Positive Solution for m-point φ-Riemann-Liouville Fractional Differential Equations with p-Laplacian Operator

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***Abstract*—We investigate an m-point p-Laplacian fractional equations with the generalized fractional derivative, the new positive solutions are found by using some theorems.**

***Index Terms*—m-point φ-Riemann-Liouville fractional derivative; Positive solution; Green function; p-Laplacian operator.**

I. Introduction

THe purpose of this paper is to find the existence of positive solution to the m-point p-Laplacian fractional equations with the generalized fractional derivative,

$$(\phi_p(D_{0^+}^{\alpha;\,\varphi}u(t)))' + f(t,u(t)) = 0, \quad 0<t<1, \quad (1)$$

$$D_{0^+}^{\alpha;\,\varphi}u(0)=0, \quad u(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad (2)$$

here $1<\alpha\leq 2$, $D_{0^+}^{\alpha;\,\varphi}$ is the φ-Riemann-Liouville fractional derivative. $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_q=(\phi_p)^{-1}$, $\dfrac{1}{p}+\dfrac{1}{q}=1$ and f,φ,α_i,ξ_i meet

(H_1) $f:[0,1]\in C[0,+\infty)\to[0,+\infty)$ and $\varphi:[0,1]\to R$ is a function which is strictly increasing and $\varphi\in C^2[0,1]$, $\varphi'(x)\neq 0$ for all $x\in[0,1]$;

(H_2) $0\leq\alpha_i<1$, $0<\xi_i<1$ $(i=1,2,\cdots,m-2)$ meet $0\leq\sum\limits_{i=1}^{m-2}\alpha_i<1$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$.

Because of the theory of fractional derivative itself and its wide application, the fractional differential equation is attracting more and more attention from many scholars. Early research on fractional differential equation focused on the solvability of linear initial fractional differential equations in terms of special functions [1-3]. Techniques of nonlinear analysis were often used by many scholars to discuss the solution of fractional differential equation [4-9]. The properties of solutions for fractional differential equation were discussed in [23]. The solution for mixed-order boundary value problem was studied in [24].

The following work of Bai [10] was the earlier paper which gave the positive solution for fractional problem with the Riemann-Liouville differentiation.

$$D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,$$

$$u(0)=u(1)=0,$$

here $1<\alpha\leq 2$.

Since then, there have been fruitful results of fractional boundary value problems, see [11-15]. Recently, some scholars have begun to study the generalized fractional order problem.

The authors of paper [16] discussed the following ψ-Caputo fractional problem

$$-{}^{C}D_{0}^{\alpha,\,\psi}u(t)=f(t,u(t)), \quad 0<t<1,$$

$$u'(0)=0, \quad \beta^{C}D_{0}^{\alpha-1,\,\psi}u(1)+u(\eta)=0,$$

here $1<\alpha\leq 2$.

[17] gave the unique results for the fractional initial value problem involving φ-Caputo fractional derivative

$${}^{C}D_{a^+}^{\alpha,\,\varphi}x(t)=f(t,x(t)), \qquad t\in[a,b],$$

$$x(a)=x_a, \quad x_{\varphi}^{[k]}(a)=x_a^k, \quad k=1,2,\cdots,n-1,$$

here $n-1<\alpha<n$.

Seemab [18] presented the positive solutions for fractional problem within φ-Riemann-Liouville operators

$$D_{0^+}^{\alpha,\,\varphi}z(x)+f(x,z(x))=0, \qquad x\in(0,1),$$

$$z(0)=0, \quad z(1)=\beta z(\eta),$$

where $1<\alpha\leq 2$, $0<(\varphi(1)-\varphi(0))^{\alpha-1}-\beta(\varphi(\eta)-\varphi(0))^{\alpha-1}<1$.

In [19], the authors established positive solutions and the Hyers-Ulam stability for Atangana-Baleanu-Caputo fractional differential equations in Banach space

$${}_{0}^{ABC}D^{\beta}[\phi_p[{}_{0}^{ABC}D^{v_0}x(t)]]=-y_1^*(t,x(t)),$$

$$\phi_p[{}_{0}^{ABC}D^{v_0}x(t)]|_{t=0}=0, \quad x(1)=0.$$

Zhang [20] presented positive solutions for Hadamard fractional integral problems

$$D^{\beta}(\phi_p(D^{\alpha}u(t)))=f(t,u(t)), \quad 1<t<e,$$

$$u(1)=u'(1)=u'(e)=0,$$

$$D^{\alpha}u(1)=0,$$

$$\phi_p(D^{\alpha}u(e))=\mu\int_1^e\phi_p(D^{\alpha}u(t))\frac{dt}{t},$$

where D^{α} is the αth Hadamard fractional derivative.

Inspired by the literature above, the problem (1), (2) is investigated in this paper.

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II. Definitions and Lemmas

Definition 2.1 [18] Now suppose that $n-1<\beta<n$. we gave a function $g\in[c,d]$ which is integrable and $\varphi\in C^n[c,d],\ \varphi'(t)\neq 0$ an increasing differentiable function. The fractional integral for function g as follows:

$$I_{c^+}^{\beta;\,\varphi}g(x)=\frac{1}{\Gamma(\beta)}\int_c^x\varphi'(s)(\varphi(x)-\varphi(s))^{\beta-1}g(s)ds. \quad (3)$$

It is obvious that when $\varphi(t)=t$, (3) is the classical Riemann-Liouville fractional integral.

Definition 2.2 [18] Now suppose that $n-1<\beta<n$. φ is given just as definition 2.1. Here's the definition for φ-Riemann-Liouville fractional derivative as follows

$$\begin{aligned}D_{a^+}^{\beta;\,\varphi}g(x)&=\left(\frac{1}{\varphi'(x)}\frac{d}{dt}\right)^nI_{a^+}^{n-\beta;\,\varphi}g(x)\\&=\frac{1}{\Gamma(n-\beta)}\left(\frac{1}{\varphi'(x)}\frac{d}{dt}\right)^n\\&\quad\int_a^x\varphi'(s)(\varphi(x)-\varphi(s))^{n-\beta-1}g(s)ds.\end{aligned}$$

here $n=[\alpha]+1$.

Let $\alpha,\beta>0$, then the relation

$$I_{a^+}^{\alpha;\,\varphi}I_{a^+}^{\beta;\,\varphi}h(x)=I_{a^+}^{\alpha+\beta;\,\varphi}h(x)$$

holds.

Definition 2.3 [18] Let $\beta>0$. $\varphi\in C^n[c,d],\ \varphi'(t)>0$ and $\varphi'(t)\neq0,\ t\in[c,d]$. Suppose $h\in C^{n-1}[c,d]$, here's the definition for φ-Caputo fractional derivative of h

$${}^CD_{c^+}^{\beta;\,\varphi}h(x)=D_{c^+}^{\beta;\,\varphi}\left[h(x)-\sum_{k=0}^{n-1}\frac{h_\varphi^{[k]}(c)}{k!}(\varphi(x)-\varphi(c))^k\right],$$

here $n=[\beta]+1$ if $\beta\notin N,\ n=\beta$ if $\beta\in N$.

Remark 2.4 [18] The relationship between the φ-Caputo and the φ-Riemann-Liouville is:

$${}^CD_{a^+}^{\alpha;\,\varphi}f(t)=D_{a^+}^{\alpha;\,\varphi}\left[f(s)-\sum_{k=0}^{n-1}\frac{f_\varphi^{[k]}(a)}{k!}(\varphi(s)-\varphi(a))^k\right](t),$$

here $t>a$ and $n=[\alpha]+1$ for $\alpha\notin N,\ n=\alpha$ for $\alpha\in N$.

Theorem 2.1 [18] Let g is a function defined on $[c,d]$. The following results are right.
1. Suppose $g\in C[c,d]$, we have ${}^CD_{c^+}^{\beta;\,\varphi}I_{c^+}^{\beta;\,\varphi}g(x)=g(x)$.
2. Suppose $g\in C^{n-1}[c,d]$, just we get

$$I_{c^+}^{\beta;\,\varphi}\,{}^CD_{c^+}^{\beta;\,\varphi}g(x)=g(x)-\sum_{k=0}^{n-1}\frac{g_\varphi^{[k]}(c)}{k!}(\varphi(x)-\varphi(c))^k.$$

Let

$$\mu=(\varphi(1)-\varphi(0))^{\alpha-1}-\sum_{i=1}^{m-2}\alpha_i(\varphi(\xi_i)-\varphi(0))^{\alpha-1}.$$

Lemma 2.1 If $0\leq\sum_{i=1}^{m-2}\alpha_i<1,\ f\in C[0,1]$ and $1<\alpha\leq2$. Consequently,

$$\begin{aligned}u(t)&=-\frac{1}{\Gamma(\alpha)}\int_0^t\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds\\&\quad+\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds\\&\quad-\frac{\sum_{i=1}^{m-2}\alpha_i(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\&\quad\int_0^{\xi_i}\varphi'(s)(\varphi(\xi_i)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds.\end{aligned}\quad (4)$$

is the unique solution for

$$(\phi_p(D_{0^+}^{\alpha;\,\varphi}u(t)))'+f(t)=0,\quad 0<t<1, \quad (5)$$

$$D_{0^+}^{\alpha;\,\varphi}u(0)=0,\quad u(0)=0,\quad u(1)=\sum_{i=1}^{m-2}\alpha_iu(\xi_i). \quad (6)$$

Proof:

$$\phi_p(D_{0^+}^{\alpha;\,\varphi}u(t))=-\int_0^tf(s)ds$$

can be got by integrating both sides of (5) on $[0,t]$, i.e.,

$$D_{0^+}^{\alpha;\,\varphi}u(t)=-\varphi_q\left(\int_0^tf(s)ds\right). \quad (7)$$

For (7), we apply the Theorem 2.1, there have

$$\begin{aligned}u(t)&=c_1(\varphi(t)-\varphi(0))^{\alpha-1}+c_2(\varphi(t)-\varphi(0))^{\alpha-2}\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds.\end{aligned}$$

Making use of the condition that $u(0)=0$, we have $c_2=0$. Thus

$$\begin{aligned}u(t)&=c_1(\varphi(t)-\varphi(0))^{\alpha-1}\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t\varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds.\end{aligned}$$

In particular,

$$\begin{aligned}u(1)&=c_1(\varphi(1)-\varphi(0))^{\alpha-1}\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^1\varphi'(s)(\varphi(1)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds\end{aligned}$$

and

$$\begin{aligned}u(\xi_i)&=c_1(\varphi(\xi_i)-\varphi(0))^{\alpha-1}\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^{\xi_i}\varphi'(s)(\varphi(\xi_i)-\varphi(s))^{\alpha-1}\\&\quad\phi_q\left(\int_0^sf(\tau)d\tau\right)ds.\end{aligned}$$

By (6), we get

$$
\begin{aligned}
c_1 &= \frac{1}{\mu\Gamma(\alpha)}\int_0^1 \varphi'(s)(\varphi(1)-\varphi(s))^{\alpha-1} \\
&\quad \phi_q\left(\int_0^s f(\tau)d\tau\right)ds \\
&\quad -\frac{\sum\limits_{i=1}^{m-2}\alpha_i}{\mu\Gamma(\alpha)}\int_0^{\xi_i}\varphi'(s)(\varphi(\xi_i)-\varphi(s))^{\alpha-1} \\
&\quad \phi_q\left(\int_0^s f(\tau)d\tau\right)ds.
\end{aligned}
$$

■

Lemma 2.2 We presumed that $0 \leq \sum_{i=1}^{m-2}\alpha_i < 1$, then the Green's function for the problem

$$
-(\phi_p(D_{0^+}^{\alpha;\,\varphi}u(t)))' = 0, \quad 0<t<1, \tag{8}
$$

$$
D_{0^+}^{\alpha;\,\varphi}u(0)=0, \quad u(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i) \tag{9}
$$

is described as

$$
G(t,s)=\begin{cases}
\dfrac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
\dfrac{-(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
\dfrac{+(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
\dfrac{-\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
0\leq t\leq 1,\ \xi_{i-1}\leq s\leq \min\{\xi_i,t\}, \\
i=1,2,\cdots,m-1; \\
\dfrac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
\dfrac{-\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
0\leq t\leq 1,\ \max\{\xi_{i-1},t\}\leq s\leq \xi_i, \\
i=1,2,\cdots,m-1.
\end{cases} \tag{10}
$$

Here, we mark $\xi_0 = 0, \xi_{m-1} = 1$ and $\sum\limits_{i=m_1}^{m_2} f_i = 0$ for $m_2 < m_1$.

Proof: If $0 \leq t \leq \xi_1$, the function $G(t,s)$ can be rewritten as

$$
\begin{aligned}
u(t) &= \int_0^t \Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg] \\
&\quad \varphi'(s)\phi_q\left(\int_0^s f(\tau)d\tau\right)ds \\
&\quad +\int_t^{\xi_1}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg] \\
&\quad \varphi'(s)\phi_q\left(\int_0^s f(\tau)d\tau\right)ds \\
&\quad +\sum_{i=2}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad \frac{(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg] \\
&\quad \varphi'(s)\phi_q\left(\int_0^s f(\tau)d\tau\right)ds \\
&\quad +\int_{\xi_{m-2}}^1\left[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\right] \\
&\quad \varphi'(s)\phi_q\left(\int_0^s f(\tau)d\tau\right)ds.
\end{aligned}
$$

If $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m-2$, the function $G(t,s)$ can be rewritten as

$$
\begin{aligned}
u(t) &= \int_0^{\xi_1}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)} \\
&\quad -\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]ds \\
&\quad \varphi'(s)\phi_q\left(\int_0^s f(\tau)d\tau\right)ds.
\end{aligned}
$$

$$
\begin{aligned}
+ & \sum_{i=2}^{r-1}\int_{\xi_{i-1}}^{\xi_i}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]ds\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\int_{\xi_{r-1}}^{t}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=r}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]ds\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\int_{t}^{\xi_r}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=r}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\sum_{i=r+1}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\int_{\xi_{m-2}}^{1}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds.
\end{aligned}
$$

If $\xi_{m-2}\le t\le 1$, the function $G(t,s)$ can be rewritten as

$$
\begin{aligned}
u(t) =& \int_0^{\xi_1}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}.\\
+ & \frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]ds\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\sum_{i=2}^{m-2}\int_{\xi_{i-1}}^{\xi_i}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{\sum\limits_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]ds\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\int_{\xi_{m-2}}^{t}\Bigg[\frac{\sum\limits_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& -\frac{(\varphi(t)-\varphi(s))^{\alpha-1}(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\\
& +\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds\\
& +\int_{t}^{1}\Bigg[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg]\\
& \varphi'(s)\phi_q\Big(\textstyle\int_0^s f(\tau)d\tau\Big)ds.
\end{aligned}
$$

Therefore,

$$u(t)=\int_0^1 G(t,s)\varphi'(s)\phi_q\Big(\int_0^s f(\tau)d\tau\Big)ds$$

is the unique solution of (5), (6). ■

Lemma 2.3 We presumed that $(H_1),(H_2)$ hold. The function $G(t,s)$ described in (10) matches the following relationship $G(t,s)>0$ for all $t,s\in(0,1)$.

Proof: For $0\le t\le 1$, $\xi_{i-1}\le s\le\min\{\xi_i,t\}, i=1,2,\cdots,m-1$,

$$
\begin{aligned}
G(t,s) &= \frac{1}{\mu\Gamma(\alpha)}\Bigg[(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1}\\
&\quad-\sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}\\
&\quad-\mu(\varphi(t)-\varphi(s))^{\alpha-1}\Bigg]ds\\
&= \frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\Bigg[(\varphi(1)-\varphi(s))^{\alpha-1}\\
&\quad-\sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}\\
&\quad-\mu\Big(\frac{\varphi(t)-\varphi(s)}{\varphi(t)-\varphi(0)}\Big)^{\alpha-1}\Bigg]ds.
\end{aligned}
$$

Consider

$$\begin{aligned} g(t) &= (\varphi(1)-\varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1} \\ &-\mu\left(\frac{\varphi(t)-\varphi(s)}{\varphi(t)-\varphi(0)}\right)^{\alpha-1}. \end{aligned}$$

Then we get

$$\begin{aligned} g'(t) &= -\mu(\alpha-1)\left(\frac{\varphi(t)-\varphi(s)}{\varphi(t)-\varphi(0)}\right)^{\alpha-2} \\ &\frac{\varphi'(t)(\varphi(s)-\varphi(0))}{(\varphi(t)-\varphi(0))^2} < 0, \end{aligned}$$

which implies that $g(t)$ is a decreasing function for $0 \le t \le 1,\ \xi_{i-1} \le s \le \min\{\xi_i, t\}, i = 1, 2, \cdots, m-1$.
Moreover, we note that

$$\begin{aligned} g(1) &= (\varphi(1)-\varphi(s))^{\alpha-1} - \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1} \\ &-\mu\left(\frac{\varphi(1)-\varphi(s)}{\varphi(1)-\varphi(0)}\right)^{\alpha-1} \\ &= \sum_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}\frac{(\varphi(1)-\varphi(s))^{\alpha-1}}{(\varphi(1)-\varphi(0))^{\alpha-1}} \\ &- \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1} \\ &= \sum_{j=1}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}\frac{(\varphi(1)-\varphi(s))^{\alpha-1}}{(\varphi(1)-\varphi(0))^{\alpha-1}} \\ &- \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(0))^{\alpha-1}\frac{(\varphi(\xi_j)-\varphi(s))^{\alpha-1}}{(\varphi(\xi_j)-\varphi(0))^{\alpha-1}}. \end{aligned}$$

From

$$\frac{(\varphi(1)-\varphi(s))^{\alpha-1}}{(\varphi(1)-\varphi(0))^{\alpha-1}} > \frac{(\varphi(\xi_j)-\varphi(s))^{\alpha-1}}{(\varphi(\xi_j)-\varphi(0))^{\alpha-1}},$$

we get $g(1) > 0$. Hence, $G(t,s) > 0$ for $0 \le t \le 1,\ \xi_{i-1} \le s \le \min\{\xi_i, t\}, i = 1, 2, \cdots, m-1$.

For $0 \le t \le 1,\ \max\{\xi_{i-1}, t\} \le s \le \xi_i, i = 1, 2, \cdots, m-1$,

$$\begin{aligned} G(t,s) &= \frac{1}{\mu\Gamma(\alpha)}\Big[(\varphi(t)-\varphi(0))^{\alpha-1}(\varphi(1)-\varphi(s))^{\alpha-1} \\ &- \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}(\varphi(t)-\varphi(0))^{\alpha-1}\Big] \\ &= \frac{1}{\mu\Gamma(\alpha)}(\varphi(t)-\varphi(0))^{\alpha-1}\Big[(\varphi(1)-\varphi(s))^{\alpha-1} \\ &- \sum_{j=i}^{m-2}\alpha_j(\varphi(\xi_j)-\varphi(s))^{\alpha-1}\Big] > 0. \end{aligned}$$

Hence, $G(t,s) > 0$ for all $t, s \in (0,1)$. ■

Lemma 2.4 [21] We presumed that K is a normal cone of the Banach space E. $< v_0, u_0 >\in E$. Furthermore, there is an increasing and completely continuous $T :< v_0, u_0 >\to< v_0, u_0 >$. We can get T has a fixed point $u^* \in< v_0, u_0 >$.

Lemma 2.5 [22] Let E be an order Banach space, $K \subset E$ is a cone, and suppose that Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : K \to K$ be a completely continuous operator such that either
(C_1) $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_2$; or
(C_2) $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap \overline{\Omega_2} \setminus \Omega_1$.

III. Existence of positive solution

We define the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Thus we have $E = C[0,1]$ is a Banach space in the above norm case. Let $K \subset E$ defined by

$$K = \{u \in E :\ u(t) \ge 0, \quad 0 \le t \le 1\}.$$

Lemma 3.1 Let $T : K \to E$ is given by the following relation

$$(Tu)(t) = \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau, u(\tau)d\tau\right)ds. \quad (11)$$

Then $T :\ K \to K$ is a completely continuous operator.

Denote

$$A^{-1} = \frac{(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}\int_0^1 \varphi'(s)(\varphi(1)-\varphi(s))^{\alpha-1}ds,$$

$$B^{-1} = \int_0^1 G(1,s)\varphi'(s)\phi_q(s)ds,$$

$$\Phi(l) = \max\{f(t,u),\ (t,u) \in [0,1]\times[0,l]\},$$

$$\varphi(l) = \min\{f(t,u),\ (t,u) \in [0,1]\times[0,l]\}.$$

Theorem 3.1 We presumed that $(H_1), (H_2)$ hold. In addition, we can find two positive constants $a > b$ such that

$$\Phi(a) \le \phi_p(aA), \qquad \varphi(b) \ge \phi_p(bB). \quad (12)$$

Thus there has at least one positive solution for equation (1),(2).

Proof: In order to apply Lemma 2.5, we divide this Theorem in two steps.

Step 1. Let $\Omega_a = \{u \in K|\ \|u\| < a\}$. For any $u \in \partial\Omega_a$, there are $\|u\| = a$ and
$f(t, u(t)) \le \Phi(a) \le \phi_p(aA)$ for $(t,u) \in [0,1]\times[0,a]$.
Thus

$$\begin{aligned} |Tu(t)| &\le \int_0^1\left|G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,u(\tau)d\tau\right)\right|ds \\ &\le \int_0^1\Big[\frac{(\varphi(t)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}(\varphi(1)-\varphi(s))^{\alpha-1} \\ &\varphi'(s)\phi_q\left(\textstyle\int_0^1 f(\tau,u(\tau)d\tau\right)\Big]ds \\ &\le \int_0^1\Big[\frac{(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)}(\varphi(1)-\varphi(s))^{\alpha-1} \\ &\varphi'(s)\phi_q\left(\textstyle\int_0^1 \phi_p(aA)d\tau\right)\Big]ds \\ &= aA\frac{(\varphi(1)-\varphi(0))^{\alpha-1}}{\mu\Gamma(\alpha)} \\ &\int_0^1(\varphi(1)-\varphi(s))^{\alpha-1}\varphi'(s)ds = a. \end{aligned}$$

So, $\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_a$.

Step 2. We define $\Omega_b = \{u \in K|\ \|u\| < b\}$. We choose $u \in \partial\Omega_b$, thus $\|u\| = b$ and
$f(t,u(t)) \ge \varphi(b) \ge \phi_p(bB)$ for $(t,u) \in [0,1]\times[0,b]$.
So

$$\begin{aligned} Tu(1) &= \int_0^1 G(1,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,u(\tau)d\tau\right)ds \\ &\ge \int_0^1 G(1,s)\varphi'(s)\phi_q\left(\int_0^s \phi_p(bB)d\tau\right)ds \\ &= bB\int_0^1 G(1,s)\varphi'(s)\varphi_q(s)ds = b. \end{aligned}$$

This is equivalent to, $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_b$. At least one fixed point $u \in \overline{\Omega_a} \setminus \Omega_b$ was got by Lemma 2.5, which implies problem (1),(2) has a positive solution u, moreover, u satisfies $b \leq \|u\| \leq a$. ■

Theorem 3.2 We presumed that $(H_1),(H_2)$ hold and $f(t,\cdot)$ was an increasing function for each $t \in [0,1]$. If we can find ν_0, ω_0 satisfying $T\nu_0 \geq \nu_0,\ T\omega_0 \leq \omega_0$ for $0 \leq \nu_0 \leq \omega_0, 0 \leq t \leq 1$. Then a positive solution u^* satisfying $\nu_0 \leq u^* \leq \omega_0$ can be got for (1),(2).

Proof: Let $\nu, \omega \in K$ satisfy $\nu \leq \omega$, then $f(t,\nu(t)) \leq f(t,\omega(t))$, there are

$$\begin{aligned} T\nu(t) &= \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,\nu(\tau)d\tau\right)ds \\ &\leq \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,\omega(\tau)d\tau\right)ds \\ &= T\omega(t). \end{aligned}$$

Hence, T is an operator and it is increasing. We can say that $T :< \nu_0, \omega_0 > \rightarrow < \nu_0, \omega_0 >$ according to $T\nu_0 \geq \nu_0,\ T\omega_0 \leq \omega_0$. Based on the completely continuous property of T, T has a fixed point $u^* \in < \nu_0, \omega_0 >$, which is the required positive solution. ■

Theorem 3.3 We presumed that $(H_1),(H_2)$ hold and $f(t,\cdot)$ is an increasing function for each $t \in [0,1]$. Moreover, if $0 < \lim_{\|u\|\to\infty} f(t,u) < \infty$ for each $t \in [0,1]$, then problem (1),(2) has a positive solution.

Proof: The assumption $0 < \lim_{\|u\|\to\infty} f(t,u) < \infty$ implies that there exist positive constants M, R such that for $\|u\| \geq R,\ f(t,u) \leq M,\ \forall t \in [0,1]$. Write

$$d = \max\{f(t,u)|0 \leq \|u\| \leq R,\quad 0 \leq t \leq 1\}. \tag{13}$$

Then $f \leq d + M$. For the problem

$$(\phi_p(D_{0^+}^{\alpha;\ \varphi}u(t)))' + (d+M) = 0, \quad 0 < t < 1, \tag{14}$$

$$D_{0^+}^{\alpha;\ \varphi}u(0) = 0, \quad u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i). \tag{15}$$

Making use of Lemma 2.2, the solution of the integral equation

$$\omega(t) = \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s (d+M)d\tau\right)ds$$

is the same as the solution of (14),(15). Thus, we can say

$$\omega(t) \geq \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,\omega(\tau))d\tau\right)ds = T\omega(t),$$

this equals to $\omega \geq T\omega$.

On the other hand, for $\nu = 0$,

$$T\nu(t) = \int_0^1 G(t,s)\varphi'(s)\phi_q\left(\int_0^s f(\tau,\nu(\tau))d\tau\right)ds \geq 0,$$

we can find $T\nu \geq \nu$. Hence, as a consequence of the Theorem 3.2, we can find a positive solution for problem (1),(2). ■

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