# A Novel Tolerantly Complete Layering Method for Fuzzy Mean-Variance-Skewness Portfolio Model within Transaction Costs

Xue Deng, Chuangjie Chen and Yanchun Wan

***Abstract*—Numerous studies points out that the return rate distribution of portfolio is generally asymmetric. To quantify the asymmetry of return distribution, the third-order central moment is introduced to generate a larger payoff in addition to expected mean and variance. This study regards the return rate as asymmetric triangular fuzzy number and then constructs a tri-objective fuzzy portfolio model, including mean, variance and skewness, within the constraint of V-shape transaction cost. To effectively solve the tri-objective model, a novel algorithm named tolerantly complete layering method is designed. This method takes investors' tolerance for each objective into account, expands the feasible region, and can provides different optimal solutions. By a practical numerical example, we present the effectiveness of our model and method. Finally, compared with the method given by Chang (2009), it is pointed out that our method can generate diverse optimal solutions, which is more flexible to satisfy investors' subjective preference for each objective.**

***Index Terms*—third-order central moment, Triangular fuzzy number, credibility theory, mean-variance-skewness, tolerantly complete layering method**

## I. Introduction

As a hot topic of modern finance theory, portfolio selection aims to obtain a kind of combination of securities which can best meet an investor's demand for return and risk. The classical mean-variance model [1]–[2] is proposed by Markowitz. This model takes expected value and variance as the measures to quantify return and risk, and assumes that asset return follows a normal distribution. This model is viewed as the pioneer for modern portfolio selection, and some typical researchers, such as Sharpe et al. [3], Merton [4], Giove et al. [5] and Gupta et al. [6], have done lots of research work to develop this model. In addition, considering that investors' demand is always complex, it is necessary to build up the multi-objective portfolio model in practical financial market, which has been studied by [7]–[11].

As mentioned above, the mean-variance model regards expected value and variance as the measures to quantify return and risk of a portfolio. Namely, the model only considers the first-order moment and the second-order moment of asset return rate. In fact, these two factors are not typically competent to explain the performance of portfolios in the practical financial market. Afterwards, many studies indicate that some higher order moments of asset return can better account for the problem. For example, Arditti et al. [12] analyzed the multi-period portfolio efficiency with the consideration of third-order moment; Bhattacharyya et al. [13] presented a fuzzy tri-objective model within third-order moment based on the interval analysis; Li et al. [14] presented a model that uses the moments of first, second and third order of fuzzy returns; Jiang et al. [15] constructed a model for assets with systematic skewness and then researched on the influence of systematic skewness. Thus, it is meaningful to build up a portfolio model that considers expected value, variance and skewness, simultaneously.

Meanwhile, many researchers also realized that transaction cost is another critical factor for the portfolio optimization problem. For example, in [16], it is pointed out that portfolio will lose efficiency if the transaction cost is not taken into account; By analyzing the empirical data, Yoshimoto [17] obtained the similar conclusion to [16]; Roy et al. [18] presented the model that quantifies the transaction cost as fuzzy number and then adopted different methods to defuzzy; Fang et al. [19] also put forward a portfolio model within the constraint of transaction cost in fuzzy environment; Liu et al. [20] took the small transaction cost into account and make the analysis about the allocation of investment ratios; Deng et al. [21] adopted skewness and entropy in an intuitionistic fuzzy model and then presented an operator named "max-min" to solve their model. Except for transaction cost, some of the portfolio applications involve some revision of an existing portfolio according to the variation of financial market and the investors' risk preference. Similarly, in our paper, we will take three criteria into account: expected value, variance and skewness, to build up the corresponding portfolio model within transaction cost which can rebalance the existing portfolio.

We also pay attention to the fact that in recent years, more and more researchers prefer to use fuzzy variable rather than

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random variable to denoted return rate. As we know, Bellman and Bellman [22] proposed the fuzzy decision theory in 1970 which can offer some theory basis for portfolio research. Since then, many researchers did the portfolio research in fuzzy environment. For example, Ramaswamy [23] found a new way for portfolio selection on the basis of fuzzy decision theory; Deng et al. [24] researched on the possibility theory and then presented a fuzzy portfolio model within borrowing constraint; [25] – [29] also studied fuzzy portfolio from different aspects. As portfolio models become more and more complex, how to solve the model becomes another hottest topic. For different kinds of models, many researchers proposed various solving algorithms, such as [30] – [32] presented several linear or nonlinear programming methods; Chang et al. [33]–[34], Dastkhan et al. [35] and Xiang et al. [36] put forward some intelligence algorithms; Deng et al. [37] presented strictly mathematical method to solve portfolio models.

The rest of this paper is organized as follows. In Section 2, the credibility theory is introduced including three basic definitions and some common properties. In Section 3, the fuzzy M-V-Sk model within transaction costs is constructed. In Section 4, the concept and process of the tolerantly complete layering method are clearly stated and then adopted to solve our proposed model. By a numerical example, we present the effectiveness of our model and method in Section 5. Finally, the conclusion of our work is given in Section 6.

## II. Mean, Variance, Skewness of Fuzzy Variables

### A. Credibility Theory

Fuzzy set, put forward by Zadeh [38], is determined by the membership function. One of the common tools used for defuzzification is the possibility theory [39]. However, this theory not only deviates from the law of truth conservation, but also deviates from the laws of excluded middle and contradiction (see Liu [40]). This problem is caused by the non-self-duality of the possibilistic measure. In fact, self-duality is essential to a measure in mathematical models and financial markets. Thus, in theory and practical application, this property should be satisfied. As we know, the maximum possibility value 1 means that we cannot obtain any information from a fuzzy event. Since in some particular case, possibility being 1 means that any value on a domain is possible. At this time, the corresponding possibility measure is then said to be vacuous. The credibility theory put forward by Liu [40] is developed on the basis of possibility theory. This theory emphasizes on self-duality and defines that the fuzzy event with credibility value 1 is inevitable. For its effectiveness, we take credibility theory as the main tool for defuzzification. That is, the return of portfolio will be denoted as fuzzy variable and the credibility measure will be used to defuzzy the variable.

**Definition 1.** [40] Let $\mu$ be a membership function, $R$ be the real number system. Then $\forall A \subset R$, $x \in R$, the credibility of fuzzy event $\xi \in A$ is

$$\mathrm{Cr}\{\xi \in \mathrm{A}\} = \frac{1}{2}\left(\sup_{x\in A}\mu(x) + 1 - \sup_{x\in A^C}\mu(x)\right). \tag{1}$$

Formula (1) is called the credibility inversion theorem because we can derive $\mu$ from the credibility of $\xi$ by the following way:

$$\mu(x) = \left(2\mathrm{Cr}\{\xi = x\}\right)\wedge 1, \quad \forall x \in R. \tag{2}$$

### B. Expected Value

Expected value can be regarded as the average credibility of a fuzzy variable, and it is usually used for the quantitative comparison of fuzzy variables.

**Definition 2.** [40] The expected value of $\xi$ is given by:

$$E[\xi] = \int_0^{+\infty}\mathrm{Cr}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathrm{Cr}\{\xi \le x\}\mathrm{d}x. \tag{3}$$

It should be noted that the above integrals are required to be finite.

Obviously, any expected value belongs to $R$. If the result is finite, that is, the expected value exists, it means that effective expected value is achieved; otherwise, if the finite result does not exist, it means non-effective expected value is achieved.

### C. Variance

Variance can describe the deviation between the distribution of a fuzzy variable and its expected value. A smaller variance will lead to a distribution closer to the expected value; conversely, it indicates that the distribution around the expected value is unsteady.

**Definition 3.** Suppose the expected value $e$ of $\xi$ is finite, then its variance is given by:

$$V[\xi] = E[(\xi - e)^2] = \int_0^{+\infty}\mathrm{Cr}\left\{(\xi - e)^2 \ge x\right\}\mathrm{d}x. \tag{4}$$

Namely,

$$V[\xi] = \int_0^{+\infty}\mathrm{Cr}\left\{(\xi \ge e + \sqrt{x}) \cup (\xi \le e - \sqrt{x})\right\}\mathrm{d}x. \tag{5}$$

According to the definition above, it is not difficult to find that variance involves the part that "$\xi$ is less than $e$ (the expected value) " and the part that "$\xi$ is greater than $e$". In most studies, it is common to take the expected value and variance as a kind of measure to quantify return and risk of portfolio.

### D. Skewness

**Definition 4.** Suppose the expected value $e$ of $\xi$ is finite, then its skewness is given by:

$$\begin{aligned} Sk[\xi] &= \mathrm{E}[(\xi - \mathrm{E}[\xi])^3] = \mathrm{E}[(\xi - e)^3] \\ &= \int_0^{+\infty}\mathrm{Cr}\left\{(\xi - e)^3 \ge r\right\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\left\{(\xi - e)^3 \le r\right\}\mathrm{d}r. \end{aligned} \tag{6}$$

It turns out that if $\mu$, the membership function of $\xi$, is symmetric, then its left tail will be symmetrical to the right tail and we have $Sk[\xi] = 0$. A negative skewness implies the left tail is stronger than the right tail; similarly, a positive value denotes that the right tail is stronger. Skewness is crucial in practical financial markets, because the asset returns distribution is usually asymmetric, in other words, the skewness of assets is usually positive or negative but not 0. By understanding the skew state of the data, we can better estimate the efficiency of a portfolio.

**Example 1.** By giving $a, b, c$ that belongs to $R$, $a < b < c$, the triangular fuzzy number $\xi$ is defined with the membership function:

$$\mu(x)=\begin{cases}\dfrac{x-a}{b-a}, & \text{if } a\le x\le b,\\ \dfrac{x-c}{b-c}, & \text{if } b\le x\le c,\\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

According to $\mathrm{Cr}\{\xi\le x\}+\mathrm{Cr}\{\xi\ge x\}=1$ and (1), the credibility of $\xi$ is

$$\mathrm{Cr}\{\xi\le x\}=\begin{cases}0, & \text{if } x\le a,\\ \dfrac{x-a}{2(b-a)}, & \text{if } a\le x\le b,\\ \dfrac{x+c-2b}{2(c-b)}, & \text{if } b\le x\le c,\\ 1, & \text{if } x\ge c,\end{cases} \tag{8}$$

$$\mathrm{Cr}\{\xi\ge x\}=\begin{cases}1, & \text{if } x\le a,\\ \dfrac{2b-a-x}{2(b-a)}, & \text{if } a\le x\le b,\\ \dfrac{c-x}{2(c-b)}, & \text{if } b\le x\le c,\\ 0, & \text{if } x\ge c.\end{cases} \tag{9}$$

**Example 2.** Suppose $\xi$ is a fuzzy number with (7), it is trivial to obtain the following results:

$$E[\xi]=\frac{a+2b+c}{4}. \tag{10}$$

$$V[\xi]=\frac{33\alpha^3+21\alpha^2\beta+11\alpha\beta^2-\beta^3}{384\alpha} \tag{11}$$

$$Sk[\xi]=\frac{(c-a)^2(c+a-2b)}{32}. \tag{12}$$

where $\alpha=\max\{c-b,b-a\},\beta=\min\{c-b,b-a\}$.

**Theorem.** If $\xi$ and $\eta$ are the independent fuzzy variables with finite expected values, let $\phi$, $\varphi$ be two numbers belonging to $R$, according to (3) –(6), we can get that

$$E[\phi\xi+\varphi\eta]=\phi E[\xi]+\varphi E[\eta]; \tag{13}$$

$$E[\xi+\phi]=E[\xi]+\phi; \tag{14}$$

$$V[\xi+\phi]=V[\xi]; \tag{15}$$

$$Sk[\xi+\phi]=Sk[\xi]. \tag{16}$$

## III. The Mean-variance-skewness Portfolio Models within/without Transaction Costs

Markowitz presented the classical portfolio model in [1]–[2] with the method to construct the optimal portfolio by maximizing the return under a fixed risk level or minimizing the risk under a fixed return level in a presupposed stochastic environment. As mentioned above, he took the expected value and variance as a kind of measures to quantify return and risk.

For convenience, let $x_i$ denote the investment ratios that will be assigned to the $i$-th asset $(i=1,2,\cdots,n)$; $x_i^0$ denote the investment ratio that have assigned to the $i$-th asset; $u_i$, $l_i$ respectively represent the upper and lower bounds of $x_i$; $k_i$ be the ratio of transaction cost of the $i$-th asset; $\xi_i$ be the return rate of the $i$-th asset and denote by fuzzy variable. Now, we use a column vector $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^T$ to represent the portfolio, then its return rate can be denoted by $\sum_{i=1}^n x_i\xi_i$. In this paper, prime ($^T$) is used to denote the transposition of a matrix. In addition, the variables $\xi_i$ are all independent.

Suppose that the investment strategy is self-financed, namely, any additional fund will not be invested in the adjusting process of portfolio. Then, for the $i$-th asset, its transaction cost is expressed by

$$C_i(x_i)=k_i\,|\,x_i-x_i^0\,|. \tag{17}$$

By (17), we can directly get that the total transaction cost of the investment strategy $\boldsymbol{x}$ is

$$C(\boldsymbol{x})=\sum_{i=1}^n C_i(x_i)=\sum_{i=1}^n k_i\,|\,x_i-x_i^0\,|. \tag{18}$$

According to (3)–(5), we can construct a M-V-Sk portfolio model without transaction costs, as shown below:

$$\begin{cases}\max\ E[\xi]=\sum_{i=1}^n x_iE[\xi_i]\\ \min\ V[\xi]=V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\\ \max\ Sk[\xi]=Sk[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\\ \text{s.t.}\quad \sum_{i=1}^n x_i=1,\\ \qquad l_i\le x_i\le u_i,\ \ i=1,2,\cdots,n,\end{cases} \tag{19}$$

In practical financial markets, it is necessary to consider the impact of transaction cost. Thus, we modify the model to introduce transaction costs, as shown below:

$$\begin{cases}\max\ E[\xi]-C(\boldsymbol{x})=\sum_{i=1}^n x_iE[\xi_i]-C(\boldsymbol{x})\\ \min\ V[\xi]=V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\\ \max\ Sk[\xi]=Sk[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]\\ \text{s.t.}\quad \sum_{i=1}^n x_i=1,\\ \qquad l_i\le x_i\le u_i,\ \ i=1,2,\cdots,n.\end{cases} \tag{20}$$

If $\xi_i$ is the triangular fuzzy number determined by the parameters $a_i,b_i,c_i$, then $\xi=\left(\sum_{i=1}^n x_ia_i,\sum_{i=1}^n x_ib_i,\sum_{i=1}^n x_ic_i\right)$. By (20), we can obtain the following specific M-V-Sk model within transaction costs:

$$\begin{cases}\max\ E[\xi]-C(\boldsymbol{x})=\dfrac{\sum_{i=1}^n x_ia_i+2\sum_{i=1}^n x_ib_i+\sum_{i=1}^n x_ic_i}{4}-C(\boldsymbol{x})\\ \min\ V[\xi]=\dfrac{33\alpha'^3+21\alpha'^2\beta+11\alpha'\beta'^2-\beta'^3}{384\alpha'}\\ \max\ Sk[\xi]=\dfrac{\left(\sum_{i=1}^n x_i\left(c_i-a_i\right)\right)^2\left(\sum_{i=1}^n x_i\left(c_i+a_i-2b_i\right)\right)}{32}\\ \text{s.t.}\quad \sum_{i=1}^n x_i=1,\\ \qquad l_i\le x_i\le u_i,\ \ i=1,2,\cdots,n.\end{cases} \tag{21}$$

Where $a'=\sum_{i=1}^n x_ia_i$, $b'=\sum_{i=1}^n x_ib_i$, $c'=\sum_{i=1}^n x_ic_i$, $\alpha'=\max\{c'-b',b'-a'\}$, $\beta'=\min\{c'-b',b'-a'\}$.

## IV. Tolerantly Complete Layering Method

### A. The Basic Ideas of Tolerantly Complete Layering Method

Suppose there is a multi-objective minimization model as follows:

$$L-\min_{x\in X}[P_s f_s(x)]_{s=1}^{m}, \tag{22}$$

where $m$ represents the number of the objective functions and $m \geq 2$ ; $P_s f_s(x)$ presents that the objective function $f_s(x)$ is located on the $s-th$ layer. Each objective function will be allocated a different priority, and they will be placed at different priority layers. It means that for each priority layer, there is only one objective function. According to the priority, the optimal solution of each layer will be gradually obtained, and that of the last layer will be taken as the optimal solution of (22). This stated idea is called complete layering method.

We pay attention to the fact that if the optimal solution of an intermediate priority layer is unique, then it must be the optimal solution of the last layer. In this case, it is not necessary to enter the next priority layer. To better handle this case, the following improvement is considered for the complete layering method: after solving each priority layer, the appropriate tolerant mounts will be given to adjust the range of optimal solutions. It means that the feasible region of the next layer function will be properly relaxed. Such improved method for solving (22) is called tolerantly complete layering method.

### B. Computation Steps of Tolerantly Complete Layering Method

**Step 1:** Define the initial feasible region. Take $X^1 = X$ and $s := 1$.

**Step 2:** Minimize the layering question. Solve the minimization question of the $s$-th priority layer function $\min_{x\in X^s} f_s(\boldsymbol{x})$ to obtain the optimal solution $\boldsymbol{x}^s$, then the corresponding value $f_s(\boldsymbol{x}^s)$ can be calculated.

**Step 3:** Check the times of iterations. In other words, check the priority of the current layer.

1) If $s = m$, output $\tilde{\boldsymbol{x}} = \boldsymbol{x}^m$;

2) If $s < m$, then go on Step 4.

**Step 4:** Construct the feasible region for the next layer. Provide the tolerant amount $\delta_s$ of the $s$-th priority layer with $\delta_s > 0$, and then the tolerant feasible region of the $(s+1)$-th priority layer will be

$$X^{s+1} = \left\{ \boldsymbol{x} \in X^s \,\middle|\, f_s(\boldsymbol{x}) \leq f_s(\boldsymbol{x}^s) + \delta_s \right\}. \tag{23}$$

Make $s := s+1$, turn to Step 2.

## V. Numerical Example

In this section, a practical dataset will be used to present the effectiveness of our M-V-Sk model and solving method. This dataset contains the return of 10 securities, which are characterized by triangular fuzzy numbers, as shown in Table I.

TABLE I
RETURNS OF 10 STOCKS AND THE CORRESPONDING EXPECTED VALUES

| asset $i$ | $a_i$ | $b_i$ | $c_i$ | $E[\xi_i]$ |
|---|---|---|---|---|
| 1 | –0.2 | 2.1 | 2.5 | 1.6250 |
| 2 | –0.1 | 1.9 | 3.0 | 1.6750 |
| 3 | –0.4 | 3.0 | 4.0 | 2.4000 |
| 4 | –0.1 | 2.0 | 2.5 | 1.6000 |
| 5 | –0.6 | 3.0 | 4.0 | 2.3500 |
| 6 | –0.2 | 2.5 | 3.0 | 1.9500 |
| 7 | –0.2 | 3.0 | 3.5 | 2.3250 |
| 8 | –0.4 | 2.5 | 4.0 | 2.1500 |
| 9 | –0.3 | 2.8 | 3.2 | 2.1250 |
| 10 | –0.3 | 2.0 | 2.5 | 1.5500 |

According to (21), we assume $k_i = 0.003, x_i^0 = 0$ and then obtain the M-V-Sk model with transaction costs (See (24)) with $\alpha = 2.3x_1 + 2.0x_2 + 3.4x_3 + 2.1x_4 + 3.6x_5 + 2.7x_6 + 3.2x_7 + 2.9x_8 + 3.1x_9 + 2.3x_{10}$ , $\beta = 0.4x_1 + 1.1x_2 + 1.0x_3 + 0.5x_4 + 1.0x_5 + 0.5x_6 + 0.5x_7 + 1.5x_8 + 0.4x_9 + 0.5x_{10}$.

### A. Using Tolerantly Complete Layering Method

Next, we will solve (24) by the tolerantly complete layering method proposed in this paper.

**Step 1:** Define the initial feasible region. Take

$$X^1 = X = \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} \sum_{i=1}^{10} x_i = 1, \\ 0.05 \leq x_i \leq 0.8, i = 1,2,\cdots,10. \end{array} \right\} \tag{25}$$

and $s := 1$.

**Step 2:** Minimize the $1-th$ layering question. After solving the minimization question of the $1-th$ priority layer objective function:

$$\begin{cases} \min\ f_1(\boldsymbol{x}) = -E[\xi] + C(\boldsymbol{x}) = -\big(1.622x_1 + 1.672x_2 + 2.397x_3 + 1.597x_4 + 2.347x_5 \\ \qquad + 1.947x_6 + 2.322x_7 + 2.147x_8 + 2.122x_9 + 1.547x_{10}\big) \\ \min\ f_2(\boldsymbol{x}) = V[\xi] = \dfrac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha} \\ \min\ f_3(\boldsymbol{x}) = -Sk[\xi] = \dfrac{\begin{pmatrix} 2.7x_1 + 3.1\,x_2 + 4.4\,x_3 + 2.6\,x_4 + 4.6\,x_5 + \\ 3.2\,x_6 + 3.7\,x_7 + 4.4x_8 + 3.5\,x_9 + 2.8x_{10} \end{pmatrix}^2 \cdot \begin{pmatrix} 1.9x_1 + 0.9x_2 + 2.4x_3 + 1.6x_4 + 2.6x_5 \\ + 2.2x_6 + 2.7x_7 + 1.4x_8 + 2.7x_9 + 1.8x_{10} \end{pmatrix}}{32} \\ \text{s.t.}\quad \sum_{i=1}^{10} x_i = 1, \\ \qquad 0.05 \leq x_i \leq 0.8, i = 1,2,\cdots,10. \end{cases} \tag{24}$$

$$\min_{x\in X^1} f_1(\boldsymbol{x}) = -(1.622x_1 + 1.672x_2 + 2.397x_3 + 1.597x_4 + 2.347x_5 + 1.947x_6 + 2.322x_7 + 2.147x_8 + 2.122x_9 + 1.547x_{10}), \tag{26}$$

the optimal solution $\boldsymbol{x}^1$ and the optimal function value $f_1(\boldsymbol{x}^1)$ are obtained as follows:

$\boldsymbol{x}^1 = (0.05, 0.05, 0.55, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05)^T$,

$f_1(\boldsymbol{x}^1) = -2.1845$.

**Step 3:** Check the times of iterations. Obviously, $s = 1 < m = 3$, thus, we go to Step 4.

**Step 4:** Construct the feasible region for the next layer. In this step, the tolerant amount of the 1-th priority layer is set to $\delta_1 = 0.10 > 0$, and then, the tolerant feasible region of the 2-th priority layer will be

$$X^2 = \left\{ \boldsymbol{x} \in X^1 \,\middle|\, f_1(\boldsymbol{x}) \le f_1(\boldsymbol{x}^1) + \delta_1 \right\} = \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} \sum_{i=1}^{10} x_i = 1, \\ 0.05 \le x_i \le 0.8, i = 1, 2, \cdots, 10, \\ f_1(\boldsymbol{x}) \le -2.1845 + 0.10. \end{array} \right\} \tag{27}$$

Make $s := s + 1$, we turn to Step 2.

**Step 2:** Minimize the $2-th$ layering question. Go on to solve the minimization question of the $2-th$ priority layer objective function

$$\min_{x\in X^2} f_2(\boldsymbol{x}) = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}. \tag{28}$$

The optimal solution $\boldsymbol{x}^2$ and the optimal function value $f_2(\boldsymbol{x}^2)$ are obtained as follows:

$\boldsymbol{x}^2 = (0.05, 0.05, 0.05, 0.1362, 0.05, 0.05, 0.4638, 0.05, 0.05, 0.05)^T$

$f_2(\boldsymbol{x}^2) = 0.8240$.

**Step 3:** Check the times of iterations. Go on to check the priority of the current layer. Since $s = 2 < m = 3$, we still need to turn to Step 4 again.

**Step 4:** Construct the feasible region of the next layer. In this step, the tolerant amount of the 2-th priority layer is set to $\delta_2 = 0.06 > 0$, and the tolerant feasible region of the 3-th priority layer is

$$X^3 = \left\{ \boldsymbol{x} \in X^2 \,\middle|\, f_2(\boldsymbol{x}) \le f_2(\boldsymbol{x}^2) + \delta_2 \right\} = \left\{ \boldsymbol{x} \,\middle|\, \begin{array}{l} x_1 + x_2 + \cdots + x_{10} = 1, \\ 0.05 \le x_i \le 0.8, i = 1, 2, \cdots, 10, \\ f_1(\boldsymbol{x}) \le -2.1845 + 0.10, \\ f_2(\boldsymbol{x}) \le 0.8240 + 0.06. \end{array} \right\} \tag{29}$$

Make $s := s + 1$, we turn to Step 2.

**Step 2:** Minimize the layering question. By solving the minimization question of the 3-th priority layer objective function

$$\min_{x\in X^3} f_3(\boldsymbol{x}) = \big(2.7x_1 + 3.1\,x_2 + 4.4\,x_3 + 2.6\,x_4 + 4.6\,x_5 + 3.2\,x_6 + 3.7\,x_7 + 4.4x_8 + 3.5\,x_9 + 2.8x_{10}\big)^2 \times \big(1.9x_1 + 0.9x_2 + 2.4x_3 + 1.6x_4 + 2.6x_5 + 2.2x_6 + 2.7x_7 + 1.4x_8 + 2.7x_9 + 1.8x_{10}\big) / 32 \tag{30}$$

the optimal solution $\boldsymbol{x}^3$ and the optimal function value $f_3(\boldsymbol{x}^3)$ are obtained as:

$\boldsymbol{x}^3 = (0.05, 0.1462, 0.05, 0.05, 0.05, 0.05, 0.4538, 0.05, 0.05, 0.05)^T$

$f_3(\boldsymbol{x}^3) = 0.8575$.

**Step 3:** Check the times of iterations. Now, we have $s = m = 3$, thus we should output the final result:

$\tilde{\boldsymbol{x}} = \boldsymbol{x}^3 = (0.05, 0.1462, 0.05, 0.05, 0.05, 0.05, 0.4538, 0.05, 0.05, 0.05)^T$.

$f_1(\tilde{\boldsymbol{x}}) = -2.0845,\ f_2(\tilde{\boldsymbol{x}}) = -0.8242,\ f_3(\tilde{\boldsymbol{x}}) = 0.8575.$

According to these results, we can see that the expected mean, variance and skewness are 2.0845, 0.8242 and -0.8575, respectively.

For further analysis, we change the tolerant amounts of the 1-th priority layer (because the 1-th priority layer will affect all the layers in the rear), and then obtain different optimal solutions of (24), as shown in Table II. Besides, the objective function values of each solution are calculated and then shown in Fig 1.

Form Table II and Fig 1, we can conclude that:

1) The expected mean increases while the variance increases.
2) The skewness value is negative at all time, which points out that the returns of 10 securities described by triangular fuzzy numbers are not symmetrical, but skewed to the left.
3) As the expected mean increases, the absolute value of skewness increases. It means that the corresponding skewness degree becomes larger along with the enlarging of the expected mean.
4) Since the expected mean, skewness degree and variance have the same trend, investors can easily adjust the portfolio strategy to meet their risk preference by choosing a suitable value for $\delta_1$. As we can see, if the value of $\delta_1$ increases, the expected mean, skewness degree and variance will decrease. Thus, if investors are risk averse, they can choose a smaller value for $\delta_1$; otherwise, a bigger value of $\delta_1$ will be more appropriate.

TABLE II
THE VALUES OF MEAN, VARIANCE AND SKEWNESS OF (24) AS TO DIFFERENT VALUES OF TOLERANT AMOUNT

| $\delta_1$ | $\delta_2$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | mean | var | sk |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.07 | 0.06 | 0.05 | 0.1000 | 0.05 | 0.05 | 0.05 | 0.05 | 0.5000 | 0.05 | 0.05 | 0.05 | 2.1145 | 0.8484 | -0.9041 |
| 0.10 | 0.06 | 0.05 | 0.1462 | 0.05 | 0.05 | 0.05 | 0.05 | 0.4538 | 0.05 | 0.05 | 0.05 | 2.0845 | 0.8242 | -0.8575 |
| 0.13 | 0.06 | 0.05 | 0.1923 | 0.05 | 0.05 | 0.05 | 0.05 | 0.4077 | 0.05 | 0.05 | 0.05 | 2.0545 | 0.8005 | -0.8121 |
| 0.16 | 0.06 | 0.05 | 0.2385 | 0.05 | 0.05 | 0.05 | 0.05 | 0.3615 | 0.05 | 0.05 | 0.05 | 2.0245 | 0.7771 | -0.7678 |
| 0.21 | 0.06 | 0.05 | 0.3154 | 0.05 | 0.05 | 0.05 | 0.05 | 0.2846 | 0.05 | 0.05 | 0.05 | 1.9745 | 0.7391 | -0.6964 |
| 0.26 | 0.06 | 0.05 | 0.3923 | 0.05 | 0.05 | 0.05 | 0.05 | 0.2077 | 0.05 | 0.05 | 0.05 | 1.9245 | 0.7022 | -0.6280 |
| 0.31 | 0.06 | 0.05 | 0.4395 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.1605 | 0.05 | 0.05 | 1.8745 | 0.6847 | -0.5615 |
| 0.36 | 0.06 | 0.05 | 0.5447 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.0553 | 0.05 | 0.05 | 1.8245 | 0.6325 | -0.4998 |
| $\ge 0.37$ | 0.06 | 0.05 | 0.55 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 1.8220 | 0.6299 | -0.4969 |

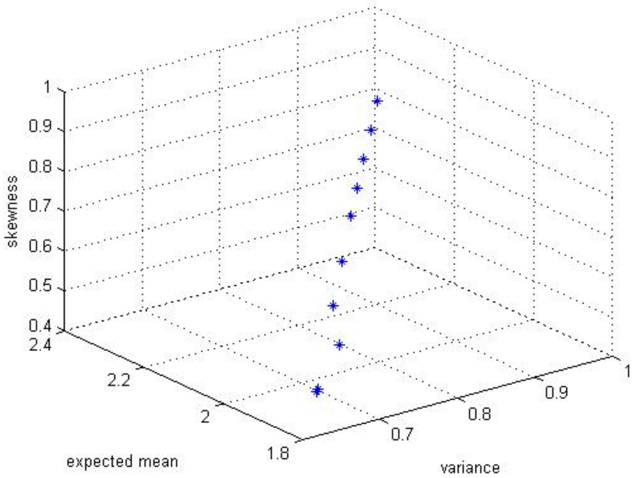

Fig. 1. The expected mean, variance and skewness of (24)

### B. Comparison with Linearly Weighted Method

To better describe the superiority of the proposed method, we also use the linearly weighted method presented by Chang [33] to solve (24). It is written as

$$\begin{cases} \min\ f_1 \\ \min\ f_2 \\ \min\ f_3 \\ \text{s.t.}\ \ \sum_{i=1}^{10} x_i = 1, \\ \qquad 0.05 \le x_i \le 0.8, i = 1, 2, \cdots, 10. \end{cases} \tag{31}$$

Where $f_1, f_2, f_3$ are the same as (24).

According to the linearly weighted method, we need to choose a pair of values $\lambda, \mu$ as the preferred parameters of $f_1, f_2$. And then, (31) is written as:

$$\begin{cases} \min \lambda f_1 + \mu f_2 + (1-\lambda-\mu) f_3 \\ \text{s.t.}\ \ \sum_{i=1}^{10} x_i = 1, \\ \qquad 0.05 \le x_i \le 0.8, i = 1, 2, \cdots, 10. \end{cases} \tag{32}$$

If we fix the values of $\lambda, \mu$, it is doubtless that the optimal solution of (32) will be unique. For instance, let $\lambda = 0.2$, $\mu = 0.4$, we have

$\tilde{\boldsymbol{x}} = (0.05, 0.55, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05, 0.05)^T$.

In this case, the values of each objective functions are $f_1(\tilde{\boldsymbol{x}}) = -1.8220$, $f_2(\tilde{\boldsymbol{x}}) = -0.6299$, $f_3(\tilde{\boldsymbol{x}}) = 0.4969$. That is, the expected mean, variance and skewness are 1.8220, 0.6299 and $-0.4969$. Compared with the tolerantly complete layering method, for the same model as (24), it should be noted our method can provide multiple groups of solutions based on the investors' tolerance amount, which really reflects the investors' preference for each objective function.

## VI. Conclusion

Considering that skewness and transaction cost are indispensable to the practical financial market, we present a M-V-Sk portfolio model within transaction costs and asymmetric triangular fuzzy return rates. Skewness can measure the degree of asymmetry and quantify the skewness degree to the center point. To solve our model, a novel algorithm called tolerantly complete layering method is proposed. For the same model, o our method can provide multiple optimal solutions which can reflect the investor's subjective preferences for each objective, while the linearly weighted method of Chang [33] can only offer one solution. The tolerantly complete layer method considers investors' subjective preference and allows them to choose the proper tolerant amounts to construct the tolerant feasible region, thus it is effective and useful in the practical financial market.

In the future, we will develop another intelligent multi-objective algorithm that not only reflects investor's preferences but also has an outstanding performance in accuracy and convergence speed..

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