The Similarity among Two Extended Shapley Values

Yan-An Hwang and Yu-Hsien Liao

Abstract—**For each of the extended Shapley values due to Derks and Peters [3] and Peters and Zank [16], we first propose a corresponding definition of marginal contributions of a potential, and further demonstrate that these solutions can be generalized as the vector of corresponding marginal contributions of a potential. We further propose several equivalent relations and related axiomatizations to demonstrate that these two solutions are almost coincident in axiomatic approach except that the loss of amount is different in the axiom of equal loss.**

Index Terms—**The Shapley value, potential, axiomatization.**

I. Introduction

In a *standard transferable-utility (TU) game*, each agent is either exhaustively concerned or not concerned at all in occasions with some other agents. However, various actions would be adopted by each agent under real-world situations always. A *multi-choice TU game* is a reasonable extension of a standard TU game in which each agent could be permitted to operate at more than one activity actions. Solutions for multi-choice TU games have been analyzed in various fields such as environmental science, economics, industrial engineering and even management science, such as Cheng et al. [2], Hwang and Liao [9], and so on. Under multi-choice TU games, we focus on the extended Shapley values due to Derks and Peters [3], and Peters and Zank [16], which we name the *D&P Shapley value* and the *P&Z Shapley value*. For each of these two solutions, we extend several overcomes of Calvo and Santos [1], Hart and Mas-Colell [4], Myerson [12] and Ortmann [13], [14] to multi-choice TU games. One interesting overcome among our characterizations is that the D&P Shapley value "almost" coincides with the P&Z Shapley value.

First, we demonstrate that each of these two solutions could be generalized as the vector of marginal contributions of a potential. The different types of marginal contributions generalize a comprehension of the difference between these two solutions.

Second, for each of these two solutions, we analyze the class of all multi-choice solutions that admit a potential, and demonstrate that any solution that admits a potential turns out to be the solution of a specific game. For each of these two solutions, we also demonstrate the equivalences among the potentializability, the axioms of balanced contributions and path independence under an axiom of equal loss.

Third, we offer characterizations of the two solutions by three different notions: **(1)** balanced contributions **(2)** path independence **(3)** consistency. We demonstrate that

- the D&P Shapley value (the P&Z Shapley value) is the only solution satisfying efficiency, upper balanced contributions and *D&P-equal loss* (*P&Z-equal loss*).
- the D&P Shapley value (the P&Z Shapley value) is the only solution satisfying efficiency, path independence and *D&P-equal loss* (*P&Z-equal loss*).
- the D&P Shapley value (the P&Z Shapley value) is the only solution satisfying weak efficiency, upper balanced contributions, consistency and weak *D&P-equal loss* (*P&Z-equal loss*).

These overcomes point out that the D&P Shapley value and the P&Z Shapley value are almost coincident in axiomatic approach. The axioms are the same, except that the loss of amount is different in the axiom of equal loss. Namely, the difference between these two solutions is only that one satisfies *D&P-equal loss* and the other satisfies *P&Z-equal loss*. Thus, the different definitions of equal loss also generalize a comprehension of the difference between these two solutions.

II. Preliminaries

Let U be the universe of agents. Suppose that each agent $i \in U$ has $b_i \in \mathbb{N}$ managing grades at which it can act. We use $B_i = \{0, 1, \cdots, b_i\}$ to be the managing grade space of agent i, where 0 means not acting. For $P \subseteq U$, let $B^P = \prod_{i\in P} B_i$ be the product set of the grade spaces for agents of P. Denote the zero vector as 0_P under $\mathbb{R}^P$.

A **multi-choice game** is denoted by (P, b, G), where $P \neq \emptyset$ is a finite set of agents, $b = (b_i)_{i\in P}$ is the vector that represents the amount of managing grades for each agent, and $G : B^P \to \mathbb{R}$ is a characteristic mapping which appoints to each $\chi = (\chi_i)_{i\in P} \in B^P$ the value that the agents can get when each agent i participates at managing grade $\chi_i \in B_i$ with $G(0_P) = 0$. (P, b, G) will be represented by G if no confusion can cause. Denote the collection of all multi-choice games by Ω. Let $(P, b, G) \in \Omega$ and $x \in B^P$, one set (P, χ, G) to be the **multi-choice subgame** given by restricting G to $\{\omega \in B^P \mid \omega_i \leq \chi_i \ \forall i \in P\}$.

Given $(P, b, G) \in \Omega$, let $K^P = \{(i, k_i) \mid i \in P, k_i \in B_i^+\}$, where $B_i^+ = B_i \setminus \{0\}$. A **solution** on Ω is a mapping τ appointing to each $(P, b, G) \in \Omega$ an element

$$\tau(P, b, G) = \big(\tau_{i,k_i}(P, b, G)\big)_{(i,k_i)\in K^P} \in \mathbb{R}^{K^P},$$

where $\tau_{i,k_i}(P, b, G)$ is the value of the agent i if i adopts grade k_i to operate G. For convenience, we suppose that $\tau_{i,0}(P, b, G) = 0$ for all $i \in P$.

Given $M \subseteq P$ and $\chi \in \mathbb{R}^P$, we define $\|\chi\| = \sum_{k\in P} \chi_k$, $Q(\chi) = \{k \in P \mid \chi_k \neq 0\}$ and χ_M to be the restriction of χ at M. Let $|M|$ be the amount of elements in M and let $\delta^M \in \mathbb{R}^P$ be the binary vector with $\delta_i^M = 1$ if $i \in T$ and $\delta_i^M = 0$ if $i \notin M$.

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Let $\chi,\omega \in \mathbb{R}^P$, one define $\omega \leq \chi$ if $\omega_i \leq \chi_i$ for all $i \in P$. The multi-choice analogue of unanimity games, **minimal effort games** (P,b,u_P^χ) with $\chi \in B^P \setminus \{0_P\}$, are defined as follows. For all $\omega \in B^P$,

$$u_P^\chi(\omega) = \begin{cases} 1 & \text{if } \omega \geq \chi, \\ 0 & \text{otherwise.} \end{cases}$$

It is also demonstrated that $v = \sum_{\chi \in B^P \setminus \{0_P\}} a^\chi(G)\ u_P^\chi$ for $(P,b,G) \in \Omega$, where $a^\chi(G) = \sum_{M \subseteq Q(\chi)} (-1)^{|M|}\ G(\chi - \delta^M)$.

Definition 1: The **P&Z Shapley value** Γ due to Peters and Zank [16] is the solution on Ω which associates with each game (P,b,G), each agent $i \in P$ and each grade $k_i \in B_i^+$ the value [1]

$$\Gamma_{i,k_i}(P,b,G) = \sum_{\chi \in B^P,\ \chi_i = k_i} \frac{a^\chi(G)}{|Q(\chi)|}.$$

Definition 2: The **D&P Shapley value** Θ due to Derks and Peters [3] is the solution on Ω which associates with each game (P,b,G), each agent $i \in P$ and each grade $k_i \in B_i^+$ the value

$$\Theta_{i,k_i}(P,b,G) = \sum_{\chi \in B^P,\ \chi_i \geq k_i} \frac{a^\chi(G)}{\|\chi\|}.$$

Without loss of generality, one could assume that $Q(b) = P$. Clearly, the dividend $a^\chi(G)$ is divided equally among the "necessary agents" under the solution Γ, and it is divided equally among the "necessary grades" under the solution Θ. Based on related overcomes of the solutions Γ and Θ, for all $(P,b,G) \in \Omega$,

$$\sum_{i \in P} \sum_{k_i=1}^{b_i} \Gamma_{i,k_i}(P,b,G) = G(b) = \sum_{i \in P} \sum_{k_i=1}^{b_i} \Theta_{i,k_i}(P,b,G).$$

III. Potential

Inspired by the potential approach due to Hart and Mas-Colell [4] on standard TU games, this section presents that for each of these two multi-choice Shapley values, there exists an extended overcome of Hart and Mas-Colell [4].

Given $i \in P$ and $\chi \in \mathbb{R}^P$, we adopt χ_{-i} to stand for $\chi_{P\setminus\{i\}}$ and let $\omega = (\chi_{-i}, k_i) \in \mathbb{R}^P$ be considered by $\omega_{-i} = \chi_{-i}$ and $\omega_i = k_i$. Further, let $j \in P$, χ_{-ij} to stand for $\chi_{P\setminus\{i,j\}}$ and (χ_{-ij}, k_i, k_j) to stand for $\big((\chi_{-i}, k_i)_{-j}, k_j\big)$.

A function $H : \Omega \longrightarrow \mathbb{R}$ is said to be *0-normalized* if $H(P, 0_P, G) = 0$ for each $P \subseteq U$. For each $(i,k_i) \in K^P$, we offer two different definitions of **marginal contribution** as follows.

- The **P&Z marginal contribution** of (i,k_i) in (P,b,G) is defined to be
$$\begin{aligned} & D_1^{i,k_i} H(P,b,G) \\ = \ & H\big(P,(b_{-i},k_i),G\big) - H\big(P,(b_{-i},k_i-1),G\big). \end{aligned}$$
- The **D&P marginal contribution** of (i,k_i) in (P,b,G) is defined to be
$$\begin{aligned} & D_2^{i,k_i} H(P,b,G) \\ = \ & H(P,b,G) - H\big(P,(b_{-i},k_i-1),G\big). \end{aligned}$$

[1]Peters and Zank [16] considered the P&Z Shapley value by imposing linearity and fixing its values on minimal effort games. In this paper we present the P&Z Shapley value by applying "dividends".

Definition 3: Let τ be a solution on Ω.

- τ admits a P&Z potential if there exists a mapping $H : \Omega \to \mathbb{R}$ such that for all $(P,b,G) \in \Omega$ and for all $(i,k_i) \in K^P$, $\tau_{i,k_i}(P,b,G) = D_1^{i,k_i} H(P,b,G)$.
- τ admits a D&P potential if there exists a mapping $H : \Omega \to \mathbb{R}$ such that for all $(P,b,G) \in \Omega$ and for all $(i,k_i) \in K^P$, $\tau_{i,k_i}(P,b,G) = D_2^{i,k_i} H(P,b,G)$.

Solutions that admit a potential present an evaluation to each game in such a way that a agent's value coincides with this evaluation. Next, we offer a corresponding definition of efficiency for each of the two marginal contributions as follows.

- H is **P&Z-efficient (P&Z-EFF)** if
$$\sum_{i \in P} \sum_{k_i=1}^{b_i} D_1^{i,k_i} H(P,b,G) = G(b)\ \forall\ (P,b,G) \in \Omega. \quad (1)$$
- H is **D&P-efficient (D&P-EFF)** if
$$\sum_{i \in P} \sum_{k_i=1}^{b_i} D_2^{i,k_i} H(P,b,G) = G(b)\ \forall\ (P,b,G) \in \Omega. \quad (2)$$

The following outcomes are generalized overcomes of Hart and Mas-Colell [4]. It demonstrates that the preceding two potential functions are corresponding to the P&Z Shapley value and the D&P Shapley value respectively. The different definitions of marginal contributions generalize a comprehension of the difference between these two extended Shapley value.

Theorem 1:

1) There exists a uniquely P&Z-EFF and 0-normalized potential H such that the P&Z Shapley value Γ admits the P&Z potential H. Furthermore, the P&Z potential of a game (P,b,G) is uniquely determined by Equation (1) applied only to the game and its subgames.
2) There exists a uniquely D&P-EFF and 0-normalized potential H such that the D&P Shapley value Θ admits the D&P potential H. Furthermore, the D&P potential of a game (P,b,G) is uniquely determined by Equation (2) applied only to the game and its subgames.

Proof: To demonstrate the overcome 1, Equation (1) coincides with

$$H(P,b,G) = \frac{1}{|P|} \cdot \Big[G(b) + \sum_{i \in P} H\big(P,(b_{-i},0),G\big)\Big]. \quad (3)$$

Starting with $H(P,0_P,G)$, it determines $H(P,b,G)$ recursively. This demonstrates the existence of the P&Z potential H, and furthermore that $H(P,b,G)$ is uniquely determined by Equation (3) applied to (P,chi,G) for all $chi \in B^P$. Let

$$H(P,b,G) = \sum_{chi \in B^P \setminus \{0_P\}} \frac{a^{chi}(G)}{|Q(chi)|}. \quad (4)$$

Clearly, Equation (1) is matched by P; hence Equation (4) defines the uniquely P&Z potential. The overcome 1 now follows for each $(i,k_i) \in K^P$, since

$$\Gamma_{i,k_i}(P,b,G) = \sum_{chi \in B^P,\ chi_i = k_i} \frac{a^{chi}(G)}{|Q(chi)|}.$$

Replacing Equation (3) by

$$H(P,b,G) = \frac{1}{\|b\|} \cdot \Big[G(b) + \sum_{i \in P} \sum_{t=1}^{b_i} H\big(P,(b_{-i},t-1),G\big)\Big],$$

it is trivial to generalize that the overcome 2; we omit it.[2] ■

IV. Equivalence and characterization (I)

The purpose of this section is to demonstrate that for each of the two multi-choice Shapley values, there exist equivalence overcomes which are analogues of the overcomes due to Calvo and Santos [1] and Ortmann [13], [14]. To state the equivalence overcomes, some more definitions will be needed. Let τ be a solution on Ω. Then τ matches

- **efficiency (EFF)** if for each $(P,b,G)\in\Omega$,

$$\sum_{i\in P}\sum_{k_i=1}^{b_i}\tau_{i,k_i}(P,b,G)=G(b).$$

- **upper balanced contributions (UBC)** if for each $(P,b,G)\in\Omega$ and for $i,j\in P, i\neq j$,

$$\begin{aligned}&\tau_{i,b_i}\big(P,(b_{-j},b_j),G\big)-\tau_{i,b_i}\big(P,(b_{-j},b_j-1),G\big)\\ =&\ \tau_{j,b_j}\big(P,(b_{-i},b_i),G\big)-\tau_{j,b_j}\big(P,(b_{-i},b_i-1),G\big).\end{aligned}$$

- **D&P equal loss (D&P-EL)** if for each $(P,b,G)\in\Omega$ and for each $(i,k_i)\in K^P$, $k_i\neq b_i$,

$$\begin{aligned}&\tau_{i,k_i}(P,b,G)-\tau_{i,k_i}\big(P,(b_{-i},b_i-1),G\big)\\ =&\ \tau_{i,b_i}(P,b,G).\end{aligned}$$

- **P&Z equal loss (P&Z-EL)** if for each $(P,b,G)\in\Omega$ and for each $(i,k_i)\in K^P$, $k_i\neq b_i$,

$$\tau_{i,k_i}(P,b,G)-\tau_{i,k_i}\big(P,(b_{-i},b_i-1),G\big)=0.$$

UBC and D&P-EL, originally introduced by Klijn et al. [10], are multi-choice analogues of the balanced contributions notion due to Myerson [12]. UBC asserts that for every (i,j) of different agents the variation in value for the managing grade b_i of agent i if agent j gets available a higher managing grade coincides with the variation in value for the managing grade b_j of agent j if agent i gets available a higher managing grade. D&P-EL asserts that whenever a agent gets available a higher managing grade the value for all initial grades varies with an amount coincide with "the value for the highest grade under new situation". "Zero" instead of "the value for the highest grade under new situation" in D&P-EL, we introduce P&Z-EL.

- A solution τ matches **independence of individual expansions (IIE)**[3] if for each $(P,b,G)\in\Omega$ and for each $(i,k_i)\in K^P$, $k_i\neq b_i$,

$$\begin{aligned}\tau_{i,k_i}\big(P,(b_{-i},k_i),G\big)&=\tau_{i,k_i}\big(P,(b_{-i},k_i+1),G\big)\\ &=\cdots\\ &=\tau_{i,k_i}(P,b,G).\end{aligned}$$

IIE says that whenever a agent gets available higher managing grade the value for all initial grades is not varied under condition that other agents are fixed. Clearly, if a solution matches IIE then it also matches P&Z-EL. Conversely, a repeated application of P&Z-EL would yield IIE. Thus, IIE and P&Z-EL are equivalent to each other.

Some weakenings of previous properties are as follows. **Weak efficiency (WEFF)**, **weak D&P equal loss (D&P-WEL)**, **weak P&Z equal loss (P&Z-WEL)** and **weak independence of individual expansions (WIIE)** assert that for all $(P,b,G)\in\Omega$, $|P|=1$, τ matches EFF, D&P-EL, P&Z-EL and IIE respectively.

Calvo and Santos [1] demonstrated that any solution would be the Shapley value of an auxiliary game if it admits a potential. One could consider a multi-choice extension of an auxiliary game.

Definition 4: Given $(P,b,G)\in\Omega$ and a solution τ on Ω. The **auxiliary multi-choice game** (P,b,G_τ) is defined by

$$G_\tau(chi)=\sum_{i\in Q(chi)}\sum_{k_i=1}^{chi_i}\tau_{i,k_i}(P,chi,G)\ \forall\ chi\in B^P.$$

Clearly, $G=G_\tau$ if τ matches EFF.

Ortmann [13], [14] provided characterizations of the potentializability by applying the path independence property under standard TU situations. To define its analogue under multi-choice situations, some more notations are needed.

An **order** for $(P,b,G)\in\Omega$ is a bijection $\rho:K^P\to\{1,\cdots,\|b\|\}$ matching [4] $\rho(i,k_i)<\rho(i,k_i+1)$ for all $i\in P$ and for all $k_i\in\{1,\cdots,b_i-1\}$. Let ρ,ρ' be two orders for (P,b,G), one say that ρ' is a *transposition* of ρ if there exist $(i,k_i),(j,k_j)\in K^P$ with $i\neq j$ and $\rho(j,k_j)=\rho(i,k_i)+1$, such that $\rho'(i,k_i)=\rho(j,k_j)$, $\rho'(j,k_j)=\rho(i,k_i)$ and $\rho'(p,k_p)=\rho(p,k_p)$ for all $(p,k_p)\in K^P\setminus\{(i,k_i),(j,k_j)\}$. Then it is not difficult to check that each order could be transformed to another order by adopting transpositions.

Now let ρ be an order and let $q\in\{1,\cdots,\|b\|\}$. The managing grade vector that is present after q steps according to ρ, denoted by $s^{\rho,q}$, is defined as follows. For each $i\in P$,

$$s_i^{\rho,q}=\max\Big(\{k_i\in B_i^+\mid\rho(i,k_i)\leq q\}\cup\{0\}\Big).$$

Definition 5: A solution τ matches **path independence (PI)** if for all $(P,b,G)\in\Omega$ and for all orders ρ,ρ',

$$\sum_{i\in P}\sum_{k_i=1}^{b_i}\tau_{i,k_i}(P,s^{\rho,(i,k_i)},G)=\sum_{i\in P}\sum_{k_i=1}^{b_i}\tau_{i,k_i}(P,s^{\rho',(i,k_i)},G).$$

Lemma 1: A solution τ matches PI if and only if τ matches UBC.

Proof: See Lemma 1 in Hwang and Liao [5]. ■

Next, we present the main overcomes of this section.

Theorem 2: Let τ be a solution on Ω. The following are equivalent:

1) τ admits a P&Z potential.
2) τ matches UBC and P&Z-EL (or IIE).
3) τ matches PI and P&Z-EL (or IIE).
4) $\tau(P,b,G)=\Gamma(P,b,G_\tau)$ for each $(P,b,G)\in\Omega$.

Proof: Let τ be a solution on Ω. Based on Lemma 1, $2\Leftrightarrow 3$. To demonstrate $1\Rightarrow 2$, assume that τ admits a P&Z potential P. Let $(P,b,G)\in\Omega$ and $(i,b_i),(j,b_j)\in K^P$, $i\neq j$,

$$\begin{aligned}&\tau_{i,b_i}(P,b,G)-\tau_{i,b_i}\big(P,(b_{-j},b_j-1),G\big)\\ =&\ [H(P,b,G)-H\big(P,(b_{-i},b_i-1),G\big)]\\ &-[H\big(P,(b_{-j},b_j-1),G\big)-H\big(P,(b_{-ij},b_i-1,b_j-1),G\big)]\\ =&\ [H(P,b,G)-H\big(P,(b_{-j},b_j-1),G\big)]\\ &-[H\big(P,(b_{-i},b_i-1),G\big)-H\big(P,(b_{-ij},b_i-1,b_j-1),G\big)]\\ =&\ \tau_{j,b_j}(P,b,G)-\tau_{j,b_j}\big(P,(b_{-i},b_i-1),G\big),\end{aligned}$$

[2] Hwang and Liao [5] provided the proof of the overcome 2.
[3] This property was initially defined by Hwang and Liao [8]
[4] For convenience, $\rho(i,k_i)$ instead of $\rho\big((i,k_i)\big)$.

i.e., τ matches UBC. To see that τ matches WIIE, we demonstrate that it matches IIE. Let $(P, b, G) \in \Omega$ and $(i, k_i) \in K^P, k_i \neq b_i$. For $k_i \leq l \leq b_i$,

$$\begin{aligned} & \tau_{i,k_i}\big(P,(b_{-i},l),G\big) \\ = \;& P\big(P,(b_{-i},k_i),G\big) - P\big(P,(b_{-i},k_i-1),G\big) \\ = \;& \tau_{i,k_i}(P,b,G), \end{aligned}$$

i.e., τ matches IIE.

To demonstrate $3 \Rightarrow 4$, assume that τ matches IIE and PI. Let $(P, b, G) \in \Omega$. The proof proceeds by induction on $\|b\|$. If $\|b\| = 1$, let $P = \{i\}$ and $b_i = 1$, then by efficiency of Γ and the definition of G_τ,

$$\tau_{i,1}(P,b,G) = G_\tau(b) = \Gamma_{i,1}(P,b,G_\tau).$$

Assume that $\tau(P, b, G) = \Gamma(P, b, G_\tau)$ for $\|b\| \leq k$, where $k \geq 1$.

The condition $\|b\| = k+1$: For every $(h, k_h) \in K^P, k_h \neq b_h$, by induction hypotheses and IIE,

$$\begin{aligned} \tau_{h,k_h}(P,b,G) &= \tau_{h,k_h}\big(P,(b_{-h},k_h),G\big) \\ &= \Gamma_{h,k_h}\big(P,(b_{-h},k_h),G_\tau\big) \\ &= \Gamma_{h,k_h}\big(P,b,G_\tau\big). \end{aligned} \tag{5}$$

It remains to demonstrate that $\tau_{h,b_h}(P, b, G) = \Gamma_{h,b_h}(P, b, G_\tau)$ for all $h \in P$. For $h \in P$, let ρ_h be an order with $\rho_h(h, b_h) = \|b\| = k + 1$. Since $\|s^{\rho_h,\rho_h(i,k_i)}\| \leq k$ for $(i, k_i) \neq (h, b_h)$, by induction hypotheses,

$$\begin{aligned} & \tau_{h,b_h}(P,b,G) - \Gamma_{h,b_h}(P,b,G_\tau) \\ = \;& \sum_{i\in P}\sum_{k_i=1}^{b_i} \tau_{i,k_i}(P,s^{\rho_h,\rho_h(i,k_i)},G) \\ & - \sum_{i\in P}\sum_{k_i=1}^{b_i} \Gamma_{i,k_i}(P,s^{\rho_h,\rho_h(i,k_i)},G) \\ = \;& C_h. \; \big(\textbf{Constant depend on } h\big) \end{aligned}$$

Similarly, for $h' \in P$, let $\rho_{h'}$ be an order with $\rho_{h'}(h', b_{h'}) = \|b\| = k + 1$

$$\begin{aligned} & \tau_{h',b_{h'}}(P,b,G) - \Gamma_{h',b_{h'}}(P,b,G_\tau) \\ = \;& \sum_{i\in P}\sum_{k_i=1}^{b_i} \tau_{i,k_i}(P,s^{\rho_{h'},\rho_{h'}(i,k_i)},G) \\ & - \sum_{i\in P}\sum_{k_i=1}^{b_i} \Gamma_{i,k_i}(P,s^{\rho_{h'},\rho_{h'}(i,k_i)},G) \\ = \;& C_{h'}. \; \big(\textbf{Constant depend on } h'\big) \end{aligned}$$

Since Γ admits a P&Z potential, Γ matches PI. Then by PI of τ and Γ, $C_h = C_{h'}$. Hence $C_h = d$ for all $h \in P$.

Let $d = \tau_{h,b_h}(P, b, G) - \Gamma_{h,b_h}(P, b, G_\tau)$ for all $h \in P$. By definition of G_τ, EFF of Γ and Equation (5),

$$\begin{aligned} |P| \cdot d &= \sum_{h\in P}\sum_{k_h=1}^{b_h} \tau_{h,k_h}(P,b,G) \\ &\quad - \sum_{h\in P}\sum_{k_h=1}^{b_h} \Gamma_{h,k_h}(P,b,G_\tau) \\ &= G_\tau(b) - G_\tau(b) \\ &= 0. \end{aligned}$$

So, $d = \tau_{h,b_h}(P, b, G) - \Gamma_{h,b_h}(P, b, G_\tau) = 0$ for all $h \in P$. That is, $\tau_{h,b_h}(P, b, G) = \Gamma_{h,b_h}(P, b, G_\tau)$ for all $h \in P$. Hence $\tau(P, b, G) = \Gamma(P, b, G_\tau)$.

To demonstrate $4 \Rightarrow 1$, assume that $\tau(P, b, G) = \Gamma(P, b, G_\tau)$ for all $(P, b, G) \in \Omega$. Since the solution Γ admits a unique P&Z potential H_Γ, we consider a function of τ as $H_\tau(P, b, G) = H_\Gamma(P, b, G_\tau)$ for all $(P, b, G) \in \Omega$. Then for every $(i, k_i) \in K^P$,

$$\begin{aligned} & H_\tau\big(P,(b_{-i},k_i),G\big) - H_\tau\big(P,(b_{-i},k_i-1),G\big) \\ = \;& H_\Gamma\big(P,(b_{-i},k_i),G_\tau\big) - H_\Gamma\big(P,(b_{-i},k_i-1),G_\tau\big) \\ = \;& \Gamma_{i,k_i}(P,b,G_\tau) \\ = \;& \tau_{i,k_i}(P,b,G). \end{aligned}$$

By Definition 3, H_τ is a P&Z potential of τ. ■

Theorem 3: Let τ be a solution on Ω. The following are equivalent:

1) τ admits a D&P potential.
2) τ matches UBC and D&P-EL.
3) τ matches PI and D&P-EL.
4) $\tau(P, b, G) = \Theta(P, b, G_\tau)$ for each $(P, b, G) \in \Omega$.

Proof: Please see Theorem 2 in Hwang and Liao [5]. ■

Klijn et al. [10] characterized the D&P Shapley value by using EFF, D&P-EL and UBC. Based on Lemma 1, Hwang and Liao [5] characterized the D&P Shapley value by means of EFF, D&P-EL and PI. P&Z-EL (or IIE) instead of D&P-EL, we present two axiomatic outcomes of the P&Z Shapley value. As we mentioned in Introduction, the two solutions are almost the same in axiomatic approach. The difference between them is only the different definitions of equal loss.

Theorem 4:

1) A solution τ matches EFF, D&P-EL and UBC if and only if $\tau = \Theta$.
2) A solution τ matches EFF, D&P-EL and PI if and only if $\tau = \Theta$.

Proof: The overcome 1 is Theorems 4.3 of Klijn et al. [10]. The overcome 2 is Theorem 3 of Hwang and Liao [5]. Clearly, the overcomes 1 and 2 follows by our Theorems 1, 3. ■

Theorem 5:

1) A solution τ matches EFF, P&Z-EL (or IIE) and UBC if and only if $\tau = \Gamma$.
2) A solution τ matches EFF, P&Z-EL (or IIE) and PI if and only if $\tau = \Gamma$.

Proof: The overcomes follows by Theorems 1 and 2. ■

V. Consistency and Characterization (II)

By applying potential approach, Hart and Mas-Colell [4] demonstrated that the Shapley value matches *consistency*. Recently, Hwang and Liao [5] offered a proof of consistency of the D&P Shapley value based on the **linearity** of the D&P Shapley value, hence, it suffices to demonstrate consistency for a **minimal effort game**. Here we demonstrate that the P&Z Shapley value also matches consistency by means of "**dividends**".

For $T \subseteq P$, we denote $T^c = P \setminus T$. Given a solution τ, a game $(P, b, G) \in \Omega$, and $T \subseteq P, T \neq \emptyset$, the **reduced game** $(T, b_T, G^\tau_{T,b})$ with respect to τ, T and b is defined as follows. For each $\chi \in B^T$,

$$G^\tau_{T,b}(\chi) = G(\chi, b_{T^c}) - \sum_{i\in T^c}\sum_{k_i=1}^{b_i} \tau_{i,k_i}\big(P,(\chi,b_{T^c}),G\big). \tag{6}$$

A solution τ is said to be consistent if it always generalizes coincident payoffs as in the original game when it is applied to arbitrary reduction,. Formally, τ matches **consistency (CON)** if for each $(P,b,G)\in\Omega$, for each $T\subseteq P, T\neq\emptyset$, for each $(i,k_i)\in K^T$,

$$\tau_{i,k_i}(T,b_T,G^{\tau}_{T,b})=\tau_{i,k_i}(P,b,G). \tag{7}$$

The following outcome relates the relation of coefficients of expressions among (P,b,G) and $(T,b_T,G^{\Gamma}_{T,b})$.

Lemma 2: Let $(P,b,G)\in\Omega$ and $T\subseteq P, T\neq\emptyset$. If $G=\sum_{\chi\in B^P\setminus\{0_P\}} a^{\chi}(G)\cdot u^{\chi}_P$, then $G^{\Gamma}_{T,b}$ can be expressed as

$$G^{\Gamma}_{T,b}=\sum_{\omega\in B^T\setminus\{0_T\}} a^{\omega}(G^{\Gamma}_{T,b})\cdot u^{\omega}_T,$$

where for each $\omega\in B^T\setminus\{0_T\}$,

$$a^{\omega}(G^{\Gamma}_{T,b})=\sum_{t\leq b_{T^c}}\frac{|Q(\omega)|}{|Q(\omega)|+|Q(t)|}\cdot a^{(\omega,t)}(G).$$

Proof: Let $(P,b,G)\in\Omega$ and $T\subseteq P, T\neq\emptyset$. For each $\omega\in B^T\setminus\{0_T\}$,

$$\begin{aligned}
&G^{\Gamma}_{T,b}(\omega)\\
=\;& G(\omega,b_{T^c})-\sum_{k\in T^c}\sum_{l=1}^{b_k}\Gamma_{k,l}\big(P,(\omega,b_{T^c}),G\big)\\
=\;& \sum_{k\in Q(\omega)}\sum_{l=1}^{\omega_k}\Gamma_{k,l}\big(P,(\omega,b_{T^c}),G\big)\\
=\;& \sum_{k\in Q(\omega)}\sum_{l=1}^{\omega_k}\sum_{\substack{\gamma\leq(\omega,b_{T^c})\\ \gamma_k=l}}\frac{a^{\gamma}(G)}{|Q(\gamma)|}\\
=\;& \sum_{k\in Q(\omega)}\Big[\sum_{\substack{\gamma\leq(\omega,b_{T^c})\\ \gamma_k=1}}\frac{a^{\gamma}(G)}{|Q(\gamma)|}+\cdots+\sum_{\substack{\gamma\leq(\omega,b_{T^c})\\ \gamma_k=\omega_k}}\frac{a^{\gamma}(G)}{|Q(\gamma)|}\Big]\\
=\;& \sum_{k\in Q(\omega)}\Big[\sum_{\substack{p\leq\omega\\ p_k=1}}\sum_{t\leq b_{T^c}}\frac{a^{(p,t)}(G)}{|Q(p)|+|Q(t)|}+\cdots\\
&\qquad+\sum_{\substack{p\leq\omega\\ p_k=\omega_k}}\sum_{t\leq b_{T^c}}\frac{a^{(p,t)}(G)}{|Q(p)|+|Q(t)|}\Big]\\
=\;& \sum_{p\leq\omega}\sum_{t\leq b_{T^c}}\frac{|Q(p)|}{|Q(p)|+|Q(t)|}\cdot a^{(p,t)}(G)\ .
\end{aligned}$$

Set $\bar{a}^{\omega}=\sum_{t\leq b_{T^c}}\frac{|Q(\omega)|}{|Q(\omega)|+|Q(t)|}\cdot a^{(\omega,t)}(G)$, we have that

$$G^{\Gamma}_{T,b}=\sum_{\omega\leq b_T\setminus\{0_T\}}\bar{a}^{\omega}\cdot u^{\omega}_T.$$

That is,

$$a^{\omega}(G^{\Gamma}_{T,b})=\bar{a}^{\omega}=\sum_{t\leq b_{T^c}}\frac{|Q(\omega)|}{|Q(\omega)|+|Q(t)|}\cdot a^{(\omega,t)}(G).$$

■

Theorem 6: The solutions Γ and Θ match CON simultaneously.

Proof: Hwang and Liao [5] demonstrated that the solution Θ matches CON. Let $(P,b,G)\in\Omega$ and $T\subseteq P, T\neq\emptyset$. By Lemma 2, for each $(i,k_i)\in k^T$,

$$\begin{aligned}
&\Gamma_{i,k_i}(T,b_T,G^{\Gamma}_{T,b})\\
=\;& \sum_{\omega\leq b_T,\ \omega_i=k_i}\frac{a^{\omega}(G^{\Gamma}_{T,b})}{|Q(\omega)|}\\
=\;& \sum_{\omega\leq b_T,\ \omega_i=k_i}\frac{1}{|Q(\omega)|}\cdot\sum_{t\leq b_{T^c}}\frac{|Q(\omega)|}{|Q(\omega)|+|Q(t)|}\cdot a^{(\omega,t)}(G)\\
=\;& \sum_{\omega\leq b_T,\ \omega_i=k_i}\sum_{t\leq b_{T^c}}\frac{a^{(\omega,t)}(G)}{|Q(\omega)|+|Q(t)|}\\
=\;& \sum_{\chi\in B^P,\ \chi_i=k_i}\frac{a^{\chi}(G)}{|Q(\chi)|}\\
=\;& \Gamma_{i,k_i}(P,b,G).
\end{aligned}$$

Hence the solution Γ matches CON. ■

In the following, we offer the parallel of Lemma 2 with respect to the D&P Shapley value without the proof.

Remark 1: Let $(P,b,G)\in\Omega$ and $T\subseteq P, T\neq\emptyset$. If $G=\sum_{\chi\in B^P\setminus\{0_P\}} a^{\chi}(G)\cdot u^{\chi}_P$, then $G^{\Gamma}_{T,b}$ can be expressed as

$$G^{\Gamma}_{T,b}=\sum_{\omega\in B^T\setminus\{0_T\}} a^{\omega}(G^{\Gamma}_{T,b})\cdot u^{\omega}_T,$$

where for each $\omega\in B^T\setminus\{0_T\}$,

$$a^{\omega}(G^{\Gamma}_{T,b})=\sum_{t\leq b_{T^c}}\frac{\|\omega\|}{\|\omega\|+\|t\|}\cdot a^{(\omega,t)}(G).$$

Lemma 3: A solution τ matches EFF if it matches WEFF and CON.

Proof: Let τ be a solution matching WEFF and CON, and $(P,b,G)\in\Omega$. It is trivial for $|P|=1$ by WEFF. Suppose that $|P|\geq 2$. Consider the reduction $\big(\{j\},b_j,G^{\tau}_{\{j\},b}\big)$ with $j\in P$. By equation (6),

$$G^{\tau}_{\{j\},b}(b_j)=G(b)-\sum_{i\in P\setminus\{j\}}\sum_{k_i=1}^{b_i}\tau_{i,k_i}(P,b,G).$$

Since τ matches CON,

$$\tau_{j,k_j}(P,b,G)=\tau_{j,k_j}\big(\{j\},b_j,G^{\tau}_{\{j\},b}\big)$$

for all $k_j\in B^+_j$. Then by WEFF,

$$\sum_{k_j=1}^{b_j}\tau_{j,k_j}(P,b,G)=G^{\tau}_{\{j\},b}(b_j).$$

So, $\sum_{i\in P}\sum_{k_i=1}^{b_i}\tau_{i,k_i}(P,b,G)=G(b)$, i.e., τ matches EFF. ■

The following outcome relates the condition of removing managing grades of a agent before passing to the reduction to the condition of removing managing grades of a agent after the passage. We demonstrate that the order does not matter.

Lemma 4: Given a solution τ, $(P,b,G)\in\Omega$, $T\subseteq P$, and $\omega\in B^T$. Then

$$\big(T,\omega,G^{\tau}_{T,b}\big)=\big(T,\omega,G^{\tau}_{T,(\omega,b_{T^c})}\big).$$

Proof: It is trivial to derive this overcome by the definitions of a reduction and a subgame, we omit it. ■

Lemma 5:

1) A solution τ matches P&Z-EL (or IIE) if it matches P&Z-WEL (or WIIE) and CON.
2) A solution τ matches D&P-EL if it matches D&P-WEL and CON.

Proof: Hwang and Liao [5] demonstrated the overcome 2. We demonstrate the overcome 1. Assume that a solution τ matches WIIE and CON. Let $(P,b,G)\in\Omega$, $i\in P$ and $k_i\in B^+_i$, $k_i\neq b_i$. Let $\omega^q=(b_{-i},k_i+q)$ for all $q=0,1,2,\cdots,b_i-k_i$. For all q, consider the reduction $\big(\{i\},k_i+q,G^{\tau}_{\{i\},\omega^q}\big)$ of the subgame (P,ω^q,G) of (P,b,G) with respect to τ, $\{i\}$ and ω^q, and the reduction $\big(\{i\},k_i,G^{\tau}_{\{i\},\omega^0}\big)$ of the subgame (P,ω^0,G) of (P,b,G) with respect to τ,$\{i\}$ and ω^0 respectively. By Lemma 4, $\big(\{i\},k_i,G^{\tau}_{\{i\},\omega^0}\big)$ is the subgame of $\big(\{i\},k_i+q,G^{\tau}_{\{i\},\omega^q}\big)$, i.e., $\big(\{i\}, k_i, G^{\tau}_{\{i\},\omega^q}\big) = \big(\{i\}, k_i, G^{\tau}_{\{i\},\omega^0}\big)$. By WIIE and CON,

$$\begin{array}{rll} & \tau_{i,k_i}\big(P,(b_{-i},k_i+q),G\big) & \\ = & \tau_{i,k_i}(P,\omega^q,G) & (\textbf{by } \omega^q=(b_{-i},k_i+q)) \\ = & \tau_{i,k_i}\big(\{i\},k_i+q,G^{\tau}_{\{i\},\omega^q}\big) & \big(\textbf{by CON}\big) \\ = & \tau_{i,k_i}\big(\{i\},k_i,G^{\tau}_{\{i\},\omega^q}\big) & \big(\textbf{by WIIE}\big) \\ = & \tau_{i,k_i}\big(\{i\},k_i,G^{\tau}_{\{i\},\omega^0}\big) & \big(\textbf{by Lemma 4}\big) \\ = & \tau_{i,k_i}(P,\omega^0,G) & \big(\textbf{by CON}\big) \\ = & \tau_{i,k_i}\big(P,(b_{-i},k_i),G\big). & \big(\textbf{by } \omega^0=(b_{-i},k_i)\big) \end{array}$$

So, τ matches IIE. ■

Theorem 7:

1) A solution τ matches WEFF, P&Z-WEL (or WIIE), UBC and CON if and only if $\tau = \Gamma$.
2) A solution τ matches WEFF, D&P-WEL, UBC and CON if and only if $\tau = \Theta$.

Proof: Hwang and Liao [5] demonstrated the overcome (2). The overcome (1) follows by Lemmas 3, 5, and Theorems 5, 6. ■

The following examples demonstrate that each of the properties adopted in the overcome (1) of Theorem 7 is logically independent of the rest of properties.

Example 1: Consider a solution τ by $\tau_{i,k_i}(P,b,G) = 0$ for all $(P,b,G) \in \Omega$ and for all $(i,k_i) \in K^P$. Clearly, τ matches P&Z-EL (or IIE), UBC and CON, but it doesn't match WEFF.

Example 2: By Theorems 4 and 6, the D&P Shapley value Θ matches EFF, UBC and CON, but it doesn't match P&Z-WEL (or WIIE).

Example 3: Consider a solution τ by for all $(P,b,G) \in \Omega$ and for all $(i,k_i) \in K^P$,

$$\tau_{i,k_i}(P,b,G) = \frac{G(b_{-i},k_i) - G(b_{-i},k_i-1)}{|Q(b_{-i},k_i)|}.$$

Clearly, τ matches WEFF, P&Z-WEL (or WIIE) and CON, but it doesn't match UBC.

Example 4: Define a solution τ by for all $(P,b,G) \in \Omega$ with $|P| = 1$ or $b_i = 1$ and for all $(i,k_i) \in K^P$,

$$\tau_{i,k_i}(P,b,G) = \Gamma_{i,k_i}(P,b,G);$$

otherwise

$$\tau_{i,k_i}(P,b,G) = \begin{cases} \Gamma_{i,k_i}(P,b,G) + \varepsilon & \text{if } k_i = b_i \\ \Gamma_{i,k_i}(P,b,G) - \frac{\varepsilon}{b_i - 1} & \text{otherwise,} \end{cases}$$

where $\varepsilon \in \mathbb{R} \setminus \{0\}$. Clearly, τ matches EFF, P&Z-WEL (or WIIE) and UBC, but it doesn't match CON.

VI. Agent-grade reduction

In this section, a different reduction are considered to characterize these two extended Shapley values.

Different from the reduction proposed in Section 5, we consider the *agent-grade reduced game* by both reducing the agents and its managing grades as follows.

Definition 6: Given $(P,b,G) \in \Omega$, $T \in 2^P \setminus \{\emptyset\}$, $\omega \in B_+^{P\setminus T}$ and a solution τ. We consider the **agent-grade reduced game** $(T, b_T, G^{T,\omega}_{\tau})$ related to T, ω and τ as follows. For all $\chi \in B^T$,

$$G^{T,\omega}_{\tau}(\chi) = G(\chi,\omega) - \sum_{i\in P\setminus T}\sum_{k_i=1}^{\omega_k} \tau_{i,k_i}(P,(\chi,\omega),G).$$

The agent-grade reduction might be described as follows. When renegotiating the value $\tau(P,b,G)$ within T, the grade vector $\omega \in B_+^{P\setminus T}$ means that the agents in $P \setminus T$ adopt nonzero grades based on the grade vector ω to operate with the coalition T lastingly.

- A solution τ matches **agent-grade consistency (AGCON)** if for all $(P,b,G) \in \Omega$, for all $T \in 2^P \setminus \{\emptyset\}$, for all $(i,k_i) \in K^T$ and for all $\omega \in B_+^{P\setminus T}$, $\tau_{i,k_i}(P,(b_T,\omega),G) = \tau_{i,k_i}(T,b_T,G^{T,\omega}_{\tau})$.

Remark 2: Given $(P,b,G) \in \Omega$, $T \in 2^P \setminus \{\emptyset\}$, $\omega = b_{P\setminus T}$ and a solution τ. Clearly, $G^{T,b_{P\setminus T}}_{\tau}(\chi) = G_{\tau}(\chi)$ for all $\chi \in B^T$. Thus, a solution matches CON if it matches AGCON.

Lemma 6: The solutions Θ and Γ match AGCON on Ω.

Proof: Let $(P,b,G) \in \Omega$, $T \in 2^P \setminus \{\emptyset\}$ and $\omega \in B_+^{P\setminus T}$. By Theorem 1, there exists an unique 0-normalized and P&Z-EFF potential H such that for all $(p,k_p) \in K^P$, $\Gamma_{p,k_p}(P,b,G) = H(P,(b_{-p},k_p),G) - H(P,(b_{-p},k_p-1),G)$. By definition of $G^{\Gamma}_{T,\omega}$ and EFF of Γ, for all $\lambda \in B^T$,

$$\begin{aligned} & G^{\Gamma}_{T,\omega}(\lambda) \\ = \;& G(\lambda,\omega) - \sum_{p\in P\setminus T}\sum_{k_p=1}^{\omega_p} \Gamma_{p,k_p}(P,(\lambda,\omega),G) \\ = \;& \sum_{p\in Q(\lambda)}\sum_{k_p=1}^{\lambda_p} \Gamma_{p,k_p}\big(P,(\lambda,\omega),G\big) \\ = \;& \sum_{p\in Q(\lambda)}\sum_{k_p=1}^{\lambda_p} \big[H\big(P,((\lambda_{-p},k_p),\omega),G\big) \\ & \qquad - H\big(P,((\lambda_{-p},k_p-1),\omega),G\big)\big]. \end{aligned} \tag{8}$$

By equation (3),

$$H(P,b,G) = \frac{1}{|P|}\cdot\big[G(b) + \sum_{i\in P} H(P,(b_{-i},0),G)\big]. \tag{9}$$

Equation (9) applied to $(T,\lambda,G^{\Gamma}_{T,\omega})$ and all its subgames determines its potential uniquely, i.e,

$$\begin{aligned} & H(T,\lambda,G^{\Gamma}_{T,\omega}) \\ = \;& \tfrac{1}{|Q(\lambda)|}\cdot\big[G^{\Gamma}_{T,\omega}(\lambda) + \sum_{i\in T(\lambda)} H(T,(\lambda_{-i},0),G^{\Gamma}_{T,\omega})\big]. \end{aligned} \tag{10}$$

Comparing this with the equations (8), (10), $H\big(P,(\lambda,\omega),G\big)$ and $H(T,\lambda,G^{\Gamma}_{T,\omega})$ may differ only by a constant $c \in \mathbb{R}$, i.e,

$$H(T,\lambda,G^{\Gamma}_{T,\omega}) \;=\; H\big(P,(\lambda,\omega),G\big) + c. \tag{11}$$

By equation (11), for all $(i,k_i) \in K^T$,

$$\begin{array}{rl} & \Gamma_{i,k_i}(T,b_T,G^{\Gamma}_{T,\omega}) \\ = & H(T,(b_{-i},k_i)_T,G^{\Gamma}_{T,\omega}) - H(T,(b_{-i},k_i-1)_T,G^{\Gamma}_{T,\omega}) \\ = & H\Big(P,\big((b_{-i},k_i)_T,\omega\big),G\Big) - H\Big(P,\big((b_{-i},k_i-1)_T,\omega\big),G\Big) \\ = & \Gamma_{i,k_i}(P,(b_T,\omega),G). \end{array}$$

Thus, the solution Γ matches AGCON. Similarly, it is easy to demonstrate that the solution Θ matches AGCON. ■

Theorem 8:

1) A solution τ matches P&Z-EL (or IIE) if it matches P&Z-WEL (or WIIE) and AGCON.
2) A solution τ matches D&P-EL if it matches D&P-WEL and AGCON.
3) A solution τ matches WEFF, P&Z-WEL (or WIIE), UBC and AGCON if and only if $\tau = \Gamma$.
4) A solution τ matches WEFF, D&P-WEL, UBC and AGCON if and only if $\tau = \Theta$.

Proof: These overcomes follow by Lemma 6, Remark 2 and Theorem 7. ■

In the following, we adopt different property to characterize these two extended Shapley values.

Definition 7: $(P, b, G) \in \Omega$ is α**-inessential** if there exists $\alpha \in \mathbb{R}^{K^P}$ such that $G(\chi) = \sum_{t\in Q(\chi)} \sum_{k_t=1}^{\chi_t} \alpha_{t,k_t}$ for all $\chi \in B^P$.

- **Inessential games (IEG)**: $\tau(P, b, G) = \alpha$ if $(P, b, G) \in \Omega$ is α-inessential. **Weak inessential games (WIEG)** asserts that τ matches IEG for all $(P, b, G) \in \Omega$ with $|P| = 1$.

Lemma 7:

1) A solution τ matches WEFF and P&Z-WEL (or WIIE) if and only if τ matches WIEG and P&Z-WEL (or WIIE).
2) A solution τ matches WEFF and D&P-WEL if and only if τ matches WIEG and D&P-WEL.

Proof: Let $(P, b, G) \in \Omega$ with $P = \{i\}$ and $k_i \in B_i$. If τ matches WEFF and P&Z-WEL (or WIIE), then $\tau_{i,k_i}(P, b_i, G) = \tau_{i,k_i}(P, k_i, G)$. Besides, $G(k_i) = \sum_{t=1}^{k_i} \big(G(k_i) - G(k_i - 1)\big)$. Thus,

$$\begin{aligned} & \tau \text{ matches P\&Z-WEL (or WIIE) and WEFF} \\ \Leftrightarrow\ & \tau_{i,k_i}(P, b_i, G) = \tau_{i,k_i}(P, k_i, G) = G(k_i) - G(k_i - 1) \\ & \text{and } G(k_i) = \sum_{t=1}^{k_i} \big(G(k_i) - G(k_i - 1)\big) \\ \Leftrightarrow\ & \tau \text{ matches P\&Z-WEL (or WIIE) and WIEG.} \end{aligned}$$

If τ matches D&P-WEL, then

$$\tau_{i,k_i}(P, b, G) - \tau_{i,k_i}\big(P, (b_{-i}, b_i - 1), G\big) = \tau_{i,b_i}(P, b, G).$$

On the other hand, $G(k_i) = \sum_{t=1}^{k_i} \frac{1}{t}\big(G(t) - G(t - 1)\big)$. Thus,

$$\begin{aligned} & \tau \text{ matches D\&P-WEL and WEFF} \\ \Leftrightarrow\ & \tau_{i,k_i}(P, b_i, G) = \sum_{t=k_i}^{b_i} \frac{1}{t}\big(G(t) - G(t - 1)\big) \\ & \text{and } G(k_i) = \sum_{t=1}^{k_i} \frac{1}{t}\big(G(t) - G(t - 1)\big) \\ \Leftrightarrow\ & \tau \text{ matches D\&P-WEL and WIEG.} \end{aligned}$$

The proof is completed. ■

Theorem 9:

1) A solution τ matches WIEG, P&Z-WEL (or WIIE), UBC and CON (or AGCON) if and only if $\tau = \Gamma$.
2) A solution τ matches WIEG, D&P-WEL, UBC and CON (or AGCON) if and only if $\tau = \Theta$.

Proof: These overcomes follow by Lemma 7 and Theorem 8. ■

VII. Conclusions

In this article, we concentrate on the extended Shapley values due to Derks and Peters [3] and Peters and Zank [16]. For these two solutions, we apply several existing concepts from standard games and reinterpret it under multi-choice games, such as potential approaches, axiomatic approaches, and so on. For the other extended Shapley values, Hwang and Liao [6], [8], [7] and Liao [11] offered similar overcomes respectively.

To sum up, each of these extended Shapley values can be generalize to be the vector of marginal contributions of a corresponding potential function. For these extended Shapley values, there also exist equivalence overcomes which are analogues of theorems due to Calvo and Santos [1] and Ortmann [13], [14]. In particular, the extended Shapley value due to Derks and Peters [3] is close to the extended Shapley value due to Peters and Zank [16]. By Theorems 2, 3, 4, 5, 7, 8, 9, these two extended Shapley values are almost the same in axiomatic approaches, except that the loss of amount is different in the axiom of equal loss.

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