High Energy Asymptotics for the One-dimensional Perturbed Harmonic Oscillator

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***Abstract*—The main goal of this paper is to study the one-dimensional perturbation $L = H+V$, where H is the harmonic oscillator and V is a decreasing scalar potential. It is well known that the eigenvalues of L can be expressed as $2k+1+\mu_k$. The main result of the paper is to describe an asymptotic formula for the fluctuation $\{\mu_k\}$ when k tends to infinity.**

***Index Terms*—Averaging method, Pseudo-differential operator, Perturbation theory, Spectrum, Eigenvalue asymptotics.**

I. Introduction

We consider in $L^2(\mathbb{R})$ the harmonic oscillator H defined by:

$$H = -\frac{d^2}{dx^2} + x^2. \tag{1}$$

We recall that H is a differential operator self-adjoint with compact resolvent [1]. Its spectrum is the sequence of simple eigenvalues $\{\lambda_k = 2k+1\}_{k\in\mathbb{N}}$.

Let $V \in \mathbb{C}^\infty(\mathbb{R},\mathbb{R})$ be a scalar potential that satisfies the following estimate:

$$|V^{(n)}(x)| \leq \frac{C_n}{\sqrt{1+x^2}},\ x\in\mathbb{R},\ n\in\mathbb{N}. \tag{2}$$

The operator $L = H + V$ is self-adjoint with compact resolvent [2]. By using the Min-Max theorem [3], we can confirm that the spectrum of L can be written in the form $\{\lambda_k + \mu_k\}$. Our main goal is to study the asymptotic behavior of the fluctuation μ_k when k tends to infinity. Our main result:

Theorem 1 [Main Theorem]:
μ_k admits the following asymptotic expansion:

$$\mu_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\sqrt{\lambda_k}\sin t)dt + O(\frac{\log^2\lambda_k}{\lambda_k^{1-\eta}}) \tag{3}$$

where η is arbitrarily chosen in $\left]0,\frac{1}{2}\right[$.

Many authors are interested in this kind of problem, among them we can qote M.Klein [4] who studied the perturbation:

$$D = -\frac{d^2}{dx^2} + x^2 + q(x), \tag{4}$$

such that: q, q' and $x \to \int_0^x q(s)ds$ are bounded.
He proved that μ_k admits the following asymptotic expansion:

$$\mu_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} q(\sqrt{\lambda}_k sin\theta)\,\mathrm{d}\theta + O(k^{\frac{-1}{3}}), \qquad k\to+\infty. \tag{5}$$

We recall that we studied in [5] the perturbation (6):

$$-\frac{d^{2m}}{dx^{2m}} + x^{2m} + V(x), \quad m\in\mathbb{N}^*, \tag{6}$$

where V is a decreasing scalar potential verifying the following estimate:

$$|V^{(n)}(x)| \leq C_n(1+x^2)^{-\frac{s}{2}},\ x\in\mathbb{R},\ n\in\mathbb{N},\ s\in\mathbb{R}_+^* - \{1\}\,. \tag{7}$$

We proved that μ_k admits the following asymptotic behavior:

$$\mu_k = \frac{1}{T}\int_{-1}^{1}\frac{V(y\lambda_k^{\frac{1}{2m}})}{(1-y^{2m})^{1-\frac{1}{2m}}}dy + \mathrm{O}(\lambda_k^{\frac{-\delta-1}{2m}}), \quad \forall m\geq 2, \tag{8}$$

where

$$T = \int_{-1}^{1}(1-u^{2m})^{\frac{1}{2m}-1}du, \tag{9}$$

and

$$\delta = \begin{cases} s & if \quad 0<s<1, \\ 1 & if \quad s>1. \end{cases} \tag{10}$$

For "$m = 1$" which is the harmonic oscillator case, we proved that:

$$\mu_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V\left(\sqrt{\lambda_k}\sin t\right)dt + O\left(\lambda_k^{-\delta+\eta}\right), \tag{11}$$

where

$$\eta \in \left]0,\frac{\delta}{2}\right[. \tag{12}$$

We established the asymptotic behavior of μ_k, but the case "s = 1" which is very classic was a borderline case. In this work we manage to give the best possible estimate for μ_k. Our main tool is the averaging method of Weinstein ([6], [7]), it consists of replacing V in $L = H + V$ by the average:

$$\overline{V} = \frac{1}{\pi}\int_0^{\pi} e^{-itH}Ve^{itH}dt.$$

Since $\overline{V}$ commutes with H, the spectrum of $\overline{L} = H + \overline{V}$ is the sequence of eigenvalues $\{\lambda_k + \overline{\mu}_k\}$, where $\overline{\mu}_k$ is exactly the k^{th} eigenvalue of $\overline{V}$.
The main advantage of this method is that the spectrum of $\overline{L} = H + \overline{V}$ and L are very close, to be precise L and $\overline{L}$ are almost unitary equivalent and $\left[H,\overline{V}\right] = 0$. We start by studying the spectrum of $\overline{L}$, then that of L. This paper is organized as follows: The first section contains auxiliary facts concerning some required proprieties of Weyl pseudo-differential operators. In the second section we will compare

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the spectrum of L and $\overline{L}$ and we will prove that L and $\overline{L}$ are almost unitary equivalent. The last section is devoted for functional calculus of the operator H in order to give the asymptotic expansion of μ_k.

II. Some well-known proprieties of pseudo-differential operators (OPD) in the Weyl's sence

In this section, we recall some well-known proprieties of pseudo-differential operators, and we will announce Proposition 3 which is the main tool in this work.

We start by introducing the definition of the temperate weight function.

Definition 1.
We call temperate weight on $\mathbb{R}^d$, ($d \in \mathbb{N}^*$), every continuous function $m : \mathbb{R}^d \to [0, +\infty[$ that checks: there exist positive constants $C_0, N_0 > 0$ such that:

$$m(x) \leq C_0.m(x_1).(1 + |x_1 - x|)^{N_0},$$

for every $x, x_1 \in \mathbb{R}^d$.

Let $\rho \in [0, 1]$, $p, q \in \mathbb{R}$. We denote by $\Gamma_\rho^{p,q}$ the space of symbols associated with the temperate weight function :

$$(x, \xi) \to (1 + x^2 + \xi^2)^{\frac{p}{2}} \log^q(2+x^2+\xi^2), \quad (x, \xi) \in \mathbb{R}\times\mathbb{R}. \tag{13}$$

Precisely the space of function $a \in C^\infty(\mathbb{R}^2)$ satisfies:
$\forall\alpha, \beta \in \mathbb{N}, \exists\, C_{\alpha,\beta} > 0$, such that:

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leq C_{\alpha,\beta}(1 + x^2 + \xi^2)^{\frac{p-\rho(\alpha+\beta)}{2}} \log^q(2+x^2+\xi^2). \tag{14}$$

We will use the standard Weyl quantization of symbols. To be precise, if $a \in \Gamma_\rho^{p,q}$, then for $u \in S(\mathbb{R})$ the operator associated is defined by :

$$op^w(a)u(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}\times\mathbb{R}} e^{i<x-y,\xi>} a(\tfrac{x+y}{2}, \xi)u(y)dyd\xi. \tag{15}$$

Let's introduce the notion of asymptotic expansion.

Definition 2.
Let $a_j \in \Gamma_\rho^{p_j,q} (j \in \mathbb{N}^*)$, we suppose that p_j is a decreasing sequence tending to $-\infty$. We say that $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ admits an asymptotic expansion: $a \sim \sum_{j=1}^{\infty} a_j$
if

$$a - \sum_{j=1}^{r-1} a_j \in \Gamma_\rho^{p_r,q} \qquad \forall r \geq 2$$

We use the notation $\Sigma_\rho^{p,q}$ for the set of operators $op^w(a)$ if $a \in \Gamma_\rho^{p,q}$.

Remark 1.
If A is an operator whose Weyl symbol is polynomial of degree m, then $A \in \Sigma_1^{m,0}$, in particular the Weyl symbol of the operator H is $\sigma_H(x, \xi) = x^2 + \xi^2$, thus $H \in \Sigma_1^{2,0}$.
In order to prove our main results, we shall recall some well-known results.

Theorem 2 [Calderon-Vailloncourt Theorem [8]].
If $a \in \Gamma_0^{0,0}$, then the operator $op^w(a)$ is bounded on $L^2(\mathbb{R})$.

Proposition 1 [Compactness].
If $a \in \Gamma_\rho^{p,q}$, $p < 0$, and $\rho \in [0, 1]$, then the operator $op^w(a)$ is compact on $L^2(\mathbb{R})$.

We will need the following proposition for the composition of pseudo-differential operators:

Proposition 2.
Let $A \in \Sigma_\rho^{p,q}$, $B \in \Sigma_\rho^{p_1,q_1}$, $\rho \in]0, 1]$, p, p_1, q and $q_1 \in \mathbb{R}$. The operator $AB \in \Sigma_\rho^{p+p_1,q+q_1}$. Its Weyl symbol admits the following asymptotic behavior:

$$c \sim \sum_{j\geq 0} c_j.$$

In particular:

$$c(x, \xi) - a(x, \xi).b(x, \xi) \in \Gamma_\rho^{p+p_1-2\rho,q+q_1}, \tag{16}$$

where

$$c_j = \frac{1}{2^j} \sum_{\alpha+\beta=j} \frac{(-1)^\beta}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b). \tag{17}$$

a and b are respectively the Weyl symbol of A and B.
In the next proposition we will give an extention of the case "$\rho = 0$", we have the following result:

Proposition 3.
If $A \in \Sigma_1^{p,q}$ and $(B_i)_{i\in\{1,...m\}}$ is the set of operators where $B_i \in \Sigma_0^{p_i,q_i}$ then:
(i) The operator $AB_1 \in \Sigma_0^{p+p_1,q+q_1}$, its Weyl symbol is given in formula (17), where $c_j \in \Gamma_0^{p+p_1-j,q+q_1}$.

(ii) $B_1B_2.....B_mH^{-\frac{p_1+p_2+....p_m}{2}} \log^{-(q_1+q_2+....q_m)}(2 + H)$ is bounded.

Proof: The proof of Proposition 3 looks like that of [proposition 1.1 (see[9])], for easy reading we will resume the demonstration. Before proving the previous proposition, we will need to use the following results concerning some functional calculus for the operator H, we recall that the functional calculus on (OPD) was studied in the case where the functions are in the Hörmander class S_1^r ($r \in \mathbb{R}$) see ([10],[11]). In our work we treat the case of the operator H where the function f is defined by:

$$f(x) = x^p \log^q(2 + x). \tag{18}$$

A direct calculation shows that :

$$\left|f^k(x)\right| \leq c_k(1 + x)^{p-k} \log^q(2 + x), \quad x > e - 2. \tag{19}$$

To prove the Proposition 3-ii/, we need the following lemma:

Lemma 1.
$f(H)$ is an OPD included in $\sum_1^{2p,q}$ and its Weyl symbol admits the following development:

$$\sigma_{f(H)} \sim \sum_{j\geq 0} \sigma_{f(H),2j},$$

$$\sigma_{f(H),2j} = \sum_{k=2j}^{3j} (-1)^k \frac{d_{j,k}}{k!} f^{(k)}(\sigma_H) \quad \forall j \geq 1,$$

where $d_{j,k} \in \Gamma_1^{2k-4j,0}$, $\sigma_{f(H),2j} \in \Gamma_1^{2p-4j,q}$.
In particular:

$$\sigma_{f(H),0} = f(\sigma_H).$$

Proof of Lemma 1: for studying $f(H)$ we follow the same strategy in [11] , we will use the Mellin transformation, which consists of:
(1) Studying the operator $(H-\lambda)^{-1}$.
(2) Studying the operator H^{-s} using its Cauchy's integral formula:

$$H^{-s} = \frac{1}{2\pi i}\int_{\Delta} \lambda^{-s}(H-\lambda)^{-1} d\lambda.$$

(3) Studying $f(H)$ using the representation formula:

$$f(H) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} M[f](s) H^{-s} ds.$$

We only change the construction of the $(H-\lambda)^{-1}$-parametrix. We prove by induction that $(H-\lambda)^{-1}$ is an OPD and its Weyl symbol admits the development $b_\lambda \sim \sum b_{j,\lambda}$ where:

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1}, \\ b_{2j+1,\lambda} = 0, \\ b_{2j,\lambda} = \sum_{k=2j}^{3j} (-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{j,k} \in \Gamma_1^{2k-4j,0}. \end{cases}$$

This yields the conclusion. ∎

Let's go back to the proof of Proposition 3.

(i) We proceed as in (appendix [9]). We denote by a the Weyl symbol of A, and b that of B_1. The Weyl symbol c of the operator AB_1 is defined by:

$$c(x,\xi) = \frac{1}{\pi^2}\int e^{-2i(\rho r-\omega\tau)} a(x+\omega,\rho+\xi) b(x+r,\tau+\xi) d\rho d\tau dr d\omega, \tag{20}$$

for every $(x,\xi) \in \mathbb{R}\times\mathbb{R}$.
We split the oscillator integral c into two parts $c^{(1)}$ and $c^{(2)}$, then we use the cutoff functions:

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho) = \chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)}\right] \text{ and } \omega_{2,\varepsilon} = 1-\omega_{1,\varepsilon},$$

where:
$\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in $[-1,1]$, $\chi \equiv 0$ in $\mathbb{R}\backslash]-2,2[$, and $\varepsilon > 0$.
Let's consider:

$$d_j(x,\xi,\omega,\tau,r,\rho) = \omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho) a(x+\omega,\rho+\xi) b(x+r,\rho+\xi) \tag{21}$$

$c^{(1)}$ (resp $c^{(2)}$) is the integral obtained in the equation (20) by replacing the amplitude by d_1 (resp d_2).

Study of $c^{(2)}$:
On the support of d_2 we have:

$$\omega^2+\rho^2+r^2+\tau^2 \geq 2\varepsilon(1+x^2+\xi^2).$$

We integrate by part using the operator:

$$M = \tfrac{1}{2}(\omega^2+\rho^2+r^2+\tau^2)^{-1}(-\rho\partial_r - r\partial_\rho + \tau\partial_\omega + \omega\partial_\tau).$$

We have for all $k \in \mathbb{N}$:

$$c^{(2)} = \frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)} ({}^tM)^k d_2 \, d\rho\, d\omega\, d\tau\, dr.$$

Thus, we obtain for all $k > 0$:

$$c^{(2)}(x,\xi) \in \Gamma_0^{p+p_1-k,q+q_1}. \tag{22}$$

Study of $c^{(1)}$:
The function: $(\omega,\tau,r,\rho) \to d_1(x,\xi,\omega,\tau,r,\rho)$ is with compact support, we deduce from [proposition II 26 (see [10])] that: for every $N \in \mathbb{N}$, we have:

$$c^{(1)}(x,\xi) = \sum_{j=0}^{N} c_j(x,\xi) + R_{N+1}(x,\xi), \tag{23}$$

where

$$c_j = \frac{1}{2^j}\sum_{|\alpha+\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(\partial_x^\alpha\partial_\xi^\beta b), \tag{24}$$

and

$$|R_{N+1}(x,\xi)| \leq c_N \left\|(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1} d_1\right\|_{H^3(\mathbb{R}^4)}. \tag{25}$$

Since $a \in \Gamma_1^{p,q}$ and $b \in \Gamma_0^{p_1,q_1}$, we have for every $j \leq N$:

$$c_j \in \Gamma_0^{p+p_1-j,q+q_1} \tag{26}$$

Let's study the rest. From inequality (25) we have :

$$|R_{N+1}(x,\xi)| \leq C_N \sum_{\substack{|\gamma|\leq 3 \\ \gamma\in\mathbb{N}^4}} \left\|(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,\tau,r,\rho} d_1\right\|_{L^2(\mathbb{R}^4)}$$

$$\leq C_N \left(1+x^2+\xi^2\right)^2 \times \sup_{\substack{[\omega^2+\rho^2+r^2+\tau^2\leq 2\varepsilon(1+x^2+\xi^2)] \\ |\gamma|\leq 3}} \left|(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,\tau,r,\rho} d_1\right|.$$

For $|\gamma| \leq 3$, we have:

$$(\partial_\omega\partial_\tau - \partial_r\partial_\rho)^{N+1}\partial^\gamma_{\omega,\tau,r,\rho} d_1 = (N+1)! \sum_{\alpha+\beta=N+1} \frac{(-1)^\beta}{\alpha!\beta!}\partial_\omega^\beta\partial_\tau^\beta\partial_r^\alpha\partial_\rho^\alpha\partial^\gamma_{\omega,\tau,r,\rho} d_1. \tag{27}$$

Since a (resp b) is independent of (τ,r) (resp (ω,ρ)), thus for $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ we have:

$$\left|\partial_\omega^\beta\partial_\tau^\beta\partial_r^\alpha\partial_\rho^\alpha\partial^\gamma_{\omega,\tau,r,\rho} d_1\right| \leq C\sum_{\Delta} \left|\partial_\omega^{i_1}\partial_\rho^{r_1} a\right| . \left|\partial_\tau^{j_1}\partial_r^{k_1} b\right| . \left|\partial_\omega^{i_2}\partial_\tau^{j_2}\partial_r^{k_2}\partial_\rho^{r_2}\omega_{1,\varepsilon}\right|, \tag{28}$$

where

$$\Delta = \begin{cases} i_1+i_2 = \beta+\gamma_1, \\ i_p \leq \beta+\gamma_1, \\ j_1+j_2 = \beta+\gamma_2, \\ j_p \leq \beta+\gamma_2 \\ k_1+k_2 = \alpha+\gamma_3, \\ k_p \leq \alpha+\gamma_3 \\ r_1+r_2 = \alpha+\gamma_4, \\ r_p \leq \alpha+\gamma_4. \end{cases}$$

For ε small enough and on the support of $\omega_{1,\varepsilon}$, we have:

$$\left|\partial_\omega^{i_2}\partial_\tau^{j_2}\partial_r^{k_2}\partial_\rho^{r_2}\omega_{1,\varepsilon}\right| \leq C\big(1+x^2+\xi^2\big)^{-\frac{1}{2}(i_2+j_2+k_2+r_2)}. \tag{29}$$

There exist positive constants c, c', C, C' such that:

$$\begin{cases} c(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}}, \\ (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}} \leq C(1+x^2+\xi^2)^{\frac{1}{2}}, \\ c'(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}}, \\ (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}} \leq C'(1+x^2+\xi^2)^{\frac{1}{2}}. \end{cases}$$

and positive constants m, m', M, M' such that:

$$\begin{cases} m\log(2+x^2+\xi^2) \leq \log(2+(x+\omega)^2+(\rho+\xi)^2), \\ \log(2+(x+\omega)^2+(\rho+\xi)^2) \leq M\log(2+x^2+\xi^2), \\ m'\log(2+x^2+\xi^2) \leq \log(2+(x+r)^2+(\tau+\xi)^2), \\ \log(2+(x+r)^2+(\tau+\xi)^2) \leq M'\log(2+x^2+\xi^2). \end{cases}$$

Due to the fact that $a \in \Gamma_1^{p,q}$ and $b \in \Gamma_0^{p_1,q_1}$, we obtain:

$$\left|\partial_\omega^\beta \partial_\tau^\beta \partial_r^\alpha \partial_\rho^\alpha \partial_{\omega,\tau,r,\rho}^\gamma d_1\right| \leq C\left(1+x^2+\xi^2\right)^{\frac{p+p_1}{2}} \times \log^{q+q_1}(2+x^2+\xi^2).\sum(1+x^2+\xi^2)^{-\frac{i_1+r_1+i_2+j_2+r_2+k_2}{2}}. \tag{30}$$

Since:

$$i_1+r_1+i_2+j_2+r_2+k_2 = N+1+\gamma_1+\gamma_4+j_2+k_2.$$

It follows then:

$$\left|\partial_\omega^\beta \partial_\tau^\beta \partial_r^\alpha \partial_\rho^\alpha \partial_{\omega,\tau,r,\rho}^\gamma d_1\right| \leq C\left(1+x^2+\xi^2\right)^{\frac{p+p_1-(N+1)}{2}} \times \log^{q+q_1}(2+x^2+\xi^2). \tag{31}$$

From equations (27) and (31), we get:

$$\left|\left(\partial_\omega\partial_\tau - \partial_r\partial_\rho\right)^{N+1} \partial_{\omega,\tau,r,\rho}^\gamma d_1\right| \leq C\left(1+x^2+\xi^2\right)^{\frac{p+p_1-(N+1)}{2}} \times \log^{q+q_1}(2+x^2+\xi^2). \tag{32}$$

Finally and from what follows, we obtain the estimate of R_{N+1}:

$$|R_{N+1}| \leq C_N\left(1+x^2+\xi^2\right)^{\frac{p+p_1-(N+1)+4}{2}} \log^{q+q_1}(2+x^2+\xi^2). \tag{33}$$

The same estimates holds for $\partial_x^\alpha \partial_\xi^\beta R_{N+1}$.
The rest of the symbole c is given by:

$$\delta_{N+1}(x,\xi) = R_{N+1}(x,\xi) + c^{(2)}(x,\xi). \tag{34}$$

By combining the equations (22), (33) and (34), we get the following estimate of δ_{N+1}:

$$\delta_{N+1}(x,\xi) = c_{N+1} + c_{N+2} + ... + c_{N+k} + \delta_{N+1+k}.$$

By choosing $k \geq 4$, we have:

$$\delta_{N+1} \in \Gamma_0^{p+p_1-(N+1),q+q_1}.$$

(ii) It is enough to do the demonstration for $m = 2$.

We notice that:

$$B_1B_2H^{-\frac{p_1+p_2}{2}}\log^{-(q_1+q_2)}(2+H) = T.S,$$

where

$$T = B_1H^{-\frac{p_1}{2}}\log^{-q_1}(2+H), \tag{35}$$

and

$$S = \log^{q_1}(2+H)H^{\frac{p_1}{2}}B_2H^{-\frac{p_1+p_2}{2}}\log^{-(q_1+q_2)}(2+H). \tag{36}$$

By applying Lemma 1, we obtain:

$$H^{-\frac{p_1}{2}}\log^{-q_1}(2+H) \in \sum\nolimits_1^{-p_1,-q_1}. \tag{37}$$

Since $B_1 \in \Sigma_0^{p_1,q_1}$, by combining the equation (37) and Proposition 3-i/, we get:

$$T = B_1H^{-\frac{p_1}{2}}\log^{-q_1}(2+H) \in \sum\nolimits_0^{0,0}. \tag{38}$$

By using the Theorem 2, we conclude that the operator T is bounded, we prove by the same way that S is also bounded. Finally we deduce that:

$$B_1B_2H^{-\frac{p_1+p_2}{2}}\log^{-(q_1+q_2)}(2+H) = T.S,$$

is bounded.
This completes the proof of Proposition 3. ■

III. The relation between the spectrum of L and $\overline{L}$

In this section we will prove the relation between μ_k and $\bar{\mu}_k$, for this reason we will use the averaging method of Weinstein. Let's first recall that the Hamiltonian flow associated to the symbol of the operator H:

$$\sigma_H(x,\xi) = x^2+\xi^2, \quad x,\xi \in \mathbb{R},$$

is a group with a parameter whose elements are square matrix of size 2:

$$\chi_t = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} \tag{39}$$

Recall again that this flow is periodic of period π, to start the averaging method we intoduce the following operators:

$$W(t) = e^{-itH}Ve^{itH},$$
$$\overline{V} = \frac{1}{\pi}\int_0^\pi W(t)dt,$$
$$\overline{\overline{V}} = \frac{1}{2\pi i}\int_0^\pi\int_0^t [W(t),W(r)]drdt.$$

Since H commute with $\overline{V}$, the spectrum of $\overline{L}$ is $\{\lambda_k + \overline{\mu}_k\}$, where $\bar{\mu}_k$ is the k^{th} eigenvalue of $\overline{V}$. To compare μ_k and $\bar{\mu}_k$ we will need the following lemmas:

Lemma 2.
$\left[H,\overline{V}\right] = 0$.

Proof: After we derive $W(t)$, we obtain:

$$\frac{dW(t)}{dt} = \frac{1}{i}\left[H, W(t)\right]. \tag{40}$$

For now we have:

$$\left[H,\overline{V}\right] = \frac{i}{\pi}\int_0^\pi \frac{dW(t)}{dt}dt = \frac{i}{\pi}(W(\pi)-W(0)). \tag{41}$$

Since $e^{\pi i H} = -id_{L^2(\mathbb{R})}$, we get $W(\pi) = W(0)$.

Finally, we deduce that $\left[H,\overline{V}\right] = 0$.

■

Lemma 3.

$$i/\overline{V} \in \Sigma_0^{-1,1}, \quad ii/\overline{\overline{V}} \in \Sigma_0^{-2+2\eta,2},$$

where $\eta \in \left]0,\frac{1}{2}\right[$.

Proof: i/ The Weyl symbol of the operator $W(t)$ is given by:

$$\sigma_{W(t)} = Vo\chi_t, \tag{42}$$

where χ_t is the flow given in (39).
This is due, on the one hand, to the fact that e^{itH} belongs to

the metaplectic group, and on the other hand, to the invariance that Weyl quantification has for this group ([12],[13]). The Weyl symbol of $\overline{V}$ is obtained by integrating the symbol of $W(t)$ uniformly with respect to t.

$$\sigma_{\overline{V}}(x,\xi)=\frac{1}{\pi}\int_0^{\pi}V(x\cos 2t+\xi\sin 2t)dt. \qquad (43)$$

Applying the inequality (2), we get the following estimate, for $\alpha,\beta\in\mathbb{N}$ and $x,\xi\in\mathbb{R}$:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{\overline{V}}(x,\xi)\right|\le C_{\alpha,\beta}\int_0^{\pi}\left[1+(x\cos 2t+\xi\sin 2t)^2\right]^{\frac{-1}{2}}dt. \qquad (44)$$

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{\bar{V}}(x,\xi)\right|\le C_{\alpha,\beta}\int_0^{\frac{\pi}{2}}\left[1+(x^2+\xi^2)\sin^2 t\right]^{\frac{-1}{2}}dt. \qquad (45)$$

We put $r=\sqrt{x^2+\xi^2}$, then we obtain:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{\bar{V}}\right|\le C_{\alpha,\beta}\left(\int_0^{\frac{\pi}{4}}\frac{dt}{\sqrt{1+r^2\sin^2 t}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dt}{\sqrt{1+r^2\sin^2 t}}\right). \qquad (46)$$

It is clair that:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dt}{\sqrt{1+r^2\sin^2 t}}=\int_0^{\frac{\pi}{4}}\frac{dt}{\sqrt{1+r^2\cos^2 t}}.$$

On $\left[0,\frac{\pi}{4}\right]$, $\cos t\ge\sin t$, so we have:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{\bar{V}}(x,\xi)\right|\le C\int_0^{\frac{\pi}{4}}\frac{dt}{\sqrt{1+r^2\sin^2 t}}. \qquad (47)$$

Finally we apply the change of variable "$u=tg(t)$",

then "$v=u\sqrt{1+r^2}$", we get:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{\bar{V}}\right|\le C\frac{\log(2+r^2)}{\sqrt{1+r^2}}\le C(1+x^2+\xi^2)^{\frac{-1}{2}}\log(2+x^2+\xi^2). \qquad (48)$$

We conclude that $\overline{V}\in\Sigma_0^{-1,1}$.

ii/ According to the previous calculations, the operator $B(t)=\displaystyle\int_0^t W(r)dr$ belongs to $\Sigma_0^{-1,1}$, its Weyl symbol $\sigma_{B(t)}$ checks:

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{B(t)}(x,\xi)\right|\le C_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{-1}{2}}\log(2+x^2+\xi^2), \qquad (49)$$

uniformly with respect to t.

Let's start by clarifying the class of the operator $\displaystyle\int_0^{\pi}W(t)B(t)dt$. For now we are interested in the operator $W(t)B(t)$, its Weyl symbol c_t is given in [10] as:

$$c_t(x,\xi)=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}\sigma_{W(t)}(x+\omega,\xi+\rho)$$
$$\times\sigma_{B(t)}(x+r,\xi+\tau)d\rho d\omega d\tau dr. \qquad (50)$$

We split the oscillator integral c_t into two parts $c_t^{(1)}$ $c_t^{(2)}$, then we use the cutoff functions:

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho)=\chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}\right] \text{ and } \omega_{2,\varepsilon}=1-\omega_{1,\varepsilon},$$

where: $\chi\in C_0^\infty(\mathbb{R})$, $\chi\equiv 1$ in $[-1,1]$, $\chi\equiv 0$ in $\mathbb{R}\backslash\,]-2,2[$, $R=\omega^2+\rho^2+r^2+\tau^2$, $\varepsilon>0$ and $\eta\in\left]0,\frac{1}{2}\right[$.

Let's consider:

$$d_j(x,\xi,\omega,\tau,r,\rho)=\omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho)\sigma_{W(t)}(x+\omega,\rho+\xi)$$
$$\times\sigma_{B(t)}(x+r,\rho+\xi) \qquad (51)$$

$c_t^{(1)}$ (resp $c_t^{(2)}$) is the integral obtained in (50) by replacing the amplitude by d_1 (resp d_2):

Study of $c_t^{(2)}$:

On the support of d_2 we have $R\ge\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}$.

We integrate by parts using the operator:

$$M=\tfrac{1}{2iR}(-\rho\partial_r-r\partial_\rho+\tau\partial_\omega+\omega\partial_\tau).$$

We have for all $k\in\mathbb{N}$:

$$c_t^{(2)}=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}({}^tM)^k d_2\,d\rho\,d\omega\,d\tau\,dr.$$

Then we obtain for all $k>0$:

$$\left|c_t^{(2)}\right|\le C_k(1+x^2+\xi^2)^{\frac{-\eta k}{4}},$$

uniformly with respect to $t\in[0,\pi]$.

Study of $c_t^{(1)}$:

On the support of d_1 we have:

$$c_t^{(1)}=\frac{1}{\pi^2}\int_{R\le 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}e^{-2i(r\rho-\omega\tau)}\sigma_{W(t)}(x+\omega,\xi+\rho)$$
$$\times\sigma_{B(t)}(x+r,\xi+\tau)\omega_{1,\varepsilon}d\rho d\omega d\tau dr. \qquad (52)$$

Then:

$$\int_0^{\pi}\left|c_t^{(1)}\right|dt\le c\int_{R\le 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}d\rho d\omega d\tau dr$$
$$\times\int_0^{\pi}\left|\sigma_{W(t)}(x+\omega,\xi+\rho)\right|dt$$
$$\times\int_0^{\pi}\left|\sigma_{B(t)}(x+r,\xi+\tau)\right|dt. \qquad (53)$$

On the support of d_1, for ε small enough and since $\eta\in\left]0,\frac{1}{2}\right[$, there exist positive constants c,c',C,C' such that:

$$\begin{cases} c(1+x^2+\xi^2)^{\frac{1}{2}}\le(1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}},\\ (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}}\le C(1+x^2+\xi^2)^{\frac{1}{2}},\\ c'(1+x^2+\xi^2)^{\frac{1}{2}}\le(1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}},\\ (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}}\le C'(1+x^2+\xi^2)^{\frac{1}{2}}. \end{cases}$$

and positive constants m,m',M,M' such that:

$$\begin{cases} m\log(2+x^2+\xi^2)\le\log(2+(x+\omega)^2+(\rho+\xi)^2),\\ \log(2+(x+\omega)^2+(\rho+\xi)^2)\le M\log(2+x^2+\xi^2),\\ m'\log(2+x^2+\xi^2)\le\log(2+(x+r)^2+(\tau+\xi)^2),\\ \log(2+(x+r)^2+(\tau+\xi)^2)\le M'\log(2+x^2+\xi^2). \end{cases}$$

Therefore:

$$\int_0^{\pi}c_t^{(1)}dt\le C(1+x^2+\xi^2)^{-1}log^2(2+x^2+\xi^2)$$
$$\times\int_{R\le 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}d\rho d\omega d\tau dr. \qquad (54)$$

Finally, we obtain:

$$\int_0^{\pi}c_t^{(1)}dt\le c(1+x^2+\xi^2)^{-1+\eta}log^2(2+x^2+\xi^2). \qquad (55)$$

At the end, and by denoting σ the Weyl symbol of the operator $\int_0^\pi W(t)B(t)dt$, we have:

$$\begin{aligned}|\sigma| &\leq \int_0^\pi \left|c_t^{(1)}\right|dt + \int_0^\pi \left|c_t^{(2)}\right|dt \\ &\leq C(1+x^2+\xi^2)^{\frac{-2+2\eta}{2}}\log^2(2+x^2+\xi^2).\end{aligned}$$

We conclude that $\overline{\overline{V}} \in \Sigma_0^{-2+2\eta,2}$.

■

In order to compare the spectrum of L and $\overline{L}$, we will need the following proposition:

Proposition 4.
There exists a skew-symmetric operator $Q \in \Sigma_0^{-1,1}$ such that the operator $(e^Q L e^{-Q} - \overline{L})H^{1-\eta}\log^{-2}(2+H)$ is bounded.

Proof: We follow the same strategy as in [proposition 5.1 [5]], we start by considering the following operator:

$$Q = Q_1 + Q_2, \tag{56}$$

where

$$Q_1 = \frac{i}{\pi}\int_0^\pi (\pi - t)W(t)dt,$$

and

$$Q_2 = \frac{-1}{2\pi}\int_0^\pi (\pi - t)\int_0^t [W(t), W(r)]drdt.$$

By following the same calculations in Lemma 3, we obtain: $Q_1 \in \Sigma_0^{-1,1}$ and $Q_2 \in \Sigma_0^{-2+2\eta,2}$, thus $Q \in \Sigma_0^{-1,1}$.
Before starting the demonstration, we will need the following lemma:

Lemma 4.
i/ $[Q_1, H] = \overline{V} - V$.
ii/ $[Q_2, H] = -\overline{\overline{V}} - \frac{1}{2}[Q_1, V]$.

Proof: i/ By using the equation (40), we have:

$$\begin{aligned}[Q_1, H] &= \frac{i}{\pi}\int_0^\pi (\pi - t)\frac{dW(t)}{dt}dt \\ &= \overline{V} - V.\end{aligned} \tag{57}$$

ii/ Again, by using the equation (40), we obtain:

$$[Q_2, H] = \frac{-1}{2\pi}\int_0^\pi (\pi - t)\int_0^t [[W(t), W(r)], H]drdt$$

$$= \frac{i}{2\pi}\int_0^\pi (\pi - t)\int_0^t \left(\left[W(t), W^{'}(r)\right] + \left[W^{'}(t), W(r)\right]\right)drdt.$$

We set:

$$F(t) = \frac{1}{\pi}\int_0^t W(r)dr.$$

On the one hand:

$$\begin{aligned}&\frac{i}{2\pi}\int_0^\pi (\pi - t)\int_0^t \left[W(t), W^{'}(r)\right]drdt \\ &= \frac{i}{2\pi}\int_0^\pi (\pi - t)\left[W(t), \int_0^t W^{'}(r)dr\right]dt \\ &= \frac{-i}{2\pi}\int_0^\pi (\pi - t)[W(t), V]dt \\ &= \frac{-1}{2}[Q_1, V].\end{aligned}$$

On the other hand:

$$\begin{aligned}&\frac{i}{2\pi}\int_0^\pi (\pi - t)\int_0^t \left[W^{'}(t), W(r)\right]drdt \\ &= \frac{i}{2}\int_0^\pi (\pi - t)\left[W^{'}(t), F(t)\right]dt \\ &= \frac{i}{2}\int_0^\pi (\pi - t)\frac{d}{dt}([W(t), F(t)])dt \\ &= \frac{i}{2}([(\pi - t)[W(t), F(t)]]_0^\pi + \int_0^\pi [W(t), F(t)]dt) \\ &= -\overline{\overline{V}}.\end{aligned}$$

At the end, we get:

$$[Q_2, H] = -\overline{\overline{V}} - \frac{1}{2}[Q_1, V]. \tag{58}$$

■

Let's go back to the proof of Proposition 4.
We set $AdQ.L = [Q, L]$. The differential equation:

$$\begin{cases} \frac{dX}{dt} = [Q, X] \\ X(0) = L, \end{cases} \tag{59}$$

admits a unique solution:

$$X(t) = e^{tADQ}.L = e^{tQ}Le^{-tQ}. \tag{60}$$

According to Lemma 4, we deduce that:

$$\begin{aligned}e^Q L e^{-Q} - \bar{L} &= \left\{-\overline{\overline{V}} + \frac{1}{2}[Q_2, V]\right\} \\ &+ \frac{1}{2}\left\{[Q, \overline{V}] + \frac{1}{2}[Q, [Q_1, V]]\right\} \\ &+ \frac{1}{2}\left\{[Q, [Q_2, V]] - \left[Q, \overline{\overline{V}}\right]\right\} \\ &+ \sum_{n\geq 2}\frac{(AdQ)^n}{(n+1)!}[Q, H] + \sum_{n\geq 3}\frac{(AdQ)^n}{n!}V.\end{aligned} \tag{61}$$

In a view of the Proposition 3, and since $H \in \Sigma_1^{2,0}$, $V \in \sum_0^{0,0}$, $\overline{V} \in \sum_0^{-1,1}$, $Q_1, Q \in \sum_0^{-1,1}$ and $Q_2, \overline{\overline{V}} \in \sum_0^{-2+2\eta,2}$,

we obtain:

$$\begin{cases}\left\|(-\overline{\overline{V}} + \frac{1}{2}[Q_2, V])H^{1-\eta}\log^{-2}(2+H)\right\| \leq c, \\ \left\|([Q, \overline{V}] + \frac{1}{2}[Q, [Q_1, V]])H\log^{-2}(2+H)\right\| \leq c, \\ \left\|\left([Q, [Q_2, V]] - \left[Q, \overline{\overline{V}}\right]\right)H^{\frac{3}{2}-\eta}\log^{-3}(2+H)\right\| \leq c, \\ \left\|(ADQ)^n[Q, H]H\log^{-2}(2+H)\right\| \leq c\|Q\|^{n-2}\ (n \geq 2), \\ \left\|(ADQ)^n VH\log^{-2}(2+H)\right\| \leq c\|Q\|^{n-2}\ (n \geq 3).\end{cases} \tag{62}$$

Combining the equations (61) and (62), we deduce that:

$$(e^Q L e^{-Q} - \overline{L})H^{1-\eta}\log^{-2}(2+H) \tag{63}$$

is bounded.

■

Now we can compare μ_k and $\overline{\mu}_k$, from what follows we deduce that there exists a constant $c > 0$ such that:

$$-cH^{-1+\eta}\log^2(2+H) \leq e^Q L e^{-Q} - \bar{L} \leq cH^{-1+\eta}\log^2(2+H) \tag{64}$$

Finally by applying the Min-Max theorem, we deduce the relation between μ_k and $\bar{\mu}_k$:

$$\mu_k = \bar{\mu}_k + O\left(\frac{\log^2\lambda_k}{\lambda_k^{1-\eta}}\right). \tag{65}$$

IV. The asymptotic behavior of μ_k

We recall that $\overline{\mu}_k$ is exactly the k^{th} eigenvalue of $\overline{V}$, in this section we start by determining the asymptotic behavior of $\overline{\mu}_k$, then by using the equation (65) we deduce that of μ_k, in polar coordinate the Weyl symbol of $\overline{V}$ is written as:

$$\sigma_{\overline{V}} = \frac{1}{2\pi}\int_0^{2\pi} V(rsint)dt, \qquad (66)$$

where $r = \sqrt{x^2+\xi^2}$.
We have :

$$\sigma_{\overline{V}} = f(\sigma_H), \qquad (67)$$

where the function f is defined by :

$$f(x) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\sqrt{x}\sin t)dt.$$

A direct calculation shows that:

$$\begin{cases} |f(x)| \leq c(1+|x|)^{\frac{-1}{2}}\log(2+|x|), \\ \left|f^{(k)}(x)\right| \leq C_k(1+|x|)^{-\frac{1}{2}-\frac{k}{2}}, \quad k \geq 1. \end{cases} \qquad (68)$$

In order to give the asymptotic behavior of $\overline{\mu}_k$ we need to use a functionnal calculus for the operator H. In this case we treat the the operator H where the function f satisfies the estimate (68). The operator $f(H)$ is defined by a functional calculus of self-adjoint operators, hence the spectrum of $f(H)$ is the sequence $\{f(\lambda_k)\}_k$, we have the following proposition:

Proposition 5.
$f(H)$ is an OPD included in $\sum_0^{-1,1}$, its Weyl symbol admits the following development:

$$\sigma_{f(H)} \sim \sum_{j\geq 0}\sigma_{f(H),2j}$$

$$\sigma_{f(H),2j} = \sum_{k=2j}^{3j}\frac{d_{j,k}}{k!}f^{(k)}(\sigma_H) \quad \forall j \geq 1,$$

where $d_{j,k} \in \Gamma_1^{2k-4j,0}$ and $\sigma_{f(H),2j} \in \Gamma_0^{-1-j,0}$.
In particular:

$$\sigma_{f(H),0} = f(\sigma_H) = \sigma_{\overline{V}}.$$

Proof: We prove the Proposition 5 by the same way as in Lemma 1, the only change is the Hörmander class to which f belongs.

$$\left|f^k(x)\right| \leq c_k(1+|x|)^{-\frac{1}{2}-\frac{1}{2}k}, \quad k \geq 1 \qquad (69)$$

∎

Now we will prove Theorem 1.

Proof of Theorem 1: By applying the Proposition 5, we have:

$$f(H) \in \sum\nolimits_0^{-1,1}, \quad \overline{V} - f(H) \in \sum\nolimits_0^{-2,0}. \qquad (70)$$

By combining the equation (70) and the Proposition 3-ii/, we deduce that: $\left(\overline{V} - f(H)\right)H$ is bounded.

Therefore, there exists $c > 0$ such that :

$$-cH^{-1} \leq \overline{V} - f(H) \leq cH^{-1}.$$

Applying the Min-Max theorem, we deduce that :

$$\bar{\mu}_k = f(\lambda_k) + O\left(\frac{1}{\lambda_k}\right). \qquad (71)$$

By combining the equations (65) and (71) we deduce:

$$\mu_k = f(\lambda_k) + O\left(\frac{\log^2\lambda_k}{\lambda_k^{1-\eta}}\right). \qquad (72)$$

Finally we have:

$$\mu_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\sqrt{\lambda_k}\sin t)dt + O(\frac{\log^2\lambda_k}{\lambda_k^{1-\eta}}), \qquad (73)$$

where $\eta \in \left]0, \frac{1}{2}\right[$.

This complete the proof of Theorem 1.

∎

V. Conclusion

The perturbed harmonic oscillator is one of the famous problems on the spectral theory, because it has many applications in physics, there are many tools to deal with this kind of problem, however we choose to use the averaging method here because the harmonic oscillator has a periodic flow, we succeeded in giving the asymptotic behavior of it's spectrum, in next works we want to go further and deal with anharmonic oscillators.

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