A Couple Mathematical Models of the Water Quality Measurement in a Stream using Upwind Implicit Methods

N. Pochai, P. Phosri

Abstract— **Two numerical simulations are being used to quality of the water in a non-uniform flow stream. The first model is a hydrodynamic model that uses the Crank-Nicolson formula to provide velocity profile and level of water. The second phase is a dispersion model, where the governing function uses advection-dispersion-reaction equations to provide the concentrations of contaminants. The first and second models are described as one-dimensional equations. The first determined flow velocity profile of the hydrodynamic model shall be the input into the dispersion model at each phase. The finite difference methods are proposed to solve the dispersion model a four points explicit upwind schemes, a third order Crank-Nicolson schemes, and the four points implicit methods, which give the approximated pollutant concentrations. Finally, we present a numerical simulation of all schemes, so as to illustrate their applicability to real-world problems. The proposed topic is related and economic to be used in real-world challenges relatively low cost of the program and the straight-forwardness of its implementations. It is also good to achieve suitable places and easier timeframes for different discharge locations.**

Index Terms— **Finite differences, water quality, one-dimensional, hydrodynamic model, advection-dispersion-reaction.**

I. Introduction

WATER pollution is a major problem; everyone should be aware of this problem. Monitoring water quality can be achieved by field measurement and calculation of data of water current in each position and time. Another way is a mathematical simulation. The model of water quality in a non-uniform flow stream shall provide velocity and elevation. Modeling used in a non-uniform flow is a hydrodynamic model of a one-dimensional (1D) shallow water structure and a dispersion model of an advection-dispersion equation (ADE).

There are several computational methods used to solve the continuous control of water pollutants in order to achieve these models. [1] introduced a hydrodynamic model and a dispersion model with a finite element approach to address minimum costs. [2], finite element methods were used in a hydrodynamic model and a dispersion model to simulate pollution in the Bay of Santander. In 2009, [26] presented a three-dimensional numerical model with the turbulent (RSM) and a non-uniform grid system Reynolds Stress Model was used to examine the effects of a double tandem obstacle cubic on the development of the incoming flow. The results obtained configuration make possible the description of the dynamic and masses features and the determination of the velocity ratio effect on the pollutant distribution. In 2010, [27] studied Padé schemes for the numerical solution of two-dimensional diffusion equations with nonlocal boundary conditions. The numerical results show that these Padé schemes are efficient and provide very accurate results. In 2015, [28] mathematical model is used to air flow and pollutant dispersion in an Urban Street Canyon with Steady Wind Boundary Conditions (SWBC) and developed the Fluctuating Wind Boundary Conditions (FWBC). Three dimensional (3D) numerical simulations are performed using Large Eddy Simulation (LES), the results of FWBC produces more realistic results when compared to the frequently employed SWBC. In 2018, [29] proposed numerical developments on the coal combustion and gasification by using CFD (Computational Fluid Dynamics) techniques with an Eddy Break Up (EBU) model. The results of the simulation show that the H2 and CH4 products from the gasification are significantly higher than those from the combustion. This particle model can thus be considered for further investigation for various coal combustion and gasification applications. [3-7] suggested computational techniques for the resolution of a uniform flow of water quality model, in particular a one-dimensional advection-dispersion-reaction equation (1D-ADRE).

Many non-uniform flow models require information related to the velocity of the flow at any point and at any time

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in the domain. Hydrodynamic models give field, flow velocity and level of the stream. In [1, 9-13], the hydrodynamics model was used to approximate the velocity of water currents, and the ADE to simulate the accumulation of toxins in bays, uniform reservoirs, and rivers. In all computational approaches, formal finite difference methods and tacit finite difference methods are often found in one-dimensional domains such as longitudinal stream systems [14, 15].

Computational calculations are being used to predict water quality using an erratic water flow process. The first is a hydrodynamic model that velocity and water height, and the second is a dispersion model that produces concentrations of contaminants. The two models formulated into 1D equations. In the hydrodynamic model, the traditional Crank-Nicolson system is also used. In every point, the flow velocity values computed from the hydrodynamic model are entered in the dispersion model as field info presented by [10-13, 16-17].

In [11] a computation method was used to approach the uneven water quality flow with 1D-ADRE using a completely implicit model: the Crank-Nicolson process for the hydrodynamic model and the BTCS for the distribution model.

The several studies of the finite difference approach found computational precision and consistency. [18] developed a mathematical dispersion by adding an upstream interpolation process, that is Quadratic Upstream Interpolation Convective Kinematics (QUICK), for a 1D unsteady flow. [4] proposed simple revisions to these schemes that make them more accurate without significant loss of computation efficiency. These people have used a third-order upwind scheme for the ADE, but the distinction [19] is used for convection terms, [20] is used for convective terms for shallow water momentum equations, and [21] is used for ADE using simple simulation spreadsheets.

The data on the water flow velocity obtained from the hydrodynamic model are used for the dispersion model with ADRE, which provides the concentration of the pollutant area. Consider the friction forces, attributable to the push on the side of the water way. The conceptual solvent for the end system of the field, that also makes sure the stability of the approximate way to solve, is given in [10-11, 16-17].

In this analysis, the hydrodynamic model and the dispersion model were used to describe the content of water flow and water contaminant. So many numerical simulations are being used to approximate the water quality of a crisis. The stream has a 1D gap, as seen in Figure 1. According to that the 1D shallow water equation and the ADRE are accessible by measuring the formula over the distance, throwing away the term due to the Coriolis force. The first concept is the hydrodynamic model, which uses the Crank-Nicolson system to provide a region of velocity and altitude of water. In each stage, the flow velocity fields calculated from the first model are contributions to the model 2 as field data. Next step is the dispersion model, which specifies the distribution of pollutants. We use a four points explicit upwind schemes, a third order Crank-Nicolson schemes, and a four points implicit schemes to approximate the concentration from the dispersion models.

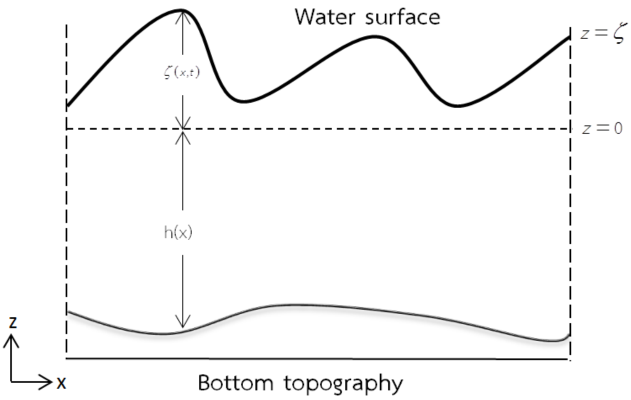

Fig. 1. The shallow water system

II. Model Formulation

A. The Hydrodynamic Model

Continuity and momentum formulae are shown by the hydrodynamic motion of the forces described in the 1D shallow water equation by removing the distribution and by removing the stress and flow conditions [1, 10-14, 22]; we have 1D shallow water equations;

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}\left[(h+\zeta)u\right]=0, \quad (1)$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0, \quad (2)$$

where x is the longitudinal distance down the river (m), t is time (s), $h(x)$ is the depth measured from the average water to the reservoir bed (m), $\zeta(x,t)$ is the elevation from the mean water level to the temporary water surface. (m/s), and $u(x,t)$ are the velocity components (m/s), for all $x\in[0,L]$. We assume that h is a constant and $\zeta \ll h$. Then (1) and (2) become;

$$\frac{\partial \zeta}{\partial t}+h\frac{\partial}{\partial x}=0, \quad (3)$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}=0. \quad (4)$$

We will transform (3) and (4) into non-dimensional [23], by letting $U=u/\sqrt{gh}$, $Y=y/l$, $X=x/l$, $Z=\zeta/h$ and $T=t\sqrt{gh}/l$. Substituting into (3) and (4) leads to;

$$\frac{\partial Z}{\partial t}+\frac{\partial U}{\partial X}=0, \quad (5)$$

$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=0. \quad (6)$$

[10 – 11, 16], introduced a damping term $-U$ into (6). We now introduce a damping term $-KU$ (6) to represent frictional forces due to the drag of sides of the stream, thus;

$$\frac{\partial Z}{\partial T}+\frac{\partial U}{\partial X}=0, \quad (7)$$

$$\frac{\partial U}{\partial T}+\frac{\partial Z}{\partial X}=-KU, \quad (8)$$

where $0<K<1$, with the initial conditions at $t=0$ and $0\le X\le 1$ being specified: $Z=0$ and $U=0$. The boundary conditions for $t>0$ are specified: $Z=e^{it}$ at $X=0$ and $\frac{\partial Z}{\partial X}=0$ at $X=1$. The system of (7) and (8) is called the damped hydrodynamic equations.

B. A Non-dimensional form of the Damped Hydrodynamic Model

In order to solve damped equation in $[0,1]\times[0,T]$, for favorable using u,d for U and Z, respectively;

$$\frac{\partial u}{\partial t}+\frac{\partial d}{\partial x}=-ku, \quad (9)$$

$$\frac{\partial d}{\partial t}+\frac{\partial u}{\partial x}=0, \quad (10)$$

with the initial conditions $u=0,d=0$ at $t=0$, and the boundary conditions $d(0,t)=f(t)$, and $\frac{\partial d}{\partial x}=0$ at $x=1$.

C. Dispersion Model

The water pollution problem can be formulated in 1D ADRE. A efficient and easy by measuring the formula over the distance is seen in [3-5, 7, 11-13] as;

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=D\frac{\partial^2 C}{\partial x^2}-KC, \quad (11)$$

where $C(x,t)$ is the concentration averaged in depth at the point x and at time t, D is the diffusion coefficient, K is the mass decay rate, and $u(x,t)$ is the velocity component, for all $x\in[0,L]$. We will consider conditions in this model following. The initial condition $C(x,0)=0$ at $t=0$ for all $x>0$. The boundary conditions $C(0,t)=C_0$ at $x=0$, and $\frac{\partial C}{\partial X}=C_0$ at $x=1$ where C_0 is the constant.

III. Crank-Nicolson Method of the Hydrodynamic Model

The hydrodynamic model allows the velocity distribution and the level of the stream. Then join the outputs of the first model in the dispersion system that provides the concentrations of the contaminant region. The mathematical equations of [10] will be pursued in this segment. In order to calculate the fluid flow and the water level of (9) and (10), we make changes to the parameters $v=e^{kt}u$ and reflect them to (9) and (10). Then we got it,

$$\frac{\partial v}{\partial t}+e^{kt}\frac{\partial d}{\partial x}=0, \quad (12)$$

$$\frac{\partial d}{\partial t}+e^{-kt}\frac{\partial v}{\partial x}=0. \quad (13)$$

From (12) and (13) can be written in the matrix form

$$\begin{pmatrix} v \\ d \end{pmatrix}_t+\begin{bmatrix} 0 & e^{kt} \\ e^{-kt} & 0 \end{bmatrix}\begin{pmatrix} v \\ d \end{pmatrix}_x=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (14)$$

That is

$$U_t+AU_x=\bar{0}, \quad (15)$$

where

$$A=\begin{bmatrix} 0 & e^{kt} \\ e^{-kt} & 0 \end{bmatrix}, \quad (16)$$

$$U=\begin{pmatrix} v \\ d \end{pmatrix},\begin{pmatrix} v \\ d \end{pmatrix}_t=\begin{pmatrix} \frac{\partial v}{\partial t} \\ \frac{\partial d}{\partial t} \end{pmatrix},\begin{pmatrix} v \\ d \end{pmatrix}_x=\begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial d}{\partial x} \end{pmatrix}, \quad (17)$$

with initial condition $d,v=0$ at $t=0$. The left boundary conditions for $x=0,t>0$ is specified: $d(0,t)=\sin t$ and $\frac{\partial v}{\partial x}=-e^{kt}\cos t$, and the right boundary conditions for $x=1,t>0$ is specified: $\frac{\partial d}{\partial x}=0$ and $v(0,t)=0$. We now discretize (15) by dividing the interval $[0,1]$ into M subintervals, such that $M\Delta x=1$, and the interval $[0,T]$ into N subintervals, such that $N\Delta t=T$. We can then approximate $d(x_i,t_n)$ by d_i^n, value of the difference approximation of $d(x,t)$ at point $x=i\Delta x$ and $t=n\Delta t$, where $0\le i\le M$ and $0\le n\le N$, and similarly defined for v_i^n and U_i^n. The grid points (x_n,t_n) are defined by $x_i=i\Delta x$ for all $i=0,1,2,\ldots,M$ and $t_n=n\Delta t$ for all $n=0,1,2,\ldots,N$, in which M and N are positive integers. Using the Crank-Nicolson method [23] with (15), the following finite difference equation can be obtained;

$$\left[I-\frac{1}{4}\kappa A[(\Delta_x+\nabla_x)]U_i^{n+1}\right]=\left[I+\frac{1}{4}\kappa A[(\Delta_x+\nabla_x)]U_i^n\right], \quad (18)$$

where

$$U_i^n=\begin{pmatrix} v_i^n \\ d_i^n \end{pmatrix}, \Delta x U_i^n=U_{i+1}^n-U_i^n, \nabla x U_i^n=U_i^n-U_{i-1}^n, \quad (19)$$

and I is the unit matrix of order 2, and $\kappa=\frac{\Delta t}{\Delta x}$. Applying the initial and boundary conditions given in (12) - (13), the general form can be obtained;

$$G^{n+1}\bar{U}^{n+1}=E^n\bar{U}^n+F^n, \quad (20)$$

where

$$G^{n+1}=\begin{bmatrix} 1 & 0 & 0 & -\frac{\kappa}{4}a_1^{n+1} & 0 & 0 & \\ \frac{\kappa}{4}a_2^{n+1} & 1 & -\frac{\kappa}{4}a_2^{n+1} & 0 & 0 & 0 & \\ 0 & \frac{\kappa}{4}a_1^{n+1} & 1 & 0 & 0 & -\frac{\kappa}{4}a_1^{n+1} & \\ \frac{\kappa}{4}a_2^{n+1} & 0 & 0 & 1 & -\frac{\kappa}{4}a_2^{n+1} & 0 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 0 & 0 & 0 & \frac{\kappa}{4}a_1^{n+1} & 1 & -\frac{\kappa}{4}a_1^{n+1} \\ & 0 & 0 & \frac{\kappa}{4}a_2^{n+1} & 0 & 0 & 1 \end{bmatrix}, \quad (21)$$

$$E^n=\begin{bmatrix} 1 & 0 & 0 & -\frac{\kappa}{4}a_1^n & 0 & 0 & \\ -\frac{\kappa}{4}a_2^n & 1 & \frac{\kappa}{4}a_2^n & 0 & 0 & 0 & \\ 0 & -\frac{\kappa}{4}a_1^n & 1 & 0 & 0 & \frac{\kappa}{4}a_1^n & \\ -\frac{\kappa}{4}a_2^n & 0 & 0 & 1 & \frac{\kappa}{4}a_2^n & 0 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 0 & 0 & 0 & -\frac{\kappa}{4}a_1^n & 1 & \frac{\kappa}{4}a_1^n \\ & 0 & 0 & -\frac{\kappa}{4}a_2^n & 0 & 0 & 1 \end{bmatrix}, \quad (22)$$

$$\bar{U}^n=\begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_3^{n+1} \end{pmatrix},$$

$$F^n=\begin{bmatrix} -\frac{\kappa}{4}a_1^{n+1}\sin(t_{n+1})-\frac{\kappa}{4}a_1^n\sin(t_n) \\ -\frac{\kappa}{4}a_2^{n+1}\Delta x e^{-t_{n+1}}\cos(t_{n+1})-\frac{\kappa}{4}a_2^n\Delta x e^{-t_n}\cos(t_n) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad (23)$$

where $a_1^n=e^{kt_n}$, $a_2^n=e^{-kt_n}$ and $t_n=n\Delta t$ for all $n=0,1,2,\cdots,N$.

IV. Finite Difference Techniques for the Model of Dispersion

In this section, we consider the computational techniques in [21] for the ADRE that is used to calculate the concentration of the pollutant. We can approximate $C(x_i,t_n)$ by C_i^n, the value of the difference approximation of $C(x,t)$ at point $x=i\Delta x$ and $t=n\Delta t$, where $1\le i\le M$, and $0\le n\le N$. The grid point (x_i,t_n) is defined by $x_i=i\Delta x$ for all $i=0,1,2,\ldots,M$, and $t_i=n\Delta t$ for all $n=0,1,2,\ldots,N$ in which M and N are positive integers.

A Third-Order Upwind Schemes

We consider the numerical techniques [21], using the forward times central space scheme for the times derivatives and the central difference for the second times derivatives, respectively, as follows;

$$\frac{\partial C}{\partial t}\cong\frac{C_i^{n+1}-C_i^n}{\Delta t}, \quad (24)$$

$$\frac{\partial^2 C}{\partial x^2}\cong\frac{1}{(\Delta x)^2}\left[C_{i-1}^n-2C_i^n+C_{i+1}^n\right], \quad (25)$$

using scheme of the third order for the spatial derivatives of ADRE, as given in [15]. At boundary points, a four-point upwind equation may well be designed in such a way that whether a point to the left or the right is assumed to be a finite difference estimation. Next, we approximate the spatial derivative for a four points explicit upwind schemes, a third order Crank-Nicolson schemes, and a four points implicit schemes at left boundary conditions, respectively, as follows;

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[-11C_i^n+18C_{i+1}^n-9C_{i+2}^n+2C_{i+3}^n\right], \quad (26)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{12\Delta x}\begin{bmatrix}-11C_i^{n+1}+18C_{i+1}^{n+1}-9C_{i+2}^{n+2}+2C_{i+3}^{n+1} \\ -11C_i^n+18C_{i+1}^n-9C_{i+2}^n+2C_{i+3}^n\end{bmatrix}, \quad (27)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[-11C_i^{n+1}+18C_{i+1}^{n+1}-9C_{i+2}^{n+2}+2C_{i+3}^{n+1}\right]. \quad (28)$$

At the interior nodes, as follows, respectively,

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[C_{i-2}^n-6C_{i-1}^n+3C_i^n+2C_{i+1}^n\right], \quad (29)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{12\Delta x}\begin{bmatrix}C_{i-2}^{n+1}-6C_{i-1}^{n+1}+3C_i^{n+1}+2C_{i+1}^{n+1} \\ +C_{i-2}^n-6C_{i-1}^n+3C_i^n+2C_{i+1}^n\end{bmatrix}, \quad (30)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[C_{i-2}^{n+1}-6C_{i-1}^{n+1}+3C_i^{n+1}+2C_{i+1}^{n+1}\right]. \quad (31)$$

At right boundary conditions, as follows, respectively,

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[-2C_{i-3}^n+9C_{i-2}^n-18C_{i-1}^n+11C_i^n\right], \quad (32)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{12\Delta x}\begin{bmatrix}-2C_{i-3}^{n+1}+9C_{i-2}^{n+1}-18C_{i-1}^{n+1}+11C_i^{n+1} \\ -2C_{i-3}^n+9C_{i-2}^n-18C_{i-1}^n+11C_i^n\end{bmatrix}, \quad (33)$$

$$\frac{\partial C}{\partial x}\cong\frac{1}{6\Delta x}\left[-2C_{i-3}^{n+1}+9C_{i-2}^{n+1}-18C_{i-1}^{n+1}+11C_i^{n+1}\right]. \quad (34)$$

A. A Four points Explicit Upwind Method

A four points explicit upwind schemes can be obtained so that the technique does not require the systems of linear equations. This methodology is a cost-effective computer implementation. Now, we're taking the explicit finite difference technique [21] into (11).

At the left boundary, substituting (24) – (25) into (11), then we obtain that;

$$\frac{C_i^{n+1}-C_i^n}{\Delta t}+\frac{u_i^n}{6\Delta x}\left[-11C_i^n+18C_{i+1}^n-9C_{i+2}^n+2C_{i+3}^n\right] =\frac{D}{(\Delta x)^2}\left[C_{i-1}^n-2C_i^n+C_{i+1}^n\right]-kC_i^n, \quad (35)$$

where $C\cong C_i^n$, $u\cong\tilde{u}_i^n$ and $\tilde{u}_i^n$ are derived using the Crank-Nicolson method in a hydrodynamic model (9), for all $1\le i\le M$ and $0\le n\le N$. Let $\beta=D\frac{\Delta t}{(\Delta x)^2}$ and $\beta_i^n=u_i^n\frac{\Delta t}{6\Delta x}$,

so (35) becomes;

$$C_i^{n+1}=\beta C_{i-1}^n+\left[1-2\beta-(\Delta t)k+11\beta_i^n\right]C_i^n +\left[\beta-18\beta_i^n\right]C_{i+1}^n-9\beta_i^nC_{i+2}^n-2\beta_i^nC_{i+3}^n. \quad (36)$$

For $i=0$, replace the known value of the left boundary by setting $C_{i-1}^n=C_0^n$ in (36) on the right-hand side. We obtain;

$$C_0^{n+1}=\left[1-\beta-(\Delta t)k+11\beta_0^n\right]C_0^n+\left[\beta-18\beta_0^n\right]C_1^n -9\beta_0^nC_2^n-2\beta_0^nC_3^n. \quad (37)$$

At interior, substituting (24) – (25) and (29) into (11). Then, we obtain;

$$C_i^{n+1}=-\beta_i^nC_{i-2}^n+\left[\beta+6\beta_i^n\right]C_{i-1}^n +\left[1-2\beta-3\beta_i^n-(\Delta t)\mathrm{k}\right]C_i^n+\left[\beta-2\beta_i^n\right]C_{i+1}^n. \quad (38)$$

At the right boundary, substituting (24) – (25) and (32) into (11). Then we obtain;

$$C_i^{n+1}=2\beta C_{i-3}^n+\left[\beta-9\beta_i^n\right]C_{i-2}^n+18\beta_i^nC_{i-1}^n +\left[1-2\beta-(\Delta t)k-11\beta_i^n\right]C_i^n+\beta_i^nC_{i+1}^n. \quad (39)$$

For $i = M$, the unknown value of the right boundary by boundary conditions, we can set $C_{M+1}^n = C_M^n$ in (39) and, by rearrangement, we can obtain;

$$\begin{aligned} C_M^{n+1} &= 2\beta_M^n C_{M-3}^n + \left[\beta - 9\beta_M^n\right] C_{M-2}^n \\ &+ 18\beta_M^n C_{M-1}^n + \left[1 - \beta - (\Delta t)k + 11\beta_M^n\right] C_M^n. \end{aligned} \tag{40}$$

B. A Third Order of Crank-Nicolson Method

Find the Crank-Nicolson process for the ADRE. Take the Crank-Nicolson scheme in [21] in (11). At the left boundary, substituting (24) – (25) and (27) into (11), then we have

$$\begin{aligned} \frac{C_i^{n+1} - C_i^n}{\Delta t} + \frac{u_i^n}{12\Delta x}\begin{bmatrix} -11C_i^{n+1} + 18C_{i+1}^{n+1} - 9C_{i+2}^{n+1} + 2C_{i+3}^{n+1} \\ -11C_i^n + 18C_{i+1}^n - 9C_{i+2}^n + 2C_{i+3}^n \end{bmatrix} \\ = \frac{D}{(\Delta x)^2}\left[C_{i-1}^n - 2C_i^n + C_{i+1}^n\right] - kC_i^n. \end{aligned} \tag{41}$$

where $C \cong C_i^n$, $u \cong \tilde{u}_i^n$ and $\tilde{u}_i^n$ are derived using the Crank-Nicolson method in a hydrodynamic model (9), for all $1 \le i \le M$ and $0 \le n \le N$. Let $\beta = D\dfrac{\Delta t}{(\Delta x)^2}$ and $\alpha_i^n = u_i^n \dfrac{\Delta t}{12\Delta x}$, so (41) becomes,

$$\begin{aligned} &\left[1 - 11\alpha_i^n\right] C_i^{n+1} + 18\alpha_i^n C_{i+1}^{n+1} - 9\alpha_i^n C_{i+2}^{n+1} + 2\alpha_i^n C_{i+3}^{n+1} \\ &= \beta C_{i-1}^n + \left[1 - 2\beta - (\Delta t)k + 11\alpha_i^n\right] C_i^n \\ &+ \left[\beta - 18\alpha_i^n\right] C_{i+1}^n + 9\alpha_i^n C_{i+2}^n - 2\alpha_i^n C_{i+3}^n. \end{aligned} \tag{42}$$

For the left boundary condition, $i = 0$, the known value on the left boundary are approximated $C_{i-1}^n = C_0^n$ in (42), we can see that;

$$\begin{aligned} &\left[1 - 11\alpha_0^n\right] C_0^{n+1} + 18\alpha_0^n C_1^{n+1} - 9\alpha_0^n C_2^{n+1} + 2\alpha_0^n C_3^{n+1} \\ &= \beta C_{-1}^n + \left[1 - 2\beta - (\Delta t)k + 11\alpha_0^n\right] C_0^n \\ &+ \left[\beta - 18\alpha_0^n\right] C_1^n + 9\alpha_0^n C_2^n - 2\alpha_0^n C_3^n. \end{aligned} \tag{43}$$

At interior, substituting (24) – (25) and (30) into (11). Then we obtain;

$$\begin{aligned} &\alpha_i^n C_{i-2}^{n+1} - 6\alpha_i^n C_{i-1}^{n+1} + \left[1 - 3\alpha_i^n\right] C_i^{n+1} + 2\alpha_i^n C_{i+1}^{n+1} \\ &= \left[\beta - \alpha_i^n\right] C_{i-2}^n + 6\alpha_i^n C_{i-1}^n \\ &+ \left[1 - 2\beta - (\Delta t)k - 3\alpha_i^n\right] C_i^n + \left[\beta - 2\alpha_i^n\right] C_{i+1}^n. \end{aligned} \tag{44}$$

Similarly, the right boundary condition**,** substituting (24) – (25) and (33) into (11). Then, we get;

$$\begin{aligned} &-2\alpha_i^n C_{i-3}^{n+1} + 9\alpha_i^n C_{i-2}^{n+1} - 18\alpha_i^n C_{i-1}^{n+1} + \left[1 + 11\alpha_i^n\right] C_i^{n+1} \\ &= 2\alpha_i^n C_{i-3}^n - 9\alpha_i^n C_{i-2}^n + \beta C_{i-1}^{n+1} \\ &+ \left[1 - 2\beta - (\Delta t)k - 11\alpha_i^n\right] C_i^n + \left[\beta + 18\alpha_i^n\right] C_{i+1}^n. \end{aligned} \tag{45}$$

For $i = M$, the known value on the right boundary condition are approximated $C_{M+1}^n = C_M^n$ in (45), and by rearranging, we obtain;

$$\begin{aligned} &-2\alpha_M^n C_{M-3}^{n+1} + 9\alpha_M^n C_{M-2}^{n+1} - 18\alpha_M^n C_{M-1}^{n+1} + \left[1 + 11\alpha_M^n\right] C_M^{n+1} \\ &= 2\alpha_M^n C_{M-3}^n - 9\alpha_M^n C_{M-2}^n + \beta C_{M-1}^{n+1} \\ &+ \left[1 - \beta - (\Delta t)k + 7\alpha_M^n\right] C_M^n. \end{aligned} \tag{46}$$

C. A Four points Implicit Upwind Method

Consider the implicit method for the ADRE. Take the implicit scheme from [21] in (11).

For the left boundary, substituting (24) – (25) and (28) into (11). Then, we have;

$$\begin{aligned} \frac{C_i^{n+1} - C_i^n}{\Delta t} + \frac{u_i^n}{6\Delta x}\left[-11C_i^{n+1} + 18C_{i+1}^{n+1} - 9C_{i+2}^{n+1} + 2C_{i+3}^{n+1}\right] \\ = \frac{D}{(\Delta x)^2}\left[C_{i-1}^n - 2C_i^n + C_{i+1}^n\right] - kC_i^n. \end{aligned} \tag{47}$$

where $C \cong C_i^n$, $u \cong \tilde{u}_i^n$ and $\tilde{u}_i^n$ are derived using the Crank-Nicolson method in a hydrodynamic model (9), for all $1 \le i \le M$, and $0 \le n \le N$. Let $\beta = D\dfrac{\Delta t}{(\Delta x)^2}$, and $\beta_i^n = u_i^n \dfrac{\Delta t}{12\Delta x}$, so (47) becomes;

$$\begin{aligned} &\left[1 - 11\beta_i^n\right] C_i^{n+1} + 18\beta_i^n C_{i+1}^{n+1} - 9\beta_i^n C_{i+2}^{n+1} + 2\beta_i^n C_{i+3}^{n+1} \\ &= \beta C_{i-1}^n + \left[1 - 2\beta - (\Delta t)k\right] C_i^n + \beta C_{i+1}^n. \end{aligned} \tag{48}$$

For $i = 0$, by substituting the approximate unknown value of the left boundary, we can get $C_{i-1}^n = C_0^n$ in (48) and by rearranging it;

$$\begin{aligned} &\left[1 - 11\beta_0^n\right] C_0^{n+1} + 18\beta_0^n C_1^{n+1} - 9\beta_0^n C_2^{n+1} + 2\beta_0^n C_3^{n+1} \\ &= \left[1 - \beta - (\Delta t)k\right] C_i^n + \beta C_1^n. \end{aligned} \tag{49}$$

At interior, substituting (24-25) and (31) into (11). Then we obtain;

$$\begin{aligned} &\beta_i^n C_{i-2}^{n+1} - 6\beta_i^n C_{i-1}^{n+1} + \left[1 - 3\beta_i^n\right] C_i^{n+1} + 2\beta_i^n C_{i+1}^{n+1} \\ &= \beta C_{i-1}^n + \left[1 - 2\beta - (\Delta t)k\right] C_i^n + \beta C_{i+1}^n. \end{aligned} \tag{50}$$

Similarly, the right boundary condition**,** substituting (24) – (25) and (34) into (11). Then we get;

$$\begin{aligned} &-2\beta_i^n C_{i-3}^{n+1} + 9\beta_i^n C_{i-2}^{n+1} - 18\beta_i^n C_{i-1}^{n+1} + \left[1 + 11\beta_i^n\right] C_i^{n+1} \\ &= \beta C_{i-1}^{n+1} + \left[1 - 2\beta - (\Delta t)k\right] C_i^n + \beta C_{i+1}^n. \end{aligned} \tag{51}$$

For $i = M$, the known value on the right boundary condition are approximated $C_{M+1}^n = C_M^n$ in (51), and by rearranging, we obtain;

$$\begin{aligned} &-2\beta_M^n C_{M-3}^{n+1} + 9\beta_M^n C_{M-2}^{n+1} - 18\beta_M^n C_{M-1}^{n+1} + \left[1 + 11\beta_M^n\right] C_M^{n+1} \\ &= \beta C_{M-1}^{n+1} + \left[1 - \beta - (\Delta t)k\right] C_M^n. \end{aligned} \tag{52}$$

V. Numerical Experiments

Assume a calculation of the concentrations of contaminants $C(x,t)$ (Kg/m^3) in a non-uniform flow stream at a t time (sec) is regarded. A stream is according with a length, 1.0 km of overall length. There is indeed a plant which discharges the sewage into the river and the intenseness of the contaminant at the disposal site is presumed to be 3 cases;

case 1; $C(0,t) = 0.4 + \sin(t)$ Kg/m^3 at $x = 0$ for all $t > 0$, and $\dfrac{\partial c}{\partial x}(1,t) = -0.5$ Kg/m^3 at $x = 1$ for all $t > 0$, and $C(x,0) = 0$ Kg/m^3 at $t = 0$.

case 2; $C(0,t)=0.4+\sin(t)$ Kg/m^3 at $x=0$ for all $t>0$, and $\frac{\partial c}{\partial x}(1,t)=0$ Kg/m^3 at $x=1$ for all $t>0$, and $C(x,0)=0$ Kg/m^3 at $t=0$

case 3; $C(0,t)=0.4+\sin(t)$ Kg/m^3 at $x=0$ for all $t>0$, and $\frac{\partial c}{\partial x}(1,t)=1$ Kg/m^3 at $x=1$ for all $t>0$, and $C(x,0)=0$ Kg/m^3 at $t=0$.

The velocity of water at the discharge point can be described as a function $d(0,t)=f(t)=0.4+\sin(t)$ for all $t>0$, and the elevation is not changed at $x=1$ km. In the analysis conducted in this study, meshes the stream, using $\Delta x=0.02$, and time increment with $\Delta t=0.002$. Using (20); for $k=0$, we obtained the elevation of water $d(x,t)$ in Table 1 and Fig. 2. For $k=-0.03$, $k=0$, and $k=0.02$, can be obtained the velocity of water $u(x,t)$ in Table 2-4, and Fig. 3-5. The comparison of elevation and velocity at $k=-0.03$, $k=0$, and $k=0.02$ are show in Fig. 6-7, respectively. The physical parameter of the stream system is the coefficient of diffusion $D=0.02$ m^2/s.

First, the estimated water velocity can be loaded into implicit and explicit methods such as four point explicit upwind methods, third order Crank-Nicolson methods, and four point implicit methods (37, 38, 40), (43, 44, 46) and (49, 50, 52) respectively. The approximation of pollutant concentration C of all schemes at $k=-1$, $k=0$, and $k=0.02$ are shown in tables 5-7, respectively. The comparison of approximated pollutant concentrations, for the above 3 cases at $k=-1$ of a four points explicit upwind methods, a third order Crank-Nicolson methods, and a four points implicit methods are show in Fig. 8-10, respectively. Similar, for $k=0$, $k=1$ are shown in Figs. 11-13, and 14-16, respectively, for the above 3 cases.

Table. I. The elevation of water flow $d(x,t)\ m/s$ where $\Delta x=0.02$, $\Delta t=0.002$ and $k=0$

t(sec), x (km)	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
4	0.611858	0.069780	-0.399759	-1.262317	-1.802668	-2.036578	0.611858	0.069780	-0.399759	-1.262317	-1.802668
8	0.998543	-0.003519	-0.286033	-0.784644	-0.763727	-0.619346	0.998543	-0.003519	-0.286033	-0.784644	-0.763727
12	0.693525	0.023939	-0.370754	-0.668179	-0.898352	-0.850130	0.693525	0.023939	-0.370754	-0.668179	-0.898352
16	0.091907	0.032294	-0.126321	-0.133233	-0.112515	-0.253000	0.091907	0.032294	-0.126321	-0.133233	-0.112515
20	0.813674	0.553929	0.094939	-0.305544	-0.572990	-0.757749	0.813674	0.553929	0.094939	-0.305544	-0.572990

Table. II. The water of velocity flow $u(x,t)\ m/s$ where $\Delta x=0.02$, $\Delta t=0.002$ and $k=-0.03$

t(sec), x (km)	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
4	2.610971	2.583252	2.390045	2.227666	1.586067	0.998144	2.610971	2.583252	2.390045	2.227666	1.586067
8	1.042248	1.056271	1.121407	1.145663	0.984761	0.678741	1.042248	1.056271	1.121407	1.145663	0.984761
12	0.080940	0.068841	-0.078776	-0.154091	0.108119	0.225758	0.080940	0.068841	-0.078776	-0.154091	0.108119
16	2.153967	2.325972	2.034130	1.777823	1.357201	0.964549	2.153967	2.325972	2.034130	1.777823	1.357201
20	1.564799	1.426789	1.206481	1.006457	0.656912	0.454190	1.564799	1.426789	1.206481	1.006457	0.656912

Table. III. The water of velocity flow $u(x,t)\ m/s$ where $\Delta x=0.02$, $\Delta t=0.002$ and $k=0$

t(sec), x (km)	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
4	2.783345	2.758803	2.555691	2.383197	1.683867	1.040888	2.783345	2.758803	2.555691	2.383197	1.683867
8	1.027512	1.043821	1.131175	1.184130	1.043308	0.723322	1.027512	1.043821	1.131175	1.184130	1.043308
12	-0.303511	-0.334005	-0.507012	-0.573429	-0.145513	0.077422	-0.303511	-0.334005	-0.507012	-0.573429	-0.145513
16	2.860197	3.145580	2.767857	2.406631	1.797891	1.207349	2.860197	3.145580	2.767857	2.406631	1.797891
20	1.952418	1.766613	1.437037	1.124076	0.655525	0.378875	1.952418	1.766613	1.437037	1.124076	0.655525

Table. IV. The water of velocity flow $u(x,t)\ m/s$ where $\Delta x=0.02$, $\Delta t=0.002$ and $k=0.02$

t(sec), x (km)	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
4	2.905499	2.883536	2.673649	2.494080	1.753150	1.070405	2.905499	2.883536	2.673649	2.494080	1.753150
8	1.005656	1.023729	1.129304	1.206807	1.086476	0.760053	1.005656	1.023729	1.129304	1.206807	1.086476
12	-0.660111	-0.708516	-0.900420	-0.955795	-0.378749	-0.059094	-0.660111	-0.708516	-0.900420	-0.955795	-0.378749
16	3.530340	3.926561	3.473736	3.013550	2.222936	1.440886	3.530340	3.926561	3.473736	3.013550	2.222936
20	2.297546	2.071018	1.632631	1.205647	0.636155	0.295197	2.297546	2.071018	1.632631	1.205647	0.636155

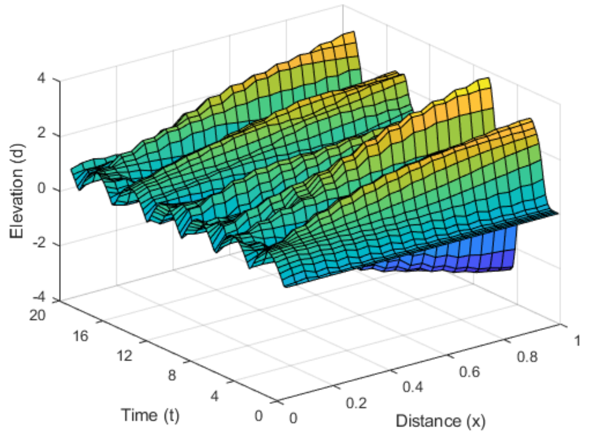

Fig. 2. The elevation of water flow $d(x,t)\ m/s$ at $k=0$ when after pass 20 s

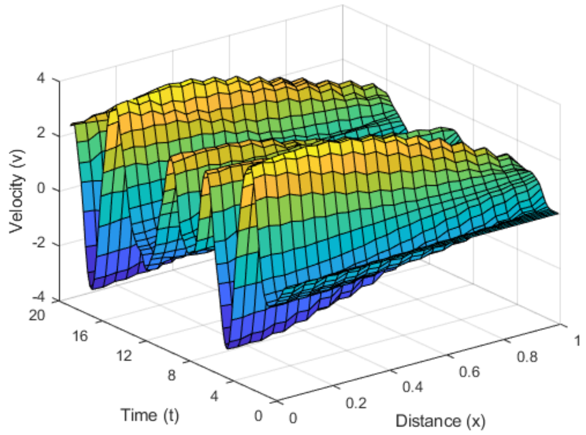

Fig. 5. The velocity of water flow $u(x,t)\ m/s$ at $k=0.02$ when after pass 20 s

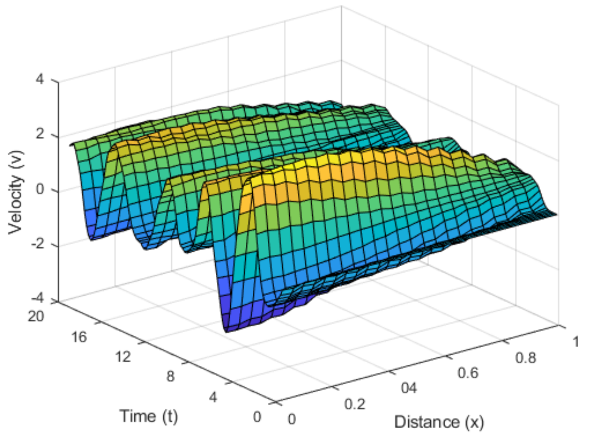

Fig. 3. The velocity of water flow $u(x,t)\ m/s$ at $k=-0.3$ when after pass 20 s

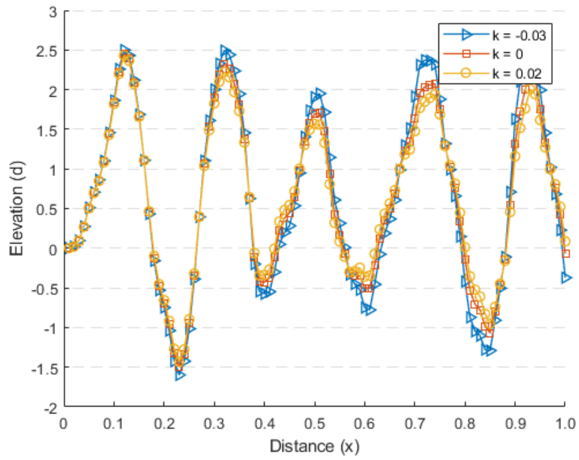

Fig. 6. The comparison elevation of water flow $u(x,t)\ m/s$ at $k=-0.03, k=0$, and $k=0.02$ when after pass 20 s

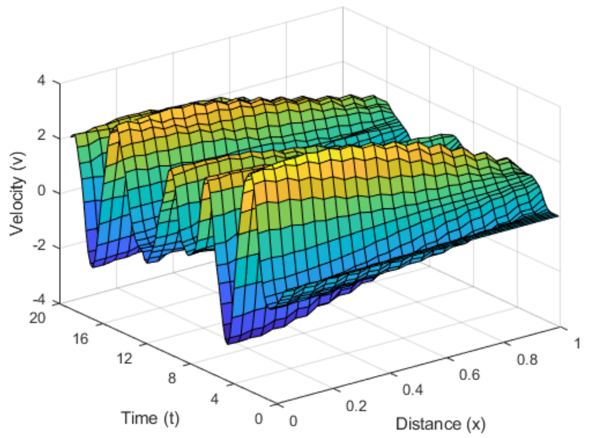

Fig. 4. The velocity of water flow $u(x,t)\ m/s$ at $k=0$ when after pass 20 s

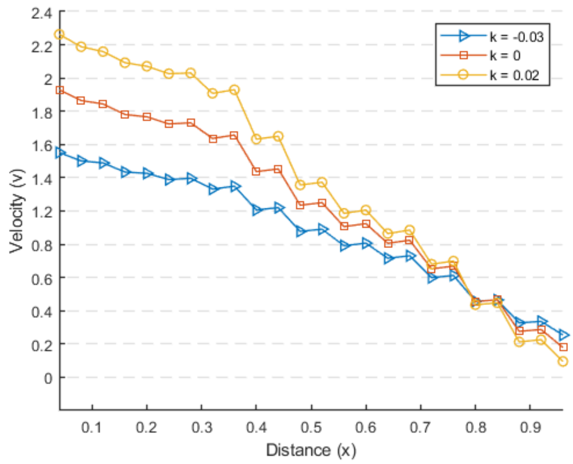

Fig. 7. The comparison velocity of water flow $u(x,t)\ m/s$ at $k=-0.03$, $k=0$, and $k=0.02$ when after pass 20 s

Table. 5. The pollutant concentration $C(x,t)\ Kg/m^3$ where $\Delta x = 0.04$, $\Delta t = 0.2$ and $k = -1$ of all methods for the above 3 cases.

t(sec), x (km)			$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
four points explicit upwind methods	case 1	4	0.59866933	0.13388683	0.01275161	0.00042256	0.00000480	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543974	-0.03342089
		8	0.78941834	0.27603853	0.06635899	0.01027764	0.00095667	0.00004886	-0.00002855	-0.00041859	-0.00345410	-0.01718876	-0.05318134
		12	0.96464247	0.41556165	0.13773746	0.03450134	0.00635247	0.00081604	-0.00024938	-0.00211319	-0.00959469	-0.03090974	-0.07178201
		16	1.11735609	0.56290780	0.22489454	0.07217117	0.01843115	0.00351607	-0.00070911	-0.00549082	-0.01828731	-0.04636889	-0.09096688
		20	1.24147098	0.71802159	0.33000427	0.12473063	0.03892169	0.00933274	-0.00122330	-0.01073596	-0.02939539	-0.06373023	-0.11170083
	case 2	4	0.59900000	0.13400000	0.01280000	0.00042300	0.00000480	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78941800	0.27603900	0.06640000	0.01030000	0.00095700	0.00005010	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464200	0.41556200	0.13773700	0.03450000	0.00636000	0.00085200	0.00007930	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11735600	0.56290800	0.22489500	0.07217500	0.01850000	0.00375000	0.00059700	0.00007410	0.00000691	0.00000047	0.00000004
		20	1.24147100	0.71802200	0.33000800	0.12475900	0.03910000	0.01020000	0.00219000	0.00038900	0.00005640	0.00000669	0.00000099
	case 3	4	0.59900000	0.13400000	0.01280000	0.00042300	0.00000480	0.00000002	0.00000014	0.00001640	0.00069600	0.01090000	0.06680000
		8	0.78941800	0.27603900	0.06640000	0.01030000	0.00095700	0.00005260	0.00006160	0.00083700	0.00691000	0.03440000	0.10600000
		12	0.96464200	0.41556200	0.13773700	0.03450000	0.00636000	0.00092300	0.00073700	0.00424000	0.01920000	0.06180000	0.14400000
		16	1.11735600	0.56290800	0.22489500	0.07218200	0.01850000	0.00423000	0.00321000	0.01120000	0.03660000	0.09270000	0.18200000
		20	1.24147100	0.71802300	0.33001600	0.12481600	0.03940000	0.01190000	0.00902000	0.02260000	0.05900000	0.12700000	0.22340500
third order Crank-Nicolson methods	case 1	4	0.59900000	0.13300000	0.01270000	0.00042300	0.00000481	0.00000002	-0.00000007	-0.00000821	-0.00034800	-0.00544000	-0.03340000
		8	0.78941800	0.27408500	0.06610000	0.01020000	0.00095500	0.00004880	-0.00002850	-0.00041900	-0.00345000	-0.01720000	-0.05320000
		12	0.96464200	0.41168500	0.13672800	0.03430000	0.00632000	0.00081300	-0.00025000	-0.00211000	-0.00959000	-0.03090000	-0.07180000
		16	1.11735600	0.55727900	0.22289700	0.07162600	0.01830000	0.00350000	-0.00071200	-0.00549000	-0.01830000	-0.04640000	-0.09100000
		20	1.24147100	0.71107600	0.32687400	0.12364400	0.03860000	0.00926000	-0.00124000	-0.01070000	-0.02940000	-0.06370000	-0.11170000
	case 2	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095519	0.00005009	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00632720	0.00084902	0.00007916	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55728000	0.22290000	0.07162900	0.01834700	0.00373340	0.00059422	0.00007387	0.00000689	0.00000047	0.00000004
		20	1.24150000	0.71108000	0.32688000	0.12367000	0.03879000	0.01011600	0.00217930	0.00038736	0.00005614	0.00000666	0.00000099
	case 3	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095525	0.00005262	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00633270	0.00092040	0.00073656	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55728000	0.22290000	0.07163700	0.01841300	0.00420850	0.00320620	0.01120400	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.71108000	0.32689000	0.12373000	0.03913700	0.01182400	0.00901190	0.02263800	0.05895900	0.12748000	0.22340000
four points implicit methods	case 1	4	0.59866900	0.13311200	0.01272900	0.00042300	0.00000500	0.00000000	0.00000000	-0.00000800	-0.00034800	-0.00544000	-0.03342100
		8	0.78941800	0.27213300	0.06574200	0.01021700	0.00095400	0.00004900	-0.00002900	-0.00041900	-0.00345400	-0.01718900	-0.05318100
		12	0.96464200	0.40781100	0.13572100	0.03411100	0.00629700	0.00081100	-0.00025000	-0.00211300	-0.00959500	-0.03091000	-0.07178200
		16	1.11735600	0.55165200	0.22090200	0.07108200	0.01819600	0.00347600	-0.00071400	-0.00549100	-0.01828700	-0.04636900	-0.09096700
		20	1.24147100	0.70413100	0.32374700	0.12256000	0.03831400	0.00919200	-0.00125100	-0.01074000	-0.02939500	-0.06373000	-0.11170100
	case 2	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095368	0.00005006	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00629940	0.00084632	0.00007900	0.00000492	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55165000	0.22090000	0.07108500	0.01822900	0.00371330	0.00059158	0.00007362	0.00000687	0.00000047	0.00000004
		20	1.24150000	0.70413000	0.32375000	0.12259000	0.03848700	0.01004600	0.00216590	0.00038529	0.00005589	0.00000664	0.00000099
	case 3	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095374	0.00005259	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00630490	0.00091770	0.00073640	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55165000	0.22090000	0.07109300	0.01829600	0.00418850	0.00320360	0.01120300	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.70413000	0.32376000	0.12265000	0.03883400	0.01175400	0.00899910	0.02263700	0.05895800	0.12748000	0.22340000

Table. 6. The pollutant concentration $C(x,t)\ Kg/m^3$ where $\Delta x = 0.04$, $\Delta t = 0.2$ and $k = 0$ of all methods for the above 3 cases.

t(sec), x (km)			$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
four points explicit upwind methods	case 1	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095667	0.00004886	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.41556000	0.13774000	0.03450100	0.00635250	0.00081604	-0.00024938	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.56291000	0.22489000	0.07217100	0.01843100	0.00351610	-0.00070911	-0.00549080	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.71802000	0.33000000	0.12473000	0.03892200	0.00933270	-0.00122330	-0.01073600	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095670	0.00005012	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.41556000	0.13774000	0.03450100	0.00635520	0.00085173	0.00007932	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.56291000	0.22489000	0.07217500	0.01846400	0.00375360	0.00059688	0.00007412	0.00000691	0.00000047	0.00000004
		20	1.24150000	0.71802000	0.33001000	0.12476000	0.03909500	0.01018600	0.00219270	0.00038945	0.00005640	0.00000669	0.00000099
	case 3	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095676	0.00005265	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.41556000	0.13774000	0.03450200	0.00636070	0.00092311	0.00073672	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.56291000	0.22490000	0.07218200	0.01853100	0.00422850	0.00320880	0.01120400	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.71802000	0.33002000	0.12482000	0.03944100	0.01189400	0.00902460	0.02264000	0.05896000	0.12748000	0.22340000
third order Crank-Nicolson methods	case 1	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095516	0.00004883	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00632450	0.00081333	-0.00024954	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.55728000	0.22290000	0.07162600	0.01831300	0.00349590	-0.00071179	-0.00549100	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.71108000	0.32687000	0.12364000	0.03861700	0.00926220	-0.00123700	-0.01073800	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095519	0.00005009	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00632720	0.00084902	0.00007916	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55728000	0.22290000	0.07162900	0.01834700	0.00373340	0.00059422	0.00007387	0.00000689	0.00000047	0.00000004
		20	1.24150000	0.71108000	0.32688000	0.12367000	0.03879000	0.01011600	0.00217930	0.00038736	0.00005614	0.00000666	0.00000099
	case 3	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095525	0.00005262	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00633270	0.00092040	0.00073656	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55728000	0.22290000	0.07163700	0.01841300	0.00420850	0.00320620	0.01120400	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.71108000	0.32689000	0.12373000	0.03913700	0.01182400	0.00901190	0.02263800	0.05895900	0.12748000	0.22340000
four points implicit methods	case 1	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095364	0.00004880	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00629670	0.00081063	-0.00024969	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.55165000	0.22090000	0.07108200	0.01819600	0.00347570	-0.00071445	-0.00549120	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.70413000	0.32375000	0.12256000	0.03831400	0.00919190	-0.00125070	-0.01074000	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095368	0.00005006	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00629940	0.00084632	0.00007900	0.00000492	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55165000	0.22090000	0.07108500	0.01822900	0.00371330	0.00059158	0.00007362	0.00000687	0.00000047	0.00000004
		20	1.24150000	0.70413000	0.32375000	0.12259000	0.03848700	0.01004600	0.00216590	0.00038529	0.00005589	0.00000664	0.00000099
	case 3	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095374	0.00005259	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00630490	0.00091770	0.00073640	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55165000	0.22090000	0.07109300	0.01829600	0.00418850	0.00320360	0.01120300	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.70413000	0.32376000	0.12265000	0.03883400	0.01175400	0.00899910	0.02263700	0.05895800	0.12748000	0.22340000

Table. 7. The pollutant concentration $C(x,t)\ Kg/\mathrm{m}^3$ where $\Delta x = 0.04$, $\Delta t = 0.2$ and $k = 1$ of all methods for the above 3 cases.

t(sec), x (km)			$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$	$x=0.6$	$x=0.7$	$x=0.8$	$x=0.9$	$x=1.0$
four points explicit upwind methods	case 1	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095667	0.00004886	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.41556000	0.13774000	0.03450100	0.00635250	0.00081604	-0.00024938	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.56291000	0.22489000	0.07217100	0.01843100	0.00351610	-0.00070911	-0.00549080	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.71802000	0.33000000	0.12473000	0.03892200	0.00933270	-0.00122330	-0.01073600	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095670	0.00005012	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.41556000	0.13774000	0.03450100	0.00635520	0.00085173	0.00007932	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.56291000	0.22489000	0.07217500	0.01846400	0.00375360	0.00059688	0.00007412	0.00000691	0.00000047	0.00000004
		20	1.24150000	0.71802000	0.33001000	0.12476000	0.03909500	0.01018600	0.00219270	0.00038945	0.00005640	0.00000669	0.00000099
	case 3	4	0.59867000	0.13389000	0.01275200	0.00042256	0.00000480	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27604000	0.06635900	0.01027800	0.00095676	0.00005265	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.41556000	0.13774000	0.03450200	0.00636070	0.00092311	0.00073672	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.56291000	0.22490000	0.07218200	0.01853100	0.00422850	0.00320880	0.01120400	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.71802000	0.33002000	0.12482000	0.03944100	0.01189400	0.00902460	0.02264000	0.05896000	0.12748000	0.22340000
third order Crank-Nicolson methods	case 1	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095516	0.00004883	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00632450	0.00081333	-0.00024954	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.55728000	0.22290000	0.07162600	0.01831300	0.00349590	-0.00071179	-0.00549100	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.71108000	0.32687000	0.12364000	0.03861700	0.00926220	-0.00123700	-0.01073800	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095519	0.00005009	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00632720	0.00084902	0.00007916	0.00000493	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55728000	0.22290000	0.07162900	0.01834700	0.00373340	0.00059422	0.00007387	0.00000689	0.00000047	0.00000004
		20	1.24150000	0.71108000	0.32688000	0.12367000	0.03879000	0.01011600	0.00217930	0.00038736	0.00005614	0.00000666	0.00000099
	case 3	4	0.59867000	0.13350000	0.01274100	0.00042268	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27408000	0.06605000	0.01024700	0.00095525	0.00005262	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.41169000	0.13673000	0.03430600	0.00633270	0.00092040	0.00073656	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55728000	0.22290000	0.07163700	0.01841300	0.00420850	0.00320620	0.01120400	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.71108000	0.32689000	0.12373000	0.03913700	0.01182400	0.00901190	0.02263800	0.05895900	0.12748000	0.22340000
four points implicit methods	case 1	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	-0.00000007	-0.00000821	-0.00034787	-0.00543970	-0.03342100
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095364	0.00004880	-0.00002855	-0.00041859	-0.00345410	-0.01718900	-0.05318100
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00629670	0.00081063	-0.00024969	-0.00211320	-0.00959470	-0.03091000	-0.07178200
		16	1.11740000	0.55165000	0.22090000	0.07108200	0.01819600	0.00347570	-0.00071445	-0.00549120	-0.01828700	-0.04636900	-0.09096700
		20	1.24150000	0.70413000	0.32375000	0.12256000	0.03831400	0.00919190	-0.00125070	-0.01074000	-0.02939500	-0.06373000	-0.11170000
	case 2	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095368	0.00005006	0.00000148	0.00000003	0.00000000	0.00000000	0.00000000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00629940	0.00084632	0.00007900	0.00000492	0.00000021	0.00000001	0.00000000
		16	1.11740000	0.55165000	0.22090000	0.07108500	0.01822900	0.00371330	0.00059158	0.00007362	0.00000687	0.00000047	0.00000004
		20	1.24150000	0.70413000	0.32375000	0.12259000	0.03848700	0.01004600	0.00216590	0.00038529	0.00005589	0.00000664	0.00000099
	case 3	4	0.59867000	0.13311000	0.01272900	0.00042280	0.00000481	0.00000002	0.00000014	0.00001641	0.00069575	0.01087900	0.06684200
		8	0.78942000	0.27213000	0.06574200	0.01021700	0.00095374	0.00005259	0.00006155	0.00083725	0.00690820	0.03437800	0.10636000
		12	0.96464000	0.40781000	0.13572000	0.03411100	0.00630490	0.00091770	0.00073640	0.00424120	0.01919000	0.06182000	0.14356000
		16	1.11740000	0.55165000	0.22090000	0.07109300	0.01829600	0.00418850	0.00320360	0.01120300	0.03659500	0.09273900	0.18193000
		20	1.24150000	0.70413000	0.32376000	0.12265000	0.03883400	0.01175400	0.00899910	0.02263700	0.05895800	0.12748000	0.22340000

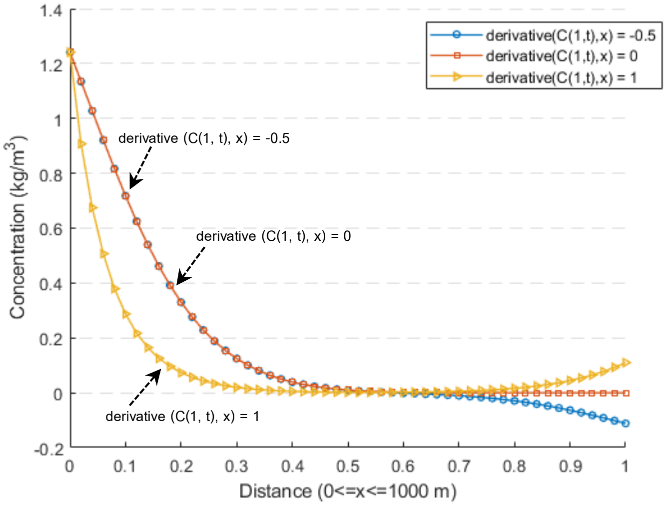

Fig. 8. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points explicit upwind method for $k=-1$ when after pass 60s.

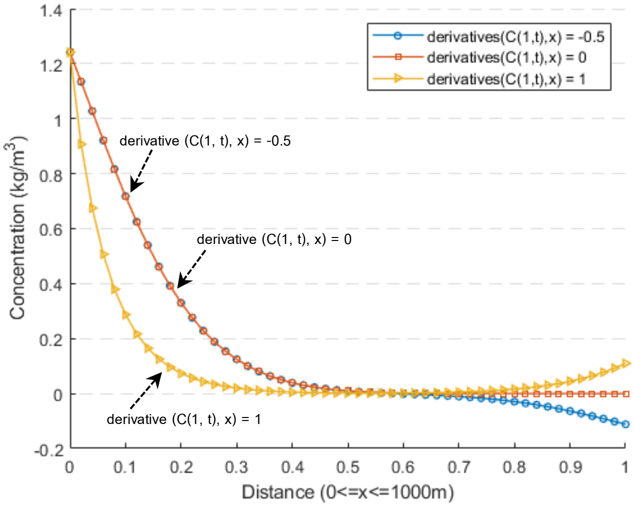

Fig. 11. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points explicit upwind method for $k=0$ when after pass 60s.

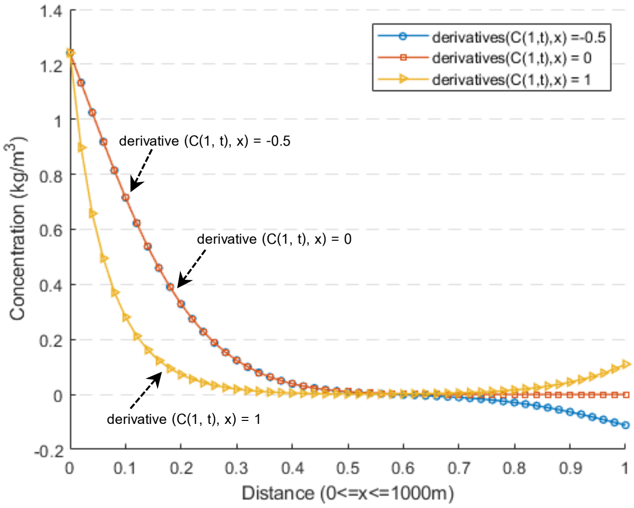

Fig. 9. The comparison of pollutant concentration at three difference $C(1,t)$ instants of third order Crank-Nicolson method for $k=-1$ when after pass 60s.

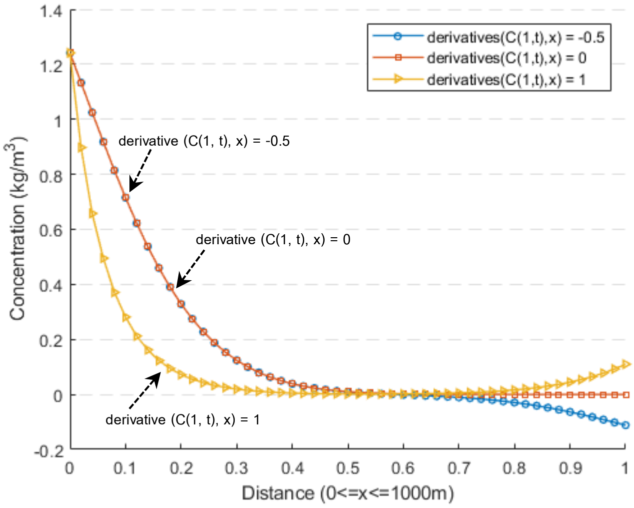

Fig. 12. The comparison of pollutant concentration at three difference $C(1,t)$ instants of third order Crank-Nicolson method for $k=0$ when after pass 60s.

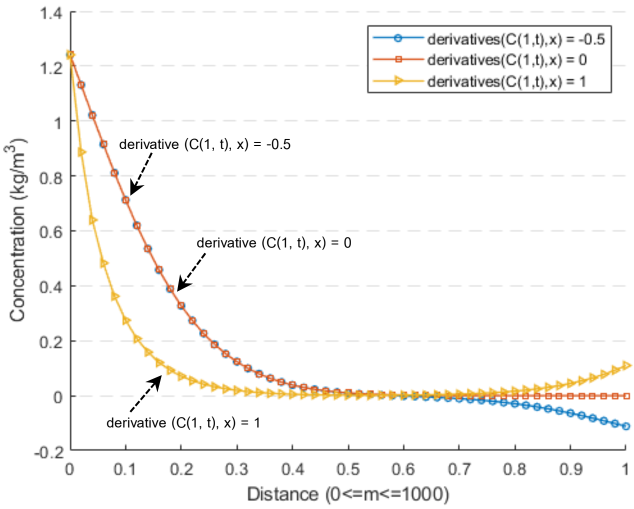

Fig. 10. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points implicit method for $k=-1$ when after pass 60s.

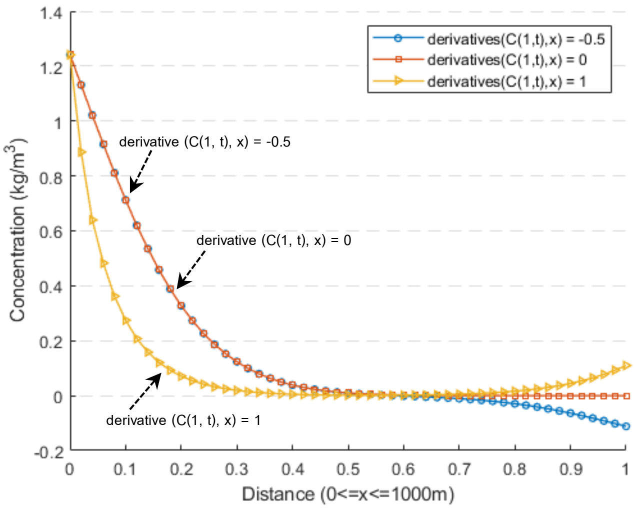

Fig. 13. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points implicit method for $k=0$ when after pass 60s.

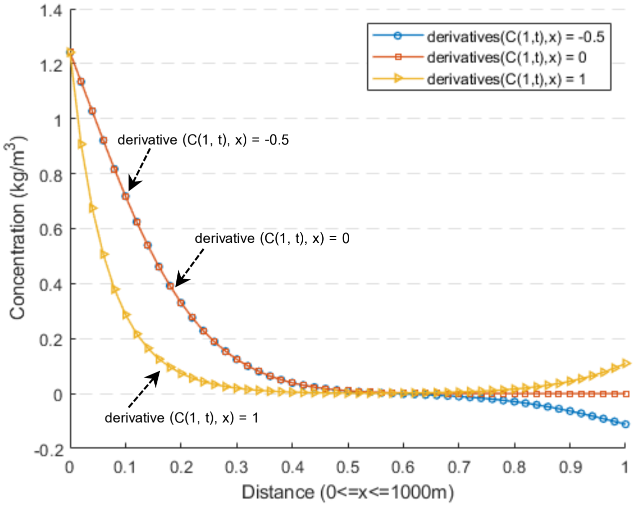

Fig. 14. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points explicit upwind method for $k=1$ when after pass 60s.

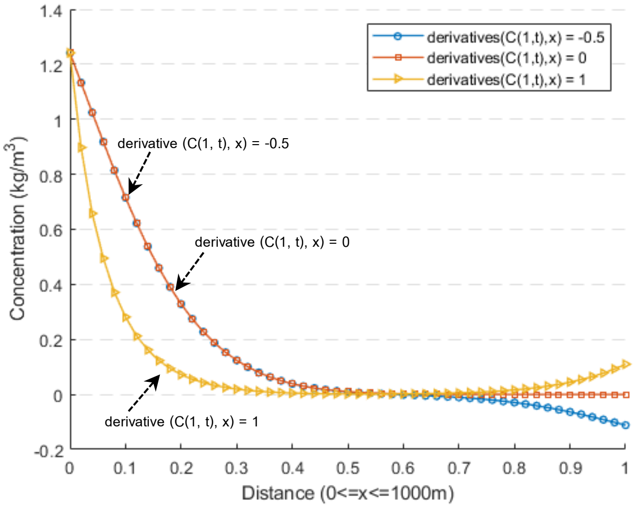

Fig. 15. The comparison of pollutant concentration at three difference $C(1,t)$ instants of third order Crank-Nicolson method for $k=1$ when after pass 60s.

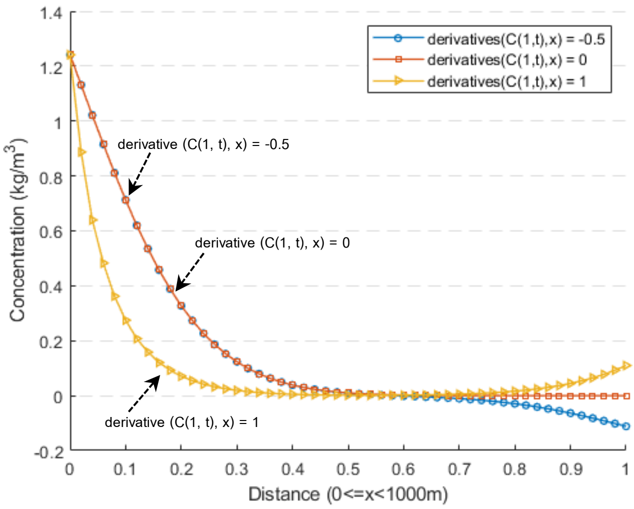

Fig. 16. The comparison of pollutant concentration at three difference $C(1,t)$ instants of four points implicit method for $k=1$ when after pass 60s.

VI. Discussion

The approximation of the pollutant concentrations of finite difference methods, a four points explicit upwind methods, third order Crank-Nicolson methods, and four points implicit methods at $k=-1$ are shown in Table 5 and, at $k=0$ are shown in Table 6, at $k=1$ are shown in Table 7, of the above 3 cases, respectively. For the comparison of the approximated pollutant concentrations of all schemes where $k=-1$, $k=0$, $k=1$, of the above 3 cases are shown in Fig. 8-10, Fig. 11-13, and Fig. 14-16, respectively. Implicit schemes exhibit excessive dispersion effects over large periods of time and space, greatly reducing the performance of implicit schemes. Moreover, implicit methods also produce a number of large systems of linear equations. The explicit upwind system of four points is economical to use. The suggested approach shows a strong agreement on consistency, and implicit schemes is less efficient than explicit schemes. Real-world challenges take a limited period of time to come up with accurate solutions. The experimental methods of the hydrodynamic model cannot, negatively, be pursued in all realms. It therefore indicates that the simulation results of the dispersion model should not be applied at any phase in the entire domain [16] and [17].

Fig. 6-8, 9-11 and 12-14 demonstrate that the decay rate of pollutants will reduce the concentration of erratic currents. When wastewater with a low degree of decomposition is discharged, the efficiency of the water is lower than the rate of decomposition of other contaminants.

VII. Conclusions

The hydrodynamic model and the ADRE have been coupled to determine the quantity of the contaminant in a stream when the current velocity is non-uniform. By using methodology in this study, the reaction of the flow to the different environmental inputs; the speed of the water and the ability to concentrate of the contaminant at the discharge point may be achieved. From the viewpoint of the explicit finite difference methods, it is assumed that the approach used is practical and appropriate. All four points of explicit upwind methods, third order Crank-Nicolson methods, and four points of implicit methods can be used in the dispersion model. We assume that the proposed solution is acceptable and economical to be used in the sense of real-world issues. Due to the simplicity of scripting and the ease of implementation. It is also capable of specifying a better location and timeline for multiple starting points.

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