0) + \alpha_k\beta I)^{-1}\beta c(G(\hat{x}) - z^\delta_{\alpha_k})\| + \\ &\quad \|(G'(x_0) + \alpha_k\beta I)^{-1}(G'(x_0) - \\ &\quad \beta c M_1)(x^\delta_{c,\alpha_k} - \hat{x})\| \\ &\leq \|\alpha_k\beta(G'(x_0) + \alpha_k\beta I)^{-1}(x_0 - \hat{x})\| + \\ &\quad \|G(\hat{x}) - z^\delta_{\alpha_k}\| + \Gamma \quad (15) \end{aligned}$$

where

$$\Gamma := \|(G'(x_0) + \alpha_k\beta I)^{-1}(G'(x_0) - \beta c M_1)(x^\delta_{c,\alpha_k} - \hat{x})\|.$$

Further by Assumption 3.4, we obtain

$$\begin{aligned}\Gamma &\leq \|(G'(x_0)+\alpha_k\beta I)^{-1}\int_0^1 [G'(x_0)-G'(\hat{x}+t(x^{\delta}_{c,\alpha_k} \\ &\quad -\hat{x}))](x^{\delta}_{c,\alpha_k}-\hat{x})dt\| + (1-\beta c)\|(G'(x_0)+\alpha_k\beta I)^{-1} \\ &\quad G'(x_0)\int_0^1 S(\hat{x}+t(x^{\delta}_{c,\alpha_k}-\hat{x}),\hat{x})(x^{\delta}_{c,\alpha_k}-\hat{x})dt\| \\ &\leq k_0 r\|x^{\delta}_{c,\alpha_k}-\hat{x}\| + (1-\beta c)k_1\|x^{\delta}_{c,\alpha_k}-\hat{x}\| \qquad (16)\end{aligned}$$

and hence by (15) and (16) we have

$$\begin{aligned}\|x^{\delta}_{c,\alpha_k}-\hat{x}\| &\leq \frac{\|\alpha_k\beta(G'(x_0)+\alpha_k\beta I)^{-1}(x_0-\hat{x})\|}{1-(1-c)k_1-k_0 r} \\ &\quad + \frac{\|G(\hat{x})-z^{\delta}_{\alpha_k}\|}{1-(1-c)k_1-k_0 r} \\ &\leq \frac{\varphi_1(\alpha_k)+(2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1-k_0 r}.\end{aligned}$$

THEOREM 3.6: Let x^{δ}_{n,α_k} be as in (10) and hypotheses of Theorem 3.2 and Theorem 3.5 hold. Then

$$\|\hat{x}-x^{\delta}_{n,\alpha_k}\| \leq Cq^n + O(\psi^{-1}(\delta)),$$

where C is as given in Theorem 3.2.

THEOREM 3.7: Suppose the assumptions in Theorem 2.2 and Theorem 3.6 hold and let

$$n_k := \min\{n : q^n \leq \frac{\delta}{\sqrt{\alpha_k}}\}.$$

Then

$$\|\hat{x}-x_{n_k}\| = O(\psi^{-1}(\delta)).$$

IV. Algorithm

Note that for $i,j \in \{0,1,2,\cdots,M\}$, $z^{\delta}_{\alpha_i}-z^{\delta}_{\alpha_j} = (\alpha_j-\alpha_i)(T^*T+\alpha_j I)^{-1}(T^*T+\alpha_i I)^{-1}[T^*(y^{\delta}-TG(x_0))]$.

The algorithm for implementing the iterative methods discussed in section 3 consists of the following steps.

- $\alpha_0=\delta^2$; and $\alpha_i=\mu^{2i}\alpha_0, \mu>1$;
- solve for v_i : $(T^*T+\alpha_i I)v_i = T^*(y^{\delta}-TG(x_0))$;
- solve for $j<i$, z_{ij} : $(T^*T+\alpha_j I)z_{ij} = (\alpha_j-\alpha_i)w_i$;
- if $\|z_{ij}\| > \frac{4}{\mu^j}$, then take $k=i-1$;
- else, repeat with $i+1$ in place of i.
- choose $n_k = \min\{n : q^n \leq \frac{\delta}{\sqrt{\alpha_k}}\}$
- solve x_{n_k} using the iteration (10).

V. Numerical Example

EXAMPLE 5.1: Let the operator $TG : H^1(0,1) \longrightarrow L^2(0,1)$ with $T: L^2(0,1)\longrightarrow L^2(0,1)$ defined by

$$T(x)(t) = \int_0^1 k(t,s)x(s)ds \qquad (17)$$

and $G: D(G)\subseteq H^1(0,1)\longrightarrow L^2(0,1)$ defined by

$$G(v) := \int_0^1 k(t,s)v^3(s)ds, \qquad (18)$$

where

$$k(t,s) = \begin{cases}(1-t)s, 0\leq s\leq t\leq 1\\(1-s)t, 0\leq t\leq s\leq 1\end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$: (see [18], section 4.3)

$$\begin{aligned}\langle G(x)-G(y), x-y\rangle &= \int_0^1\left[\int_0^1 k(t,s)(x^3-y^3)(s)ds\right] \\ &\quad \times(x-y)(t)dt \geq 0.\end{aligned}$$

Hence G is a monotone operator. Let G' be the Fréchet derivative of G that is,

$$G'(v)w = 3\int_0^1 k(t,s)(v(s))^2 w(s)ds.$$

So for any $v\in B_r(x_0), x_0(s)\geq k_3>0, \forall s\in(0,1)$, we have

$$G'(v)w = G'(x_0)S(v,x_0)w,$$

where

$$S(v,x_0) = (\frac{v}{x_0})^2.$$

For the computation, we take

$$y(t) = \frac{1}{720}(26-36t+15t^4-6t^5-t^6)$$

and $y^{\delta}=y+\delta$. Then, the actual solution is

$$\hat{x}(t) = \frac{1}{\sqrt{2}}(1-t).$$

With $\varphi_1(\lambda)=\lambda$ the function $x_0-\hat{x}$ satisfies the source condition

$$x_0-\hat{x} = \varphi_1(G'(x_0))1.$$

Thus, an accuracy of order at least $O(\delta^{\frac{1}{2}})$ is expected.

We use the Gauss-Legendre quadrature formula:

$$\int_0^1 f(t)dt \approx \sum_{j=1}^n w_j f(t_j),$$

where the abscissa t_j and the weight w_j for n = 25 are as in [19].

The discretized form of (10) is as follows: $x^{\delta}_{n+1,\alpha_k}(t_i) = x^{\delta}_{n,\alpha_k}(t_i) - \beta[G(x^{\delta}_{n,\alpha_k})(t_i) - z^{\delta}_{\alpha_k} + \frac{\alpha_k}{c}(x^{\delta}_{n,\alpha_k}(t_i)-x_0(t_0))]$ where

$$z^{\delta}_{\alpha} = (T^*T+\alpha_k I)^{-1}T^*(y^{\delta}-TG(x_0)(t_0)) + G(x_0)(t_0)$$

and

$\sum_{j=1}^{25} a_{ij}x(t_j)^3$ with

$$a_{ij} = \begin{cases} w_j t_j(1-t_i) & \text{if } j\leq i\\ w_j t_i(1-t_j) & \text{if } i<j\end{cases}.$$

We choose $a=1.5$, $\alpha_0=(1.3)\delta$, and $\mu=1.1$.

The relative error $\frac{||x_k-\hat{x}||}{||\hat{x}||}$ and the residual error $\frac{\|TG(x_k)-\hat{x}\|}{\delta^{1/2}}$ for $\beta=0.5$ are given in Table I below.

TABLE I
Iterations and corresponding error estimates

δ	α_k	$\frac{\|\|x_k-\hat{x}\|\|}{\|\|\hat{x}\|\|}$	$\frac{\|TG(x_k)-y^{\delta}\|}{\|y^{\delta}\|}$
0.01	0.014641	0.60548222632	1.13715579036
0.005	0.0073205	0.60548222632	1.209913876288
0.001	0.0014641	0.58876640464	1.212214660751

References

[1] Argyros, I. K. Convergence and Application of Newton-type Iterations, Springer, 2008.

[2] Argyros, I. K. A Semilocal convergence for directional Newton methods, Math.Comput.(AMS). **80**, 327-343, 2011.

[3] Bakushinskii, A. B. The problem of convergence of the iteratively regularized Gauss-Newton method, Comput. Math. Math. Phys., **32**, 1353-1359, 1992.

[4] Blaschke, B., Neubauer, A. and Scherzer, O. On convergence rates for the iteratively regularized Gauss-Newton method IMA J.Numer.Anal., **17**, 421-436, 1997.

[5] Engl, H. W., Kunisch, K. and Neubauer, A. Convergence rates for Tikhonov regularization of nonlinear ill-posed problems, Inverse Problems, **5**, 523-540, 1989.

[6] George, S. Newton-Tikhonov regularization of ill-posed Hammerstein operator equation, J. Inverse and Ill-Posed Problems, **14**(2), 135-146, 2006.

[7] George, S. and Thamban Nair, M. A modified Newton-Lavrentiev regularization for nonlinear ill-posed Hammerstein operator equations, J. Complexity and Ill-Posed Problems, **24**, 228-240, 2008.

[8] George, S. and Kunhanandan, M. An iterative regularization method for Ill-posed Hammerstein type operator equation, J.Inv.Ill-Posed Problems **17**, 831-844, 2009.

[9] Santhosh George and Monnanda Erappa Shobha, Newton type iteration for Tikhonov regularization of non-linear ill-posed Hammerstein type equations, J Appl Math Comput (2014) 44:69–82, DOI 10.1007/s12190-013-0681-1. ISSN: 1598-5865 (print version)ISSN: 1865-2085 (electronic version) , 2014.

[10] Monnanda Erappa Shobha and Santhosh George, On Improving the Semilocal Convergence of Newton-Type Iterative Method for Ill-posed Hammerstein Type Operator Equations, IAENG International Journal of Applied Mathematics, vol. 43, no.2, pp.64-70, 2013

[11] Groetsch, C. W. Theory of Tikhonov regularization for Fredholm Equation of the first kind, Pitmann Books, 1984.

[12] Kaltenbacher, B. A posteriori parameter choice strategies for some Newton-type methods for the regularization of nonlinear ill-posed problems, Numer. Math., **79**, 501-528, 1998.

[13] Krasnoselskii, M. A., Zabreiko, P. P., Pustylnik, E. I. and Sobolevskii, P. E. Integral operators in spaces of summable functions (Translated by T.Ando, Noordhoff International publishing, Leyden, 1976).

[14] Langer, S. and Hohage, T. Convergence analysis of an inexact iteratively regularized Gauss-Newton method under general source conditions, J.Inverse Ill-Posed Probl., **15**, 19-35, 2007.

[15] Mahale, P. and Nair, M. T. A simplified generalized Gauss-Newton method for nonlinear ill-posed problems, Math. Comp., **78**(265), 171-184, 2009.

[16] Pereverzev, S. and Schock, E. On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM. J. Numer. Anal., **43**(5), 2060-2076, 2005.

[17] Ramm, A. G., Smirnova, A. B. and Favini, A. Continuous modified Newton's-type method for nonlinear operator equations. Ann.Mat.Pura Appl. **182**, 37-52, 2003.

[18] Semenova, E.V. Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators, Comput. Methods Appl. Math., **4**, 444-454, 2010.

[19] Sreedeep, C.D. Iterative Regularization Theory for Non-Linear Ill-Posed Problems, Doctoral Thesis NITK, 2019.

Modification:
1) Modified on 3rd of May 2021.
2) Affiliation of the first author was MAHE earlier, which is elaborated in the revised form as "Manipal Academy of Higher Education"

Shobha M Erappa recieved Ph.D from department of Mathematical and Computational Sciences, National Institute of Technology Karantaka under the supervision of Prof. Santhosh George At present she is a favulty in the department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India. Her research topics are inverse and ill-posed problems.

Santhosh George pursued his Ph.D in Mathematics from Goa University. He is presently working as a Professor in the department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India. He has several publications in the reputed international journals and proceedings of international conferences. His research area includes Functional Analysis, Inverse and Ill-posed problems and its applications. He has guided nine PhDs and two are ongoing.