A Copula-based Partial Dependent System Reliability Model and Its Application

Chunping Li, Haiqing Zhao, and Huibing Hao

Abstract—In this paper, some new partial dependent reliability models are obtained by using copula function under the different complex systems. Some explicit expressions of the series and parallel dependent system reliability model are given respectively, such as the complete dependence, complete independence and partial dependence. Then, the different reliability functions of dependent systems are compared, and some specific methods are obtained in order to improve system reliability under the given parameters: (1) Engineers can improve the system reliability by changing the dependency between components, (2) Engineers can improve the system reliability by changing the number of dependent components, (3) Engineers can improve the system. Finally, some numerical examples are presented to illustrate the results obtained in this article.

Index Terms—System reliability, Copula function, Series system, Parallel system, Partial dependence

I. INTRODUCTION

Reliability is subject of many research in recent years due to its importance in complex systems. The system is composed of multiple components through series, parallel or hybrid. The traditional system reliability researches almost assume that components were independent of each other, and the failure of a component was not affected the other components. However, the assumptions may not accord with the engineering practice. Due to the common load impact, common working environment, size, material and other factors, the dependence of components is inevitable. When those components are dependent, it creates a new problem to analyze the reliability of system.

In previous studies, some dependence systems have also been studied. For example, some papers have studied the dependence of a k out of n system, such as Ge et al.^[1], Papastavridis et al.^[2], Xiao et al.^[3] and Yun^[4]. Some other

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H. B. Hao is with the School of Mathematics and Computation Science, Lingnan Normal University¹, Zhanjiang, 524048, Guangdong, China, and with the Department of Mathematics, Hubei Engineering University², Hubei, 432000, China. e-mail: haohuibing@163.com. researches have considered the dependence situation of common cause failures, such as Kotz et al. ^[5], Vaurio ^[6], Xie et al. ^[7], Navarro and Balakrishan^[8]. But all above the studies have only discussed the special dependent structure.

Considering that copula function can describe dependence structure of random variables, and copula method has been widely used to obtain the dependent systems reliability. For example, Ran et al.^[9] analyzed the availability and cost of a parallel redundant complex system by using copula function. Noh et al.^[10] studied reliability optimization problems of dependence system by using Gaussian copula. Jia et al.^[11-13] considered different dependent reliability based on copula function.

Different from the previous studies, some new partial dependence reliability models are presented in this paper, and the corresponding properties are discussed. The rest of the paper is organized as follows, section II introduces copula method. In sections III and IV, reliability model for the partial dependent series system and the partial dependent parallel system are studied, respectively. In section V, a numerical example is given. Finally, some conclusions are provided in section VI.

II. THE BASIC THEORY OF COPULA FUNCTION

Copulas are powerful functions to describe the dependence among multiple variables. By using copula function and marginal distributions, a multivariate distribution can be obtained.

Definition A two-dimensional Copula is a distribution C $([0,1]^2 \rightarrow [0,1])$ with the following properties:

(i)
$$C(u,0) = C(0,v) = 0$$
, $C(u,1) = u$, $C(1,v) = v$

(ii)
$$\forall 0 \le u_1 \le u_2 \le 1, 0 \le v_1 \le v_2 \le 1$$
, we have

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$$

Theorem Let (X_1, X_2, \dots, X_n) be an n-dimensional random vector with margin distributions $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$. Then, there exists an n-dimensional Copula function $C(u_1, u_2, \dots, u_n)$ such that for all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we have

$$P(X_1 \le x_1, \cdots, X_n \le x_n) = C(F_1(x_1), \cdots, F_n(x_n))$$
(1)

In this paper, an n-dimension Gumbel copula is used to describe the dependence among multiple variables

$$C(u_1, u_2, \cdots, u_n; \theta) = \exp\left(-\left[\sum_{i=1}^n (-\ln u_i)^\theta\right]^{1/\theta}\right)$$
(2)

where θ is the Gumbel copula parameter with $\theta \in [1, +\infty)$.

III. RELIABILITY MODEL FOR PARTIAL DEPENDENT SERIES SYSTEM

A. Modeling for the complete independent series system

Suppose that a series system contains four independent components as A_1, A_2, A_3, A_4 . Let T_i be the lifetime variable of the component A_i with the distribution function

$$F_i(t) = P\{T_i \le t\}, i = 1, 2, 3, 4$$

Let *T* be the lifetime variable of the series system, then we can obtain $T = \min(T_1, T_2, T_3, T_4)$, and the independent system reliability can be obtained as

$$\begin{aligned} R_{I}^{(s)}(t) &= P\{T > t\} \\ &= P\{\min(T_{1}, T_{2}, T_{3}, T_{4}) > t\} \\ &= P\{T_{1} > t, T_{2} > t, T_{3} > t, T_{4} > t\} \\ &= P\{T_{1} > t\} \cdot P\{T_{2} > t\} \cdot P\{T_{3} > t\} \cdot P\{T_{4} > t\} \\ &= [1 - F_{1}(t)][1 - F_{2}(t)][1 - F_{3}(t)][1 - F_{4}(t)] \end{aligned}$$
(3)

B. Modeling for the complete dependent series system

Suppose that a series system contains four dependent components as A_1 , A_2 , A_3 , A_4 , and the dependent relationship can be described by copula function. Let *T* be the lifetime variable of the dependent series system, then the series system reliability can be obtained as

$$\begin{aligned} R_D^{(s)}(t) &= P\{T > t\} \\ &= P\{\min(T_1, T_2, T_3, T_4) > t\} \\ &= P\{T_1 > t, T_2 > t, T_3 > t, T_4 > t\} \\ &= 1 - \sum_{i=1}^{4} P\{T_i \le t\} + \sum_{1 \le i < j \le 4} P\{T_i \le t, T_j \le t\} \\ &- \sum_{1 \le i < j < k \le 4} P\{T_i \le t, T_j \le t, T_k \le t\} + P\{T_1 \le t, T_2 \le t, T_3 \le t, T_4 \le t\} \\ &= 1 - \sum_{i=1}^{4} F_i(t) + \sum_{1 \le i < j \le 4} C(F_i(t), F_j(t)) \\ &- \sum_{1 \le i < j < k \le 4} C(F_i(t), F_j(t), F_k(t)) + C(F_1(t), F_2(t), F_3(t), F_4(t)) \end{aligned}$$
(4)

C. Modeling for the partial dependent series system

In some systems, not all the components are dependent, that is to say, some components are dependent and some of them are independent. Considering the complexity of the partial dependent series system, the system reliability is discussed with the following three situations.

Case I Suppose that the three components, A_1 , A_2 , A_3 , are dependent of each other, and they are independent of the component A_4 , then the system reliability is:

$$\begin{split} R^{(3)}_{PD1}(t) &= P\{T > t\} \\ &= P\{\min(T_1, T_2, T_3, T_4) > t\} \\ &= P\{T_1 > t, T_2 > t, T_3 > t, T_4 > t\} \\ &= P\{T_1 > t, T_2 > t, T_3 > t\} \cdot P\{T_4 > t\} \\ &= [1 - P\{T_4 \le t\}] \cdot [1 - \sum_{i=1}^3 p\{T_i \le t\} + \sum_{1 \le i < j \le 3} P\{T_i \le t, T_j \le t\} \\ &- P\{T_1 \le t, T_2 \le t, T_3 \le t\}] \end{split}$$

$$= [1 - F_4(t)] \cdot [1 - \sum_{i=1}^{3} F_i(t) + \sum_{1 \le i < j \le 3} C(F_i(t), F_j(t)) - C(F_1(t), F_2(t), F_3(t))]$$
(5)

Case II Suppose that the components, A_1 , A_2 , are dependent, they are independent of the other components, A_3 , A_4 , and the components, A_3 , A_4 , are independent, then the system reliability can be obtained as follows: $P_{(x)}^{(s)}(A) = P(T > A)$

$$\begin{aligned} R_{PD2}^{(s)}(t) &= P\{T > t\} \\ &= P\{\min(T_1, T_2, T_3, T_4) > t\} \\ &= P\{T_1 > t, T_2 > t, T_3 > t, T_4 > t\} \\ &= P\{T_1 > t, T_2 > t\} \cdot P\{T_3 > t\} \cdot P\{T_4 > t\} \\ &= [1 - P\{T_1 \le t\} - P\{T_2 \le t\} \\ &\quad + P\{T_1 \le t, T_2 \le t\}][1 - P\{T_3 \le t\}][1 - P\{T_4 \le t\}] \\ &= [1 - F_1(t) - F_2(t) + C(F_1(t), F_2(t))][1 - F_3(t)][1 - F_4(t)](6) \end{aligned}$$

Case III Suppose that the components, A_1 and A_2 are dependent; the components A_3 and A_4 are dependent, but the two subsystems are independent, then the system reliability is:

$$\begin{split} R_{PD3}^{(s)}(t) &= P\{T > t\} \\ &= P\{\min(T_1, T_2, T_3, T_4) > t\} \\ &= P\{T_1 > t, T_2 > t, T_3 > t, T_4 > t\} \\ &= P\{T_1 > t, T_2 > t\} \cdot P\{T_3 > t, T_4 > t\} \\ &= [1 - P\{T_1 \le t\} - P\{T_2 \le t\} + P\{T_1 \le t, T_2 \le t\}] \\ &\times [1 - P\{T_3 \le t\} - P\{T_4 \le t\} + P\{T_3 \le t, T_4 \le t\}] \\ &= [1 - F_1(t) - F_2(t) + C(F_1(t), F_2(t))] \\ &\times [1 - F_3(t) - F_4(t) + C(F_3(t), F_4(t))] \end{split}$$
(7)

IV. RELIABILITY MODEL FOR PARTIAL DEPENDENT PARALLEL SYSTEM

A. Modeling for the complete independent parallel system

Similarly, suppose that a parallel system contains four independent components as A_1 , A_2 , A_3 , A_4 . Let T be the lifetime variable of the parallel system, then we can get $T = \max(T_1, T_2, T_3, T_4)$. Then, the reliability of independent parallel system can be obtained as

$$\begin{aligned} R_{I}^{(p)}(t) &= P\{T > t\} \\ &= P\{\max(T_{1}, T_{2}, T_{3}, T_{4}) > t\} \\ &= 1 - P\{\max(T_{1}, T_{2}, T_{3}, T_{4}) \le t\} \\ &= 1 - P\{T_{1} \le t, T_{2} \le t, T_{3} \le t, T_{4} \le t\} \\ &= 1 - P\{T_{1} \le t\} \cdot P\{T_{2} \le t\} \cdot P\{T_{3} \le t\} \cdot P\{T_{4} \le t\} \\ &= 1 - F_{1}(t)F_{2}(t)F_{3}(t)F_{4}(t) \end{aligned}$$

$$(8)$$

B. Modeling for the complete dependent parallel system

Suppose that a parallel system contains four dependent components as A_1 , A_2 , A_3 , A_4 , and the dependent relationship can be described by copula function. Let *T* be the lifetime variable of the dependent system, then the reliability of parallel system can be given as $P^{(p)}(t) = P(T > t)$

$$\begin{aligned} R_D^{(r)}(t) &= P\{I > t\} \\ &= P\{\max(T_1, T_2, T_3, T_4) > t\} \\ &= 1 - P\{\max(T_1, T_2, T_3, T_4) \le t\} \\ &= 1 - P\{T_1 \le t, T_2 \le t, T_3 \le t, T_4 \le t\} \end{aligned}$$

$$=1-C(F_{1}(t),F_{2}(t),F_{3}(t),F_{4}(t))$$
(9)

C. Modeling for the partial dependent parallel system

Similarly, considering the complexity of the partial dependent parallel system, the system reliability is discussed with the following three situations.

Case IV Suppose that the three components, A_1 , A_2 , A_3 , are dependent of each other, and they are independent of the component A_4 , then the system reliability can be given as follows:

$$\begin{aligned} R_{PD1}^{(p)}(t) &= P\{T > t\} \\ &= P\{\max(T_1, T_2, T_3, T_4) > t\} \\ &= 1 - P(T_1 \le t, T_2 \le t, T_3 \le t, T_4 \le t) \\ &= 1 - P\{T_1 \le t, T_2 \le t, T_3 \le t\} \cdot P\{T_4 \le t\} \\ &= 1 - C(F_1(t), F_2(t), F_3(t))F_4(t) \end{aligned}$$
(10)

Case V Suppose that the components, A_1 , A_2 , are dependent, they are independent of the other components, A_3 , A_4 , and the components, A_3 , A_4 , are independent, then the system reliability is:

$$\begin{aligned} R_{PD2}^{(p)}(t) &= P\{T > t\} \\ &= P\{\max(T_1, T_2, T_3, T_4) > t\} \\ &= 1 - P(T_1 \le t, T_2 \le t, T_3 \le t, T_4 \le t) \\ &= 1 - P(T_1 \le t, T_2 \le t) P(T_3 \le t) P(T_4 \le t) \\ &= 1 - C(F_1(t), F_2(t)) F_3(t) F_4(t) \end{aligned}$$
(11)

Case VI Suppose that the components, A_1 , A_2 , are dependent, and the components, A_3 , A_4 , are dependent, but the two subsystems are independent, then the system reliability is obtained as:

$$\begin{aligned} R_{PD3}^{(\nu)}(t) &= P\{T > t\} \\ &= P\{\max(T_1, T_2, T_3, T_4) > t\} \\ &= 1 - P(T_1 \le t, T_2 \le t, T_3 \le t, T_4 \le t) \\ &= 1 - P(T_1 \le t, T_2 \le t) \times P(T_3 \le t, T_4 \le t) \\ &= 1 - C(F_1(t), F_2(t)) \cdot C(F_3(t), F_4(t)) \end{aligned}$$
(12)

V. RELIABILITY ANALYSIS BASED ON NUMERICAL EXAMPLE

In order to illustrate the results, some numerical examples are given in the following. Considering that the correlation between components is generally positive and the system reliability is easily to construct, the multivariate Gumbel copulas function is used to describe the dependent of different components in this paper.

A. Reliability analysis for the dependent series system

Supposes that four components has the same distribution function

$$F_i(t) = P\{T_i \le t\} = 1 - \exp(-\lambda t), \ i = 1, 2, 3, 4$$

with $\lambda = 1.5$ and $\theta = 5$.

Then, the system reliability under the different dependent situations can be obtained as follow

$$R_l^{(s)}(t) = \exp(-4\lambda t) \tag{13}$$

$$R_{D}^{(s)}(t) = 4\exp(-\lambda t) - 3 + 6\exp(-[2(-\ln(1-e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}}) -4\exp(-[3(-\ln(1-e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}}) +\exp(-[4(-\ln(1-e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}})$$
(14)

$$R_{PD1}^{(s)}(t) = \exp(-\lambda t)[3\exp(-\lambda t) - 2 +3\exp(-[2(-\ln(1 - e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}}) -\exp(-[3(-\ln(1 - e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}})]$$
(15)
$$R^{(s)}(t) = \exp(-2\lambda t)[2\exp(-\lambda t) - 1]$$

$$+\exp(-[2(-\ln(1-e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}})]$$
(16)

$$R_{PD3}^{(s)}(t) = [2\exp(-\lambda t) - 1]$$



By using Matlab software, the reliability curves of different dependent situation are plotted in Fig.1. When the components are completely independent, the series system reliability is plotted as curve 1. Then, when the components are partial dependent, the series system reliability is plotted as curves 2-4, where the curve 2 is the Case I, the curve 3 is the Case II, and the curve 4 is the Case III. At last, when the components are completely dependent, the series system reliability is plotted as curve 5. From the Fig.1, we can get the following conclusions under the given parameters $\lambda = 1.5$ and $\theta = 5$:

(1) When the series system components exists partial dependency, we can find

$$R_{I}^{(s)}(t) < R_{PD}^{(s)}(t) < R_{D}^{(s)}(t)$$
(18)

That is to say, before the system is put into operation, the reliability of the series system can be improved by combining dependent components together. In other words, engineers can improve the series system reliability by increasing the dependency between components. (2) From the Fig. 1, we can find

$$R_{I}^{(s)}(t) < R_{PD1}^{(s)}(t) < R_{PD2}^{(s)}(t) < R_{D}^{(s)}(t)$$
(19)

That is to say, the system contains the more numbers of dependency components, series system has higher reliability. In other words, in practical engineering applications, engineers can also improve series system reliability by increasing the number of dependent components in the system.

(3) From the Fig. 1, we can also find

$$R_{PD2}^{(s)}(t) < R_{PD3}^{(s)}(t) < R_D^{(s)}(t)$$
(20)

That is to say, due to the change of practical running environment, the series system components can not be completely dependent. At this point, the dependent system by a new construction method can improve the series system reliability.

B. Reliability analysis for the dependent parallel system

Similarly, let $\lambda = 1.5$ and $\theta = 5$, the system reliability under the different dependent parallel situations can be obtained as follow

$$R_{I}^{(p)}(t) = 1 - \left[\left[1 - \exp(-\lambda t) \right] \right]^{4}$$
(21)

$$R_{D}^{(p)}(t) = 1 - \exp(-[4(-\ln(1 - e^{-\lambda t}))^{\theta}]^{\overline{\theta}})$$
(22)

$$R_{PD1}^{(p)}(t) = 1 - [1 - \exp(-\lambda t)] \exp(-[3(-\ln(1 - e^{-\lambda t}))^{\theta}]^{\frac{1}{\theta}})$$
(23)

$$R_{PD2}^{(p)}(t) = 1 - [1 - \exp(-\lambda t)]^2 \exp(-[2(-\ln(1 - e^{-\lambda t}))^{\theta}]^{\overline{\theta}})$$
(24)

$$R_{PD3}^{(p)}(t) = 1 - [\exp(-[2(-\ln(1-e^{-\lambda t}))^{\theta}]^{\overline{\theta}})]^2$$
(25)



Similarly, by using Matlab software, the reliability curves of different dependent situation are plotted in Fig.2. When the components are completely independent, the parallel system reliability is plotted as curve 1. When the components are partial dependent, the parallel system reliability is plotted as curves 2-4, where the curve 4 is the Case IV, the curve 2 is the Case V, and the curve 3 is the Case VI. At last, when the components are completely dependent, the parallel system reliability is plotted as curve 5. From the Fig.2, we can obtain the following conclusions under the given parameters $\lambda = 1.5$ and $\theta = 5$:

(1) When the parallel system components exists partial dependency, from the Fig. 2, we can find

$$R_{I}^{(s)}(t) > R_{PD}^{(s)}(t) > R_{D}^{(s)}(t)$$
(26)

That means the dependent of the system components will reduce the parallel system reliability. In other words, engineers can improve the parallel system reliability by decreasing the dependency between components.

(2) From the Fig. 2, we can find

$$R_{I}^{(s)}(t) > R_{PD2}^{(s)}(t) > R_{PD1}^{(s)}(t) > R_{D}^{(s)}(t)$$
 (27)

That is to say, the system contains the fewer number of dependency components, the parallel system has higher reliability. In other words, engineers can also improve the parallel system reliability by decreasing the number of dependent components in the system.

VI. CONCLUSION

Copula function is a useful tool for constructing the dependence structure. In this paper, the dependent series or parallel system reliability models are presented by using copula functions and traditional reliability theory, some formulas are presented to compute the system reliability. Some explicit expressions of system reliability are given under the different dependent situations, such as the complete dependence, complete independence and partial dependence. Then, the different reliability functions of dependant systems are compared, and some specific methods are obtained in order to improve system reliability under the given parameters.

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