A New Explicit Iteration Method for Common Solutions to Fixed Point Problems, Variational Inclusion Problems and Null Point Problems

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Abstract—**In this paper, we present a new viscosity technique for finding a common element of the set of common solutions of the variational inclusion problems, the set of common fixed points of an infinite family of demimetric mappings, and the set of solutions of the null point problems in Banach spaces. Under suitable assumptions, strong convergence of the sequence generated by the iterative algorithm is proven to the unique solution of the above problems. Furthermore, the main result is extended to the 2-generalized hybrid mappings and strict pseudo-contractions. A numerical example is also given to demonstrate the results.**

Index Terms—**Banach space, demimetric mapping, variational inclusion problem, fixed point, null point problems**

I. Introduction

Let H be a real Hilbert space, C be a nonempty closed convex subset of H,T be a mapping on C and $F(T) := \{x \in C : Tx = x\}$. Let $A : C \to H$ be a mapping. The metric (nearest point) projection from H onto a nonempty closed convex subset C of H is defined as follows: for each point $x \in H$, there exists a unique point $P_C x \in C$ with the property

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C,$$

that is, for any point $x \in H$, $\bar{x} = P_C x$ if and only if $\bar{x} \in C$ and $\|x - \bar{x}\| = \inf\{\|x - y\| : y \in C\}$.

The metric (nearest point) projection in the setting of Hilbert spaces has been extensively studied in the literature; see, for instance, [1], [2], [3], [4], [5]. The following lemma is a well-known result about approximation or projection.

Lemma I.1. *([6]) Let $P_C : H \to C$ be a metric projection from H on a nonempty closed convex subset C of H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Definition I.2. A mapping $T : C \to H$ is said to be:

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(1) a k-strict pseudo-contraction if there exist $k \in [0, 1)$ such that for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\| + k \|x - Tx - (y - Ty)\| ;$$

(2) a 2-generalized hybrid mapping if there $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ exist such that for all $x, y \in C$

$$\begin{aligned} & \delta_1 \left\|T^2x - Ty\right\|^2 + \delta_2 \|Tx - Ty\|^2 \\ & +(1 - \delta_1 - \delta_2) \|x - Ty\|^2 \\ \leq\ & \varepsilon_1 \left\|T^2x - y\right\|^2 + \varepsilon_2 \|Tx - y\|^2 \\ & +(1 - \varepsilon_1 - \varepsilon_2) \|x - y\|^2 . \end{aligned}$$

The class of 2-generalized hybrid mappings contains the classes of nonexpansive mappings, nonspreading mappings, hybrid mappings and generalized hybrid mappings in a Hilbert space; see [7], [8]. In general, 2-generalized hybrid mappings are not continuous; see [9]. Hence, the class of k-strict pseudo-contractions does not contain the class of 2-generalized hybrid mappings by the fact that k-strict pseudo-contractions are continuous.

The following example is a 2-generalized hybrid mapping, but it is not a k-strict pseudo-contraction.

Example I.3. ([10]) Let $S : [0, 2] \to \mathbb{R}$ be defined as

$$Sx = \begin{cases} 0, & x \in [0, 2); \\ 1, & x = 2. \end{cases}$$

Then S is a 2-generalized hybrid mapping and $F(S) = \{0\}$. However, it is not a k-strict pseudo-contraction.

On the other hand, the class of 2-generalized hybrid mappings does not contain the class of k-strict pseudo-contractions. We give an example for a k-strict pseudo-contraction which is not a 2-generalized hybrid mapping.

Example I.4. Let $S : \mathbb{R} \to \mathbb{R}$ be defined as

$$Sx = -3x.$$

Then S is a k-strict pseudo-contraction but not a 2-generalized hybrid mapping (check for instance the condition of 2-generalized hybrid mapping for $x = 0$ and $y = -1$).

Recently, Takahashi [11] introduced a broader class of nonlinear mappings in a Banach space called k-demimetric mapping. This class mappings contains the classes of 2-generalized hybrid mappings, k-strict pseudo-contractions, firmly-quasi-nonexpansive mappings and quasi-nonexpansive mappings.

Definition I.5. Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let k be a real number with $k \in (-\infty, 1)$. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(T)$,

$$\langle x - q, J(x - Tx)\rangle \geq \frac{1-k}{2} \|x - Tx\|^2 .$$

k-demimetric mapping may not be strictly pseudo-contractive. The following example (a k-demimetric mapping) is not pseudo-contractive. Then it is not strictly pseudo-contractive.

Example I.6. ([12]) Let H be the real line and $C = [-1, 1]$. Define T on C by $T(x) = \frac{2}{3}x\sin\frac{1}{x}$ if $x \neq 0$ and $T(0) = 0$. Clearly, 0 is the only fixed point of T. Also, for $x \in C$, $|T(x) - 0|^2 = |T(x)|^2 = |\frac{2}{3}x\sin\frac{1}{x}|^2 \leq |\frac{2x}{3}|^2 \leq |x|^2 \leq |x - 0|^2 + k|T(x) - x|^2$ for any $k \in [0, 1)$. Thus T is demimetric. We show that T is not pseudo-contractive. Let $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$. Then $|T(x) - T(y)| = \frac{256}{81\pi^2}$. However,

$$|x - y|^2 + |(I - T)x - (I - T)y|^2 = \frac{160}{81\pi^2}.$$

Takahashi [13] use Halpern type iteration to prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert. More precisely, Takahashi [13] introduced and studied the following iterative algorithm:

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(1 - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)), \ \forall n \in \mathbb{N}. \end{cases}$$

where $\{T_j\}_{j=1}^{M} : C \to H$ is a finite family of k_j-demimetric and demiclosed mappings, and $\{B_i\}_{i=1}^{N} : C \to H$ is a finite family of μ_i-inverse strongly monotone mappings. Then a strong convergence theorem is obtained under some mild restrictions on the parameters.

On the other hand, in order to finding a common fixed point of an infinite family of demimetric mappings in a Hilbert space, Akashi and Takahashi [14] introduced the following Mann's type iteration without assuming that demimetric mappings are commutative:

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ x_{n+1} = P_C(\alpha_n x_n + (1-\alpha_n)z_n), \qquad \forall n \in \mathbb{N}, \end{cases}$$

where $\{T_j\}_{j=1}^{\infty} : C \to H$ is an infinite family of k_j-demimetric and demiclosed mappings. A weak convergence theorem is presented under certain appropriate assumptions on the parameters.

Very recently, Takahashi [15] introduced the following iteration process for finding a common element of the set of common fixed points of an infinite family of demimetric mappings and the set of common solutions of variational inequality problems for an infinite family of inverse strongly monotone mappings in a Hilbert space:

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{\infty} \sigma_i J_{\eta_n}(1 - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)), \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{T_j\}_{j=1}^{\infty} : C \to H$ is an infinite family of k_j-demimetric and demiclosed mappings, $\{B_i\}_{i=1}^{\infty} : C \to H$ is an infinite family of μ_i-inverse strongly monotone mappings. Then a strong convergence result is proposed under some mild restrictions on the parameters.

Inspired by Akashi and Takahashi [14], Takahashi [13] and Takahashi [15], we present a new iterative scheme for finding a common element of the set of common solutions of the variational inclusion problems, the set of common fixed points of an infinite family of demimetric mappings, and the set of solutions of the null point problems in Banach spaces. The main results presented in this paper improve the corresponding results in [14], [13], [15], to a certain extent. Furthermore, some other results are also extended to some extent; see e.g., [16], [6], [17], [18], [19], [20], [8], [21], [22], [23], [24], [25], [26].

II. Preliminaries

Throughout this paper, we denote E the real Banach space, E^* the dual of E, I the identity mapping on E, H the real Hilbert space, and $\mathbb{N}$ the set of nonnegative integers. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively. The (normalized) duality mapping of E is denoted by J, that is,

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\}$$

for all $x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. If E is a Hilbert space, then $J = I$, where I is the identity mapping on H.

The norm of a Banach space E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. In this case, we say that E is smooth.

Let C be a nonempty closed convex subset of H and let $T : C \to H$ be a mapping. We say that

(i) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(ii) T is firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$.

It is easily found that T is firmly nonexpansive if and only if $T = (I + V)/2$ for some nonexpansive mapping V. Hence a firmly nonexpansive mapping must be nonexpansive. We also notice that if T is nonexpansive, then the fixed point set of T, $F(T)$, is closed and convex [20].

Lemma II.1. *([27]) Let $\{\alpha_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{n_i} < \alpha_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\alpha_{m_k} \leq \alpha_{m_k+1} \quad \text{and} \quad \alpha_k \leq \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : \alpha_j < \alpha_{j+1}\}$.

Lemma II.2. *([28]) Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property:*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}, \{b_n\}, \{c_n\}$ satisfy the restrictions:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \lim_{n\to\infty} \gamma_n = 0,$
(ii) $b_n \geq 0, \quad \sum_{n=1}^{\infty} b_n < \infty,$

(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma II.3. *In a Hilbert space H, it holds for all $x,y \in H$ and $\lambda \in [0,1]$ that*

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2,$$

which can be extended to the more general situation: For all $x_1,x_2,...,x_n \in H$, $\lambda_i \in [0,1]$, and $\sum_{i=1}^n \lambda_i = 1$, we have

$$\|\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_nx_n\|^2 = \lambda_1\|x_1\|^2+\lambda_2\|x_1\|^2 +\ldots+\lambda_n\|x_n\|^2 - \sum_{1\leq i\leq j\leq n}\lambda_i\lambda_j\|x_i-x_j\|^2.$$

Lemma II.4. *([14]) Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty,1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

Lemma II.5. *([29]) Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in N\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty \alpha_n = 1$. Then a mapping T on C defined by*

$$Tx = \sum_{n=1}^\infty \alpha_n T_n x,$$

for $x \in C$, is well defined, nonexpansive, and $F(T) = \bigcap_{n=1}^\infty F(T_n)$ holds.

Recall that a mapping $A : C \to H$ is said to be α-inverse-strongly monotone (ism) if there exists a constant $\alpha > 0$ such that

$$\langle Ax-Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

We collect some basic properties of inverse strongly monotone operators in the following proposition.

Proposition II.6. *([1]) We have:*

(i) *If $A : C \to H$ is α-ism and λ is any constant in $(0,2\alpha]$, then the mapping $I-\lambda A$ is nonexpansive;*
(ii) *A mapping $T : C \to H$ is nonexpansive if and only if $I-T$ is $\frac{1}{2}$-ism;*
(iii) *If A is α-ism, then for $\gamma > 0$, γA is $\frac{\alpha}{\gamma}$-ism.*

III. Some new lemmas

We also need the following lemmas, which are fundamental for our main theorem.

Lemma III.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$. Let $A : C \to H$ be α-ism. Suppose that $M^{-1}0\bigcap A^{-1}0 \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I-rA)z$;
(ii) $z \in (M+A)^{-1}0$;
(iii) $z \in M^{-1}0\bigcap A^{-1}0$.

Consequently, $F(J_\lambda(1-rA)) = (M+A)^{-1}0 = M^{-1}0\bigcap A^{-1}0$.

Proof: Since $M^{-1}0\bigcap A^{-1}0 \neq \emptyset$, there exists $z_0 \in D(M)$ such that $0 \in Mz_0$ and $0 = Az_0$.

$(i) \Rightarrow (ii)$. Assuming $z = J_\lambda(I-rA)z$ yields

$$-\frac{r}{\lambda}Az \in Mz.$$

Since M is monotone and $0 \in Mz_0$, we obtain

$$\langle Az, z-z_0\rangle \leq 0.$$

This together with $Az_0 = 0$ implies that

$$\alpha\|Az\|^2 = \alpha\|Az-Az_0\|^2 \leq \langle Az-Az_0, z-z_0\rangle \leq 0.$$

Therefore, $Az = 0$. This reduces the fixed point equation $z = J_\lambda(I-rA)z$ to the fixed point equation $z = J_\lambda z$ that is equivalent to $0 \in Mz$. Consequently, $0 \in Mz+Az$. This means $z \in (M+A)^{-1}0$.

$(ii) \Rightarrow (iii)$. The assumption $z \in (M+A)^{-1}0$ can be rewritten as $-Az \in Mz$. The monotonicity of M then implies (note $0 \in Mz_0$) that

$$\langle Az, z-z_0\rangle \leq 0. \tag{III.1}$$

Noticing $Az_0 = 0$, we obtain that

$$\alpha\|Az\|^2 = \alpha\|Az-Az_0\|^2 \leq \langle Az-Az_0, z-z_0\rangle \leq 0.$$

It shows that $Az = 0$. Now the assumption $z \in (M+A)^{-1}0$ is reduced to the relation $0 \in Mz$. Consequently, we have $z \in M^{-1}0\bigcap A^{-1}0$.

$(iii) \Rightarrow (i)$. Since $z \in M^{-1}0\bigcap A^{-1}0$, we have that $z \in M^{-1}0$ and $z \in \bigcap A^{-1}0$. It follows that $z = J_\lambda z$ and $Az = 0$. Thus we have

$$z = J_\lambda(I-rA)z.$$

The proof is completed. ■

Lemma III.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{A_i\}_{i=1}^\infty : C \to H$ be an infinite family of α_i-ism mappings such that $\bigcap_{i=1}^\infty A_i^{-1}0 \neq \emptyset$. Let $\{\lambda_i\}_{i=1}^\infty$ be a positive sequence such that $\sum_{i=1}^\infty \lambda_i = 1$. Then $\sum_{i=1}^\infty \lambda_iA_i : C \to H$ is an α-ism mapping with $\alpha = \inf\{\alpha_i : i = 1,2,...\}$, and $(\sum_{i=1}^\infty \lambda_iA_i)^{-1}0 = \bigcap_{i=1}^\infty A_i^{-1}0$ holds.*

Proof: Setting $S_i = I-2\alpha A_i$, from Proposition II.6(i), we know S_i is nonexpansive. Since $\bigcap_{i=1}^\infty A_i^{-1}0 \neq \emptyset$, noticing $F(S_i) = A_i^{-1}0$, we have that $\bigcap_{i=1}^\infty F(S_i) = \bigcap_{i=1}^\infty A_i^{-1}0 \neq \emptyset$. It follows from Lemma II.5 that $\sum_{i=1}^\infty \lambda_iS_i$ strongly converges. Noticing that $A_i = \frac{1}{2\alpha}(I-S_i)$, we deduce that $\sum_{i=1}^\infty \lambda_iA_i$ strongly converges. Letting

$$S = \sum_{i=1}^\infty \lambda_iS_i \quad \text{and} \quad A = \sum_{i=1}^\infty \lambda_iA_i,$$

then we have $A = \frac{1}{2\alpha}(I-S)$. Since S is nonexpansive due to Lemma II.5, we deduce $I-S$ is $\frac{1}{2}$-ism by Proposition II.6 (ii). Hence, we get A is α-ism by Proposition II.6 (iii). Taking into consideration that $F(S) = A^{-1}0$ and noticing the fact that $\bigcap_{i=1}^\infty F(S_i) = \bigcap_{i=1}^\infty A_i^{-1}0$, we deduce that $(\sum_{i=1}^\infty \lambda_iA_i)^{-1}0 = \bigcap_{i=1}^\infty A_i^{-1}0$. ■

For every $i = 1,2,...$, let $A_i : C \to H$ and $M : C \supseteq Dom(M) \to 2^H$ be nonlinear mappings. We introduce the combination of variational inclusion problem in Hilbert spaces as follows: find a point $x^* \in C$ such that

$$0 \in (M+\sum_{i=1}^\infty \lambda_iA_i)x^*,$$

where λ_i is a real positive number for all $i = 1,2,...$ with $\sum_{i=1}^\infty \lambda_i = 1$.

Lemma III.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{A_i\}_{i=1}^{\infty}: C \to H$ be an infinite family of α_i-ism mappings with $\alpha = \inf\{\alpha_i : i = 1,2,...\}$, M be maximal monotone in H with $Dom(M) \subseteq C$ and $J_r = (I+rM)^{-1}$ be the resolvent of M for $r > 0$. Let $\{\lambda_i\}$ be a real number sequence in $(0,1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\bigcap_{i=1}^{\infty}(M + A_i)^{-1}0 \neq \emptyset$. Then,*

$$(M+\sum_{i=1}^{\infty}\lambda_i A_i)^{-1}0=\bigcap_{i=1}^{\infty}(M+A_i)^{-1}0.$$

Proof: We can obtain the desired result due to Lemma III.1 and Lemma III.2. ■

Lemma III.4. *([30]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}: C \to H$ be an infinite family of k_i-demimetric mappings with $\sup\{k_i : i = 1,2,...\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then, the following conclusions hold*

1. *$\sum_{i=1}^{\infty} \eta_i T_i : C \to H$ is a k-demimetric mapping with $k = \sup\{k_i : i = 1,2,...\}$;*
2. *$F(\sum_{i=1}^{\infty} \eta_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$;*
3. *if T_i is demiclosed for each $i \in \mathbb{N}$, then $\sum_{i=1}^{\infty} \eta_i T_i : C \to H$ is demiclosed.*

Lemma III.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$. Given $0 < s \leq r$ and $x \in H$, it holds that*

$$\|J_s x - J_r x\| \leq \left|1-\frac{s}{r}\right| \|x - J_r x\|$$

and

$$\|x - J_s x\| \leq 2\|x - J_r x\|.$$

Proof: Note that $(x - J_\lambda x)/\lambda \in M(J_\lambda x)$. Since M is monotone, we have

$$\langle\frac{x-J_s x}{s}-\frac{x-J_r x}{r}, J_s x - J_r x\rangle \geq 0.$$

It turns out that

$$\begin{aligned}\|J_s x - J_r x\|^2 &\leq \frac{r-s}{r}\langle x - J_r x, J_s x - J_r x\rangle \\ &\leq |1-\frac{s}{r}|\,\|x - J_r x\|\,\|J_s x - J_r x\|.\end{aligned}$$

This along with the triangle inequality yields that

$$\begin{aligned}\|x - J_s x\| &\leq \|x - J_r x\| + \|J_s x - J_r x\| \\ &\leq \|x - J_r x\| + |1-\frac{s}{r}|\,\|x - J_r x\| \\ &\leq 2\|x - J_r x\|.\end{aligned}$$

This completes the proof. ■

IV. Main Results

NOW, we can prove the main theorem.

Theorem IV.1. *Let E be a smooth, strictly convex, and reflexive Banach space and let J be the duality mapping on E. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{\infty}: C \to H$ be an infinite family of μ_i-ism mappings with $\mu = \inf\{\mu_i : i = 1,2,...\}$. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$ and let $f : C \to H$ be a contraction with coefficient $v \in [0,1)$. Let $B : H \to E$ be a bounded linear operator such that $B \neq 0$ and let B^* be the adjoint operator of B. Let $\{T_i\}_{i=1}^{\infty}: C \to H$ be an infinite family of k_i-demimetric and demiclosed mappings with $k = \sup\{k_i : i = 1,2,...\} < 1$, $S : E \to E$ be a $\hat{k}$-demimetric and demiclosed mapping. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} u_n = J_{\eta_n}(I-\lambda_n\sum_{i=1}^{\infty}\sigma_i A_i)x_n, \\ y_n = \zeta_n(\sum_{i=1}^{\infty}\delta_i T_i u_n - \tau B^* J(I-S)Bu_n) \\ \quad +(1-\zeta_n)u_n, \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n),\ \forall n \in \mathbb{N}, \end{cases} \tag{IV.1}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\zeta_n\} \subset (0,1)$ and $\{\delta_n\},\{\sigma_n\},\{\eta_n\},\{\lambda_n\} \subset (0,+\infty)$ satisfy the following conditions:

- (i) *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*
- (ii) *$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$,*
- (iii) *$0 < c \leq \eta_n$,*
- (iv) *$\sum_{i=1}^{\infty} \delta_i = 1$ and $\sum_{i=1}^{\infty} \sigma_i = 1$,*
- (v) *$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\mu$,*
- (vi) *$0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < \min\{\frac{1-k}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.*

Assume $\Gamma := F(\sum_{i=1}^{\infty}\delta_i T_i)\bigcap(M+\sum_{i=1}^{\infty}\sigma_i A_i)^{-1}0\bigcap B^{-1}F(S) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (IV.1) strongly converges to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f z_0$.

Proof: Set $T = \sum_{i=1}^{\infty} \delta_i T_i$ and $A = \sum_{i=1}^{\infty} \sigma_i A_i$. We get by Lemma III.3 and Lemma III.4 that $\Gamma = \bigcap_{i=1}^{\infty} F(T_i)\bigcap(\bigcap_{i=1}^{\infty}(M + A_i)^{-1}0)\bigcap B^{-1}F(S)$. Taking any $z \in \Gamma$, we have that

$$\begin{aligned} &\langle u_n - Tu_n + \tau B^* J(I-S)Bu_n, u_n - z\rangle \\ =\ &\langle u_n - Tu_n, u_n - z\rangle + \tau\langle B^* J(I-S)Bu_n, u_n - z\rangle \\ =\ &\langle u_n - Tu_n, u_n - z\rangle + \tau\langle J(I-S)Bu_n, Bu_n - Bz\rangle \\ \geq\ &\frac{1-k}{2}\|u_n - Tu_n\|^2 + \frac{\tau(1-\hat{k})}{2}\|Bu_n - SBu_n\|^2. \end{aligned} \tag{IV.2}$$

It follows from (IV.1) that

$$\begin{aligned} &\|y_n - z\|^2 \\ =\ &\|(1-\zeta_n)u_n + \zeta_n(Tu_n - \tau B^* J(I-S)Bu_n) - z\|^2 \\ =\ &\|u_n - z - \zeta_n(u_n - Tu_n + \tau B^* J(I-S)Bu_n)\|^2 \\ =\ &\|u_n - z\|^2 - 2\zeta_n\langle u_n - Tu_n + \tau B^* J(I-S)Bu_n, u_n - z\rangle \\ &+\zeta_n^2\|u_n - Tu_n + \tau B^* J(I-S)Bu_n\|^2 \\ \leq\ &\|u_n - z\|^2 - 2\zeta_n\langle u_n - Tu_n + \tau B^* J(I-S)Bu_n, u_n - z\rangle \\ &+\zeta_n^2(\|u_n - Tu_n\| + \|\tau B^* J(I-S)Bu_n\|)^2 \\ \leq\ &\|u_n - z\|^2 - 2\zeta_n\langle u_n - Tu_n + \tau B^* J(I-S)Bu_n, u_n - z\rangle \\ &+\zeta_n^2(2\|u_n - Tu_n\|^2 + 2\tau^2\|B\|^2\|(I-S)Bu_n\|^2). \end{aligned}$$

This together with (IV.2) implies that

$$\begin{aligned} &\|y_n - z\|^2 \\ \leq\ &\|u_n - z\|^2 - \zeta_n(1-k)\|u_n - Tu_n\|^2 \\ &-\zeta_n\tau(1-\hat{k})\|Bu_n - SBu_n\|^2 \\ &+\zeta_n^2(2\|u_n - Tu_n\|^2 + 2\tau^2\|B\|^2\|Bu_n - SBu_n\|^2) \\ =\ &\|u_n - z\|^2 - \zeta_n(1-k-2\zeta_n)\|u_n - Tu_n\|^2 \\ &-\zeta_n\tau(1-\hat{k}-2\zeta_n\tau\|B\|^2)\|Bu_n - SBu_n\|^2. \end{aligned} \tag{IV.3}$$

By condition (vi), we have

$$\|y_n - z\| \leq \|u_n - z\|. \tag{IV.4}$$

In view of Lemma III.2, we get A is μ-ism. Hence we obtain $J_{\eta_n}(I-\lambda_n A)$ is nonexpansive due to Proposition II.6. Now put $J_{\eta_n}=\frac{1}{2}I+\frac{1}{2}V_n$ for all $n\in\mathbb{N}$. Since J_{η_n} is firmly nonexpansive, then we know that V_n is nonexpansive and $F(J_{\eta_n})=F(V_n)$. Noticing $z\in\bigcap_{i=1}^{\infty}(M+A_i)^{-1}0$, we get by Lemma III.1 and Lemma III.2 that $z\in M^{-1}0\bigcap(\bigcap_{i=1}^{\infty}A_i^{-1}0)=M^{-1}0\bigcap A^{-1}0$. It follows that

$$\begin{aligned}
&\|u_n-z\|^2\\
=\;&\left\|J_{\eta_n}(I-\lambda_n A)x_n-J_{\eta_n}z\right\|^2\\
=\;&\left\|(\frac{1}{2}I+\frac{1}{2}V_n)(I-\lambda_n A)x_n-(\frac{1}{2}I+\frac{1}{2}V_n)z\right\|^2\\
\leq\;&\frac{1}{2}\|(I-\lambda_n A)x_n-z\|^2+\frac{1}{2}\|V_n(I-\lambda_n A)x_n-z\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
\leq\;&\|(I-\lambda_n A)x_n-z\|^2-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
=\;&\|x_n-z\|^2-2\lambda_n\langle Ax_n,x_n-z\rangle+\lambda_n^2\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
=\;&\|x_n-z\|^2-2\lambda_n\langle Ax_n-Az,x_n-z\rangle+\lambda_n^2\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
\leq\;&\|x_n-z\|^2-2\lambda_n\mu\|Ax_n-Az\|^2+\lambda_n^2\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
=\;&\|x_n-z\|^2-2\lambda_n\mu\|Ax_n\|^2+\lambda_n^2\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
\leq\;&\|x_n-z\|^2-\lambda_n(2\mu-\lambda_n)\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2. \qquad \text{(IV.5)}
\end{aligned}$$

It follows from (IV.4), (IV.5) and (v) that

$$\begin{aligned}
&\|x_{n+1}-z\|\\
\leq\;&\|\alpha_n(fx_n-z)+\beta_n(u_n-z)+\gamma_n(y_n-z)\|\\
\leq\;&\alpha_n\|fx_n-z\|+\beta_n\|u_n-z\|+\gamma_n\|y_n-z\|\\
\leq\;&\alpha_n\|fx_n-z\|+(1-\alpha_n)\|x_n-z\|\\
\leq\;&\alpha_n\|fx_n-fz\|+\alpha_n\|fz-z\|+(1-\alpha_n)\|x_n-z\|\\
\leq\;&\alpha_n v\|x_n-z\|+\alpha_n\|fz-z\|+(1-\alpha_n)\|x_n-z\|\\
=\;&(1-\alpha_n(1-v))\|x_n-z\|+\alpha_n\|fz-z\|\\
\leq\;&\max\{\|x_n-z\|,\frac{\|fz-z\|}{1-v}\}.
\end{aligned}$$

By induction, we have

$$\|x_n-z\|\leq\max\{\|x_0-z\|,\frac{\|fz-z\|}{1-v}\},\quad\forall n\in\mathbb{N},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$.

In terms of (IV.3), (IV.4), (IV.5) and Lemma II.3, we have

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\leq\;&\|\alpha_n(fx_n-z)+\beta_n(u_n-z)+\gamma_n(y_n-z)\|^2\\
\leq\;&\alpha_n\|fx_n-z\|^2+\beta_n\|u_n-z\|^2+\gamma_n\|y_n-z\|^2\\
&-\beta_n\gamma_n\|u_n-y_n\|^2\\
\leq\;&\alpha_n\|fx_n-z\|^2+\beta_n(\|x_n-z\|^2-\lambda_n(2\mu-\lambda_n)\|Ax_n\|^2\\
&-\frac{1}{4}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2)\\
&+\gamma_n(\|u_n-z\|^2-\zeta_n(1-k-2\zeta_n)\|u_n-Tu_n\|^2\\
&-\zeta_n\tau(1-\hat{k}-2\zeta_n\tau\|B\|^2)\|Bu_n-SBu_n\|^2)\\
&-\beta_n\gamma_n\|u_n-y_n\|^2\\
\leq\;&\alpha_n\|fx_n-z\|^2+\|x_n-z\|^2-\beta_n\lambda_n(2\mu-\lambda_n)\|Ax_n\|^2\\
&-\frac{1}{4}\beta_n\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
&-\gamma_n\zeta_n(1-k-2\zeta_n)\|u_n-Tu_n\|^2\\
&-\gamma_n\zeta_n\tau(1-\hat{k}-2\zeta_n\tau\|B\|^2)\|Bu_n-SBu_n\|^2\\
&-\beta_n\gamma_n^2\|u_n-y_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\beta_n\lambda_n(2\mu-\lambda_n)\|Ax_n\|^2+\gamma_n\zeta_n(1-k-2\zeta_n)\|u_n-Tu_n\|^2\\
&+\frac{1}{4}\beta_n\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|^2\\
&+\gamma_n\zeta_n\tau(1-\hat{k}-2\zeta_n\tau\|B\|^2)\|Bu_n-SBu_n\|^2\\
&+\beta_n\gamma_n^2\|u_n-y_n\|^2\\
\leq\;&\alpha_n\|fx_n-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2. \qquad \text{(IV.6)}
\end{aligned}$$

Case 1. Assume there exists some integer $m>0$ such that $\{\|x_n-z_0\|\}$ is decreasing for all $n\geq m$. In this case, we know that $\lim_{n\to\infty}\|x_n-z_0\|$ exists. From (IV.6) and conditions (i)-(vi), we deduce

$$\lim_{n\to\infty}\|u_n-y_n\|=0, \qquad \text{(IV.7)}$$

$$\lim_{n\to\infty}\|Ax_n\|=0, \qquad \text{(IV.8)}$$

$$\lim_{n\to\infty}\|u_n-Tu_n\|=0, \qquad \text{(IV.9)}$$

$$\lim_{n\to\infty}\|Bu_n-SBu_n\|=0 \qquad \text{(IV.10)}$$

and

$$\lim_{n\to\infty}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|=0. \qquad \text{(IV.11)}$$

Observe that

$$\begin{aligned}
&\|u_n-x_n\|\\
=\;&\left\|J_{\eta_n}(I-\lambda_n A)x_n-x_n\right\|\\
=\;&\left\|(\frac{1}{2}I+\frac{1}{2}V_n)(I-\lambda_n A)x_n-x_n\right\|\\
\leq\;&\frac{1}{2}\|(I-\lambda_n A)x_n-x_n\|+\frac{1}{2}\|V_n(I-\lambda_n A)x_n-x_n\|\\
\leq\;&\frac{1}{2}\|(I-\lambda_n A)x_n-x_n\|+\frac{1}{2}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|\\
&+\frac{1}{2}\|(I-\lambda_n A)x_n-x_n\|\\
\leq\;&\lambda_n\|Ax_n\|+\frac{1}{2}\|V_n(I-\lambda_n A)x_n-(I-\lambda_n A)x_n\|.
\end{aligned}$$

This together with (IV.8) and (IV.11) implies that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \qquad \text{(IV.12)}$$

Let $T_c = J_c(I - 2\mu A)$, where $J_c = (I + cM)^{-1}$. Then we deduce from (IV.8), (IV.12), Lemma III.5 and (iii) that

$$\begin{aligned}
& \|T_c x_n - x_n\| \\
= & \|J_c(I-2\mu A)x_n - x_n\| \\
= & \|J_c(I-2\mu A)x_n - (I-2\mu A)x_n\| + \|(I-2\mu A)x_n - x_n\| \\
\leq & 2\left\|J_{\eta_n}(I-2\mu A)x_n - (I-2\mu A)x_n\right\| \\
& + \|(I-2\mu A)x_n - x_n\| \\
\leq & 2\left\|J_{\eta_n}(I-2\mu A)x_n - J_{\eta_n}(I-2\lambda_n A)x_n\right\| \\
& +2\left\|J_{\eta_n}(I-2\lambda_n A)x_n - x_n\right\| + 2\|x_n - (I-2\mu A)x_n\| \\
& + \|(I-2\mu A)x_n - x_n\| \\
\leq & 4|\mu - \lambda_n|\,\|Ax_n\| + 2\|u_n - x_n\| + 6\mu\|Ax_n\| \\
& \to 0. \qquad \text{(IV.13)}
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying $x_{n_i} \rightharpoonup \tilde{x} \in C$. Without loss of generality, we may also assume

$$\lim_{i\to\infty}\langle fz_0 - z_0, x_{n_i} - z_0\rangle = \limsup_{n\to\infty}\langle fz_0 - z_0, x_n - z_0\rangle. \quad \text{(IV.14)}$$

Since T_i is demiclosed for each $i \in \mathbb{N}$, noticing (IV.9), (IV.12) and Lemma III.4, we have $\tilde{x} \in F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. Because B is bounded and linear, we also see that $\{Bu_{n_i}\}$ converges weakly to $B\tilde{x}$. Using this and (IV.10), we have $\tilde{x} \in B^{-1}F(S)$. Noting (IV.13) and the fact that $T_c = J_c(I-2\mu A)$ is nonexpansive, we get $\tilde{x} \in (M+A)^{-1}0$ due to Lemma III.1.

It follows from (IV.14) and Lemma I.1 that

$$\begin{aligned}
& \limsup_{n\to\infty}\langle fz_0 - z_0, x_n - z_0\rangle \\
= & \lim_{i\to\infty}\langle fz_0 - z_0, x_{n_i} - z_0\rangle \\
= & \langle fz_0 - z_0, \tilde{x} - z_0\rangle \\
= & \langle fz_0 - P_\Gamma fz_0, \tilde{x} - P_\Gamma fz_0\rangle \\
\leq & 0. \qquad \text{(IV.15)}
\end{aligned}$$

Setting $h_n = \alpha_n f x_n + \beta_n u_n + \gamma_n y_n$ for all $n \in \mathbb{N}$, we have from (IV.1) that $x_{n+1} = P_C h_n$. It follows from (IV.4), (IV.5) and Lemma I.1 that

$$\begin{aligned}
& \|x_{n+1} - z_0\|^2 \\
= & \langle P_C h_n - h_n, P_C h_n - z_0\rangle + \langle h_n - z_0, P_C h_n - z_0\rangle \\
\leq & \langle \alpha_n f x_n + \beta_n u_n + \gamma_n y_n - z_0, x_{n+1} - z_0\rangle \\
\leq & \|\beta_n(u_n - z_0) + \gamma_n(y_n - z_0)\|\,\|x_{n+1} - z_0\| \\
& +\alpha_n\langle fx_n - z_0, x_{n+1} - z_0\rangle \\
\leq & \beta_n\|u_n - z_0\|\,\|x_{n+1} - z_0\| + \gamma_n\|y_n - z_0\|\,\|x_{n+1} - z_0\| \\
& +\alpha_n\langle fx_n - fz_0, x_{n+1} - z_0\rangle + \alpha_n\langle fz_0 - z_0, x_{n+1} - z_0\rangle \\
\leq & (1-\alpha_n)\|x_n - z_0\|\,\|x_{n+1} - z_0\| \\
& +\alpha_n v\|x_n - z_0\|\,\|x_{n+1} - z_0\| \\
& +\alpha_n\langle fz_0 - z_0, x_{n+1} - z_0\rangle \\
\leq & (1-\alpha_n(1-v))\|x_n - z_0\|^2 + \alpha_n\langle fz_0 - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

This together with Lemma II.2 and (IV.15) implies $x_n \to z_0$ as $n \to \infty$.

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - z_0\| < \|x_{n_i+1} - z_0\|$ for all $i \in \mathbb{N}$. Then by Lemma II.1, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that

$$\begin{aligned}
\left\|x_{m_j} - z_0\right\| &\leq \left\|x_{m_j+1} - z_0\right\|, \\
\left\|x_j - z_0\right\| &\leq \left\|x_{m_j+1} - z_0\right\|. \qquad \text{(IV.16)}
\end{aligned}$$

Following a similar argument as in the proof of Case 1, we have that

$$\lim_{n\to\infty}\left\|u_{m_j} - y_{m_j}\right\| = 0, \qquad \text{(IV.17)}$$

and

$$\lim_{n\to\infty}\left\|u_{m_j} - x_{m_j}\right\| = 0. \qquad \text{(IV.18)}$$

We want to show that

$$\limsup_{j\to\infty}\langle fz_0 - z_0, x_{m_j} - z_0\rangle \leq 0, \qquad \text{(IV.19)}$$

where $z_0 = P_\Gamma fz_0$. Without loss of generality, there exists a subsequence $\{x_{m_{j_k}}\}$ of $\{x_{m_j}\}$ such that $x_{m_{j_k}} \rightharpoonup \omega$ for some $\omega \in C$ and

$$\lim_{k\to\infty}\langle fz_0 - z_0, x_{m_{j_k}} - z_0\rangle = \limsup_{j\to\infty}\langle fz_0 - z_0, x_{m_j} - z_0\rangle.$$

Thus we deduce that

$$\begin{aligned}
& \limsup_{j\to\infty}\langle fz_0 - z_0, x_{m_j} - z_0\rangle \\
= & \lim_{k\to\infty}\langle fz_0 - z_0, x_{m_{j_k}} - z_0\rangle \\
= & \langle fz_0 - P_\Gamma fz_0, \omega - P_\Gamma fz_0\rangle \\
\leq & 0. \qquad \text{(IV.20)}
\end{aligned}$$

Taking into consideration that

$$\begin{aligned}
& \left\|x_{m_j+1} - x_{m_j}\right\| \\
\leq & \left\|\alpha_{m_j} f x_{m_j} + \beta_{m_j} u_{m_j} + \gamma_{m_j} y_{m_j} - x_{m_j}\right\| \\
\leq & \alpha_{m_j}\left\|fx_{m_j} - x_{m_j}\right\| + \beta_{m_j}\left\|u_{m_j} - x_{m_j}\right\| \\
& +\gamma_{m_j}\left\|y_{m_j} - x_{m_j}\right\| \\
\leq & \alpha_{m_j}\left\|fx_{m_j} - x_{m_j}\right\| + \beta_{m_j}\left\|u_{m_j} - x_{m_j}\right\| \\
& +\gamma_{m_j}\left\|y_{m_j} - u_{m_j}\right\| + \gamma_{m_j}\left\|u_{m_j} - x_{m_j}\right\| \\
= & \alpha_{m_j}\left\|fx_{m_j} - x_{m_j}\right\| + (1-\alpha_{m_j})\left\|u_{m_j} - x_{m_j}\right\| \\
& +\gamma_{m_j}\left\|y_{m_j} - u_{m_j}\right\|,
\end{aligned}$$

we deduce from (IV.17), (IV.18) and (i) that

$$\lim_{j\to\infty}\left\|x_{m_j+1} - x_{m_j}\right\| = 0. \qquad \text{(IV.21)}$$

Letting $h_{m_j} = \alpha_{m_j} f x_{m_j} + \beta_{m_j} u_{m_j} + \gamma_{m_j} y_{m_j}$ for all $j \in \mathbb{N}$, we get by LemmaI.1, (IV.4), (IV.5) and (IV.16) that

$$\begin{aligned}
& \left\|x_{m_j+1}-z_0\right\|^2 \\
= \; & \langle P_C h_{m_j}-h_{m_j}, P_C h_{m_j}-z_0\rangle+\langle h_{m_j}-z_0, P_C h_{m_j}-z_0\rangle \\
\leq \; & \langle \alpha_{m_j} f x_{m_j}+\beta_{m_j} u_{m_j}+\gamma_{m_j} y_{m_j}-z_0, x_{m_j+1}-z_0\rangle \\
\leq \; & \left\|\beta_{m_j}(u_{m_j}-z_0)+\gamma_{m_j}(y_{m_j}-z_0)\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\alpha_{m_j}\langle f x_{m_j}-z_0, x_{m_j+1}-z_0\rangle \\
\leq \; & \beta_{m_j}\left\|u_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\gamma_{m_j}\left\|y_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\alpha_{m_j}\langle f x_{m_j}-f z_0, x_{m_j+1}-z_0\rangle \\
& +\alpha_{m_j}\langle f z_0-z_0, x_{m_j+1}-x_{m_j}\rangle \\
& +\alpha_{m_j}\langle f z_0-z_0, x_{m_j}-z_0\rangle \\
\leq \; & (1-\alpha_{m_j})\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\alpha_{m_j} v\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\alpha_{m_j}\left\|f z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\| \\
& +\alpha_{m_j}\langle f z_0-z_0, x_{m_j}-z_0\rangle \\
\leq \; & (1-\alpha_{m_j}(1-v))\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
& +\alpha_{m_j}\left\|f z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\| \\
& +\alpha_{m_j}\langle f z_0-z_0, x_{m_j}-z_0\rangle \\
\leq \; & (1-\alpha_{m_j}(1-v))\left\|x_{m_j+1}-z_0\right\|^2 \\
& +\alpha_{m_j}\left\|f z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\| \\
& +\alpha_{m_j}\langle f z_0-z_0, x_{m_j}-z_0\rangle,
\end{aligned}$$

which means that

$$\begin{aligned}
\left\|x_{m_j+1}-z_0\right\|^2 \;\leq\; & \frac{1}{1-v}\left\|f z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\| \\
& +\frac{1}{1-v}\langle f z_0-z_0, x_{m_j}-z_0\rangle.
\end{aligned}$$

In view of (IV.20) and (IV.21), we have

$$\lim_{j\to\infty}\left\|x_{m_j+1}-z_0\right\|=0.$$

By (IV.16), we obtain

$$0\leq\left\|x_j-z_0\right\|\leq\left\|x_{m_j+1}-z_0\right\|.$$

Consequently, we get $x_j\to z_0$ as $j\to\infty$. The proof is completed. ■

Remark IV.2. Theorem IV.1 extends, improves and develops Theorem 3.1 of Akashi and Takahashi [14], Theorem 3.1 of Takahashi [13] and Theorem 3.1 of Takahashi [15] in the following aspects.

(i) Theorem IV.1 strengthens the corresponding results in [14] from weak convergence analysis to strong convergence analysis.

(ii) Compared with the corresponding results in [14], [13] and [15], Theorem IV.1 solves the more general and challenging problem for finding a common element of the set of common fixed points of an infinite family of demimetric mappings, the set of common solutions of the variational inclusion problems and the set of solutions of the null point problems in Banach spaces.

(iii) The proof of Theorem IV.1 is based on the novel results (Lemma III.1 to Lemma III.5). That is very different from the proof of Akashi and Takahashi [14], Theorem 3.1 of Takahashi [13] and Theorem 3.1 of Takahashi [15].

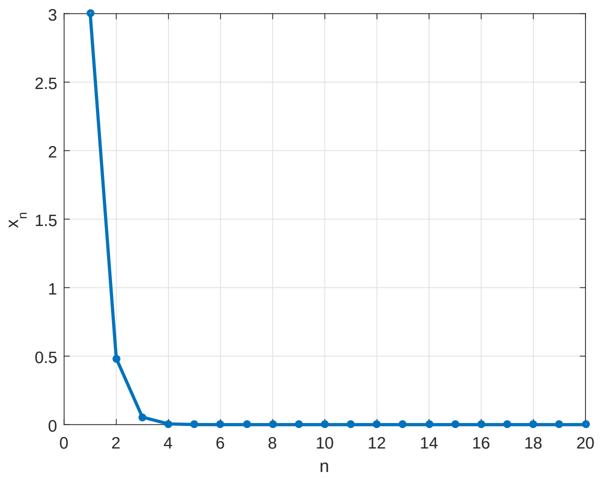

Fig. 1: The convergence of $\{x_n\}$ with initial values $x_1=3$.

(iv) The algorithm (IV.1) is more flexible than the ones given in [14], [13], [15].

Therefore, the new algorithm can be expected to be widely applicable.

Example IV.3. Let $C=H=\mathbb{R}$ with the inner product defined by $\langle x,y\rangle=xy$ for all $x,y\in\mathbb{R}$ and the standard norm $|\cdot|$. Letting $M,B,f:H\to H$ be defined as $Mx=3x$, $Bx=-\frac{3}{2}x$, $fx=\frac{1}{6}x$ for all $x\in H$, we then see $M:H\to 2^H$ be a maximal monotone operator with $dom(M)\subset H$ and B is a bounded linear operator and f be contractive. Let $S:H\to H$ be defined as $Sx=-3x$ for all $x\in H$, $T_i:H\to H$ be defined as $T_ix=-2x$, and $A_i:H\to H$ be defined as $A_ix=6x$ for all $i\in\mathbb{N}$ and $x\in H$. It is easy to check that $\Gamma=\{0\}$. Also, it is easy to check S is $\frac{1}{2}$-demimetric and demiclosed, A_i are $\frac{1}{6}$-ism, and T_i are $\frac{1}{3}$-demimetric and demiclosed for all $i\in\mathbb{N}$.

Let us choose $\alpha_n=\frac{1}{6n}$, $\beta_n=\frac{n+1}{3n}$, $\gamma_n=\frac{4n-3}{6n}$, $\lambda_n=10$, $\tau=\frac{1}{3}$, $\eta_n=\zeta_n=\frac{1}{6}$, $\eta_n=\zeta_n=\frac{1}{6}$, $\delta_i=\sigma_i=\frac{1}{2^i}$ for all $n,i\in\mathbb{N}$. Then α_n, β_n, γ_n,λ_n,τ, η_n,ζ_n,δ_i and σ_i satisfy all the conditions of Theorem IV.1. Therefore iterative scheme (IV.1) becomes

$$x_{n+1}=\frac{12n+17}{180n}x_n,\quad \forall n\in\mathbb{N}.$$

The numerical results are reported in Table I(where e-k denotes 10^{-k}) and Figure IV demonstrate Theorem IV.1.

TABLE I: The values of the sequence $\{x_n\}$

n			x_n		
1–5	3.0000	0.4833	0.0550	0.0054	0.0005
6–10	4.17e-5	3.44e-6	2.76e-7	2.16e-8	1.67e-9
11–15	1.27e-10	9.56e-12	7.13e-13	5.27e-14	3.87e-15
16–20	2.82e-16	2.05e-17	1.48e-18	1.06e-19	7.62e-21

V. An Extension of Our Main Results

By using Theorem IV.1, we have the following strong convergence results for computing the common solution of fixed point problems of nonlinear mappings, variational inclusion problems and null point problems in Banach spaces.

Theorem V.1. *Let E be a smooth, strictly convex, and reflexive Banach space and let J be the duality mapping on E. Let*

H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{\infty}: C \to H$ be an infinite family of μ_i-ism mappings with $\mu = \inf\{\mu_i : i = 1,2,...\}$. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$. Let $f : C \to H$ be a contraction with coefficient $v \in [0,1)$. Let $B : H \to E$ be a bounded linear operator such that $B \neq 0$ and let B^* be the adjoint operator of B. Let $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of 2-generalized hybrid and demiclosed mappings, $S : E \to E$ be $\hat{k}$-demimetric and demiclosed mapping. Assume $\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \bigcap (\bigcap_{i=1}^{\infty}(M + A_i)^{-1}0) \bigcap B^{-1}F(S)$ is nonempty. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} u_n = J_{\eta_n}(I - \lambda_n \sum_{i=1}^{\infty} \sigma_i A_i)x_n, \\ y_n = (1-\zeta_n)u_n + \zeta_n(\sum_{i=1}^{\infty} \delta_i T_i u_n - \tau B^* J(I-S)Bu_n), \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{V.1}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\} \subset (0,1)$ and $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\lambda_n\} \subset (0,+\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$,
(iii) $0 < c \le \eta_n$,
(iv) $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\sum_{i=1}^{\infty} \sigma_i = 1$,
(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\mu$,
(vi) $0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < \min\{\frac{1}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.

Then the sequence $\{x_n\}$ generated by (V.1) strongly converges to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f z_0$.

Proof: Note that the 2-generalized hybrid mapping T with $F(T) \neq \emptyset$ is 0-demimetric. Therefore, Theorem IV.1 implies the conclusion. ∎

Theorem V.2. *Let E be a smooth, strictly convex, and reflexive Banach space and let J be the duality mapping on E. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of μ_i-ism mappings with $\mu = \inf\{\mu_i : i = 1,2,...\}$. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$. Let $f : C \to H$ be a contraction with coefficient $v \in [0,1)$. Let $B : H \to E$ be a bounded linear operator such that $B \neq 0$ and let B^* be the adjoint operator of B. Let $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of k_i-strict pseudo-contractions with $k = \sup\{k_i : i = 1,2,...\} < 1$. Assume $\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \bigcap (\bigcap_{i=1}^{\infty}(M + A_i)^{-1}0) \bigcap B^{-1}F(S)$ is nonempty. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} u_n = J_{\eta_n}(I - \lambda_n \sum_{i=1}^{\infty} \sigma_i A_i)x_n, \\ y_n = (1-\zeta_n)u_n + \zeta_n(\sum_{i=1}^{\infty} \delta_i T_i u_n - \tau B^* J(I-S)Bu_n), \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{V.2}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\} \subset (0,1)$ and $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\lambda_n\} \subset (0,+\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$,
(iii) $0 < c \le \eta_n$,
(iv) $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\sum_{i=1}^{\infty} \sigma_i = 1$,
(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\mu$,
(vi) $0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < \min\{\frac{1-k}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.

Then the sequence $\{x_n\}$ generated by (V.2) strongly converges to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f z_0$.

Proof: Noticing that k-strict pseudocontraction T with $F(T) \neq \emptyset$ is k-demimetric and demiclosed; see [6], we have the desired result due to Theorem IV.1. ∎

Theorem V.3. *Let E be a smooth, strictly convex, and reflexive Banach space and let J be the duality mapping on E. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of μ_i-ism mappings with $\mu = \inf\{\mu_i : i = 1,2,...\}$. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$ and let $f : C \to H$ be a contraction with coefficient $v \in [0,1)$. Let $B : H \to E$ be a bounded linear operator such that $B \neq 0$ and let B^* be the adjoint operator of B. Let $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of nonexpansive mappings, $S : E \to E$ be a $\hat{k}$-demimetric and demiclosed mapping. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} u_n = J_{\eta_n}(I - \lambda_n \sum_{i=1}^{\infty} \sigma_i A_i)x_n, \\ y_n = \zeta_n(\sum_{i=1}^{\infty} \delta_i T_i u_n - \tau B^* J(I-S)Bu_n) \\ \quad +(1-\zeta_n)u_n, \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{V.3}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\} \subset (0,1)$ and $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\lambda_n\} \subset (0,+\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$,
(iii) $0 < c \le \eta_n$,
(iv) $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\sum_{i=1}^{\infty} \sigma_i = 1$,
(v) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\mu$,
(vi) $0 < \liminf_{n\to\infty} \zeta_n \le \limsup_{n\to\infty} \zeta_n < \min\{\frac{1}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.

Assume $\Gamma := F(\sum_{i=1}^{\infty} \delta_i T_i) \bigcap (M + \sum_{i=1}^{\infty} \sigma_i A_i)^{-1}0 \bigcap B^{-1}F(S) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (V.3) strongly converges to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f z_0$.

Theorem V.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{A_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of μ_i-ism mappings with $\mu = \inf\{\mu_i : i = 1,2,...\}$. Let $M : H \to 2^H$ be a maximal monotone operator with $dom(M) \subset C$. Let J_λ be the resolvent of M for $\lambda > 0$ and let $f : C \to H$ be a contraction with coefficient $v \in [0,1)$. Let $B : H \to E$ be a bounded linear operator such that $B \neq 0$ and let B^* be the adjoint operator of B. Let $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of k_i-demimetric and demiclosed mappings with $k = \sup\{k_i : i = 1,2,...\} < 1$, $S : H \to H$ be a $\hat{k}$-demimetric and demiclosed mapping. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} u_n = J_{\eta_n}(I - \lambda_n \sum_{i=1}^{\infty} \sigma_i A_i)x_n, \\ y_n = \zeta_n(\sum_{i=1}^{\infty} \delta_i T_i u_n - \tau B^*(I-S)Bu_n) \\ \quad +(1-\zeta_n)u_n, \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{V.4}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\zeta_n\} \subset (0,1)$ and $\{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\lambda_n\} \subset (0,+\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$,

(iii) $0 < c \leq \eta_n$,
(iv) $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\sum_{i=1}^{\infty} \sigma_i = 1$,
(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\mu$,
(vi) $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < \min\{\frac{1-k}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.

Assume $\Gamma := F(\sum_{i=1}^{\infty} \delta_i T_i) \bigcap (M + \sum_{i=1}^{\infty} \sigma_i A_i)^{-1} 0 \bigcap B^{-1} F(S) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated by* (V.4) *strongly converges to a point* $z_0 \in \Gamma$, *where* $z_0 = P_\Gamma f z_0$.

Theorem V.5. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $\{A_i\}_{i=1}^{\infty} : C \to H$ *be an infinite family of* μ_i*-ism mappings with* $\mu = \inf\{\mu_i : i = 1,2,...\}$. *Let* $M : H \to 2^H$ *be a maximal monotone operator with* $dom(M) \subset C$. *Let* J_λ *be the resolvent of* M *for* $\lambda > 0$ *and let* $f : C \to H$ *be a contraction with coefficient* $v \in [0,1)$. *Let* $B : H \to E$ *be a bounded linear operator such that* $B \neq 0$ *and let* B^* *be the adjoint operator of* B. *Let* $\{T_i\}_{i=1}^{\infty} : C \to H$ *be an infinite family of nonexpansive mappings,* $S : H \to H$ *be a* $\hat{k}$*-demimetric and demiclosed mapping. For* $x_0 \in C$, *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} u_n = J_{\eta_n}(I - \lambda_n \sum_{i=1}^{\infty} \sigma_i A_i) x_n, \\ y_n = \zeta_n(\sum_{i=1}^{\infty} \delta_i T_i u_n - \tau B^*(I-S)Bu_n) \\ \quad +(1-\zeta_n)u_n, \\ x_{n+1} = P_C(\alpha_n f x_n + \beta_n u_n + \gamma_n y_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{V.5}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\zeta_n\} \subset (0,1)$ *and* $\{\delta_n\},\{\sigma_n\},\{\eta_n\},\{\lambda_n\} \subset (0,+\infty)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$,
(iii) $0 < c \leq \eta_n$,
(iv) $\sum_{i=1}^{\infty} \delta_i = 1$ *and* $\sum_{i=1}^{\infty} \sigma_i = 1$,
(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\mu$,
(vi) $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < \min\{\frac{1}{2}, \frac{1-\hat{k}}{2\tau\|B\|^2}\}$.

Assume $\Gamma := F(\sum_{i=1}^{\infty} \delta_i T_i) \bigcap (M + \sum_{i=1}^{\infty} \sigma_i A_i)^{-1} 0 \bigcap B^{-1} F(S) \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated by* (V.5) *strongly converges to a point* $z_0 \in \Gamma$, *where* $z_0 = P_\Gamma f z_0$.

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