On a Stochastic Budworm Growth Model

Famei Zheng, Ruizhuang Zhang and Guixin Hu

***Abstract*—In this report, we dissect a budworm growth model with multi-perturbations . We first show that the model has a unique global positive solution for any positive initial data. Then we explore the persistence and extinction of the species, and test the growth rate and the global asymptotic stability of the solution of the model. In addition, we give the *explicit* density function of the stationary probability distribution of the model, which can be utilized to test the growth of the budworm species more accurately. Finally, we use the theoretical findings to explore the growth of spruce budworm (*Choristoneura fumiferana* Clemens) in eastern North America.**

***Index Terms*—population model, stochasticity, explicit density function.**

I. Introduction

THE SBW (spruce budworm, *Choristoneura fumiferana* Clemens) has become the most harmful indigenous pest of spruces in Eastern North America [11]. The outbreak of the SBW has led to great losses, for example, by 2010, owning to SBW outbreaks, Quebec province lost \$12.5 billion [10]. Additionally, Eastern North America is currently subjected to an outbreak starting in 2006. By 2018, it had dispersed about 8.1 million hectares [1].

In order to portray the growth law of the SBW, Ludwig et al. [9] put forward the following deterministic population model which has been attracted much attention (see, e.g, [5], [12]):

$$\frac{\mathrm{d}S}{\mathrm{d}t} = S\bigg(b - \theta S - \frac{\sigma S}{\varpi + S^2}\bigg), \tag{1}$$

where $S(t)$ represents the population abundance; $b > 0$, $\theta > 0$, $\sigma > 0$ and $\varpi > 0$ measure the growth rate, the strength of the intra-specific competition, the predators' consumption rate and the saturate effect, respectively. However, model (1) does not consider the environmental stochasticity. Actually, the evolution of SBW has close relationships with temperature and humidity that are of high stochasticity [2]. Accordingly, many authors (see, e.g., [7], [8], [14], [15]) introduced environmental perturbations into model (1) and delved into the following stochastic model

$$\mathrm{d}S = S\bigg(b - \theta S - \frac{\sigma S}{\varpi + S^2}\bigg)\mathrm{d}t + \lambda S \mathrm{d}\psi(t), \tag{2}$$

where $\lambda > 0$ characterizes the intensity of the environmental perturbations, $\psi(t)$ stands for a Wiener process.

For model (2), some interesting results have been obtained.

(i) ([7], [8], [14], [15]) if $\alpha := b - \lambda^2/2 < 0$, then $\lim_{t\to+\infty} S(t) = 0$;

(ii) ([7]) if $\alpha > 0$, then Eq. (2) possesses a unique invariant measure concentrated on $(0, +\infty)$.

Model (2) supposes that the growth rate is affected by a single white noise. Nevertheless, the growth rate may be affected by several white noises simultaneously. Therefore, one need to test the following model:

$$\mathrm{d}S = S\bigg(b - \theta S - \frac{\sigma S}{\varpi + S^2}\bigg)\mathrm{d}t + \sum_{i=1}^{n} \lambda_i S \mathrm{d}\psi_i(t), \tag{3}$$

where $\psi_1(t)$, $\psi_2(t)$,..., $\psi_n(t)$ are independent Wiener processes.

The objectives of this report are to explore the dynamical properties of model (3). The rest of this report is arranged as follows. In Section II, we show that the model has a unique global positive solution for any positive initial data. In Section III, we explore the extinction, non-persistence in the mean, weak persistence and stochastic permanence of the species. In Section IV, we estimate the growth rate of the solution. In Section V, we focus on the global asymptotic stability (GAS) of the solution. In Section VI, the explicit density function of the invariant measure of model (3) is given. In Section VII, we extend some findings to cover model (3) with regime-switching. In Section VIII, we use the findings to explore the growth of spruce budworm (*Choristoneura fumiferana* Clemens) in eastern North America.

II. Existence and uniqueness of the solution

Theorem 1. *For any $S(0) > 0$, model (3) has a unique global positive solution $S(t)$ almost surely (a.s.).*

Proof: We first focus on the following model

$$\begin{aligned} \mathrm{d}y(t) \;&= \bigg[b - \sum_{i=1}^{n} \lambda_i^2/2 - \theta e^{y(t)} - \frac{\sigma e^{y(t)}}{\varpi^2 + e^{2y(t)}}\bigg]\mathrm{d}t \\ &\quad + \sum_{i=1}^{n} \lambda_i \mathrm{d}\psi_i(t) \end{aligned} \tag{4}$$

with $y(0) = \ln S(0)$. The coefficients of Eq. (4) are locally Lipschitz continuous, therefore Eq. (4) has a unique solution on $[0, \tau_e)$, where $\tau_e \le +\infty$. According to Itô's formula, $S(t) = e^{y(t)}$ is the unique positive solution of (3).

Now let us show that $\tau_e = +\infty$. Choose an integer $k_0 > S(0)$. For every integer m, define

$$\tau_m = \inf\{t \in [0, \tau_e) : S(t) \ge m\}.$$

Let $\tau_\infty = \lim_{k\to+\infty} \tau_m$. It then follows that $\tau_\infty \le \tau_e$. If $\tau_e < +\infty$, then there are two constants $\hat{T} > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathrm{P}\{\tau_\infty \le \hat{T}\} > \epsilon.$$

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In other words, there is an integer $m_1 \geq m_0$ such that for arbitrary $m \geq m_1$,

$$\mathrm{P}\{\Omega_m\} \geq \epsilon, \tag{5}$$

where $\Omega_m = \{\omega : \tau_m \leq \hat{T}\}$. Define

$$U_1(S) = S^{\delta},\ S > 0,\ 0 < \delta < 1.$$

In view of Itô's formula

$$\begin{aligned} \mathrm{d}U_1(S) &= \delta S^{\delta}\left[b - \theta S - \frac{\sigma S}{\varpi^2 + S^2} + \frac{\delta - 1}{2}\sum_{i=1}^{n}\lambda_i^2\right]\mathrm{d}t \\ &\quad + \delta\sum_{i=1}^{n}\lambda_i S^{\delta}\mathrm{d}\psi_i(t) \\ &\leq \delta b U_1(S)\mathrm{d}t + \delta\sum_{i=1}^{n}\lambda_i U_1(S)\mathrm{d}\psi_i(t). \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathbb{E}U_1(S(\tau_m \wedge \hat{T})) \\ &\leq U_1(S(0)) + \delta b\int_0^{\tau_m\wedge\hat{T}} \mathbb{E}U_1(S(s))\mathrm{d}s \\ &\leq U_1(S(0)) + \delta b\int_0^{\hat{T}} \mathbb{E}U_1(S(\tau_m \wedge s))\mathrm{d}s. \end{aligned}$$

By Gronwall's inequality, one has

$$\mathbb{E}U_1(S(\tau_m \wedge \hat{T})) \leq U_1(S(0))e^{\delta b\hat{T}}. \tag{6}$$

For $\omega \in \Omega_m$, $U_1(S(\tau_m, \omega)) \geq m^{\delta}$. According to (5) and (6), we have

$$U_1(S(0))e^{\delta b\hat{T}} \geq \mathbb{E}[1_{\Omega_m}(\omega)U_1(S(\tau_m, \omega))] \geq \epsilon m^{\delta}.$$

We then obtain a contradiction by letting $m \to +\infty$:

$$\infty > U_1(S(0))e^{\delta b\hat{T}} = \infty.$$

Consequently, $\tau_e = +\infty$. ■

III. Extinction and persistence

In this part, we pay attention to the extinction and persistence of the species.

Theorem 2. *If $\bar{\alpha} < 0$, then $\lim_{t\to+\infty} S(t) = 0$, a.s., namely, the species goes to extinction, where*

$$\bar{\alpha} = b - \frac{1}{2}\sum_{i=1}^{n}\lambda_i^2.$$

Proof: By Itô's formula,

$$\mathrm{d}\ln S = \left[\bar{\alpha} - \theta S - \frac{\sigma S}{\varpi^2 + S^2}\right]\mathrm{d}t + \sum_{i=1}^{n}\lambda_i\mathrm{d}\psi_i(t).$$

Therefore,

$$\ln S(t)/S(0) = \bar{\alpha}t - \int_0^t\left[\theta S + \frac{\sigma S(s)}{\varpi^2 + S^2(s)}\right]\mathrm{d}s + \Lambda(t), \tag{7}$$

where

$$\Lambda(t) = \sum_{i=1}^{n}\lambda_i\psi_i(t).$$

Notice that

$$\lim_{t\to+\infty} t^{-1}\Lambda(t) = 0, \quad a.s.. \tag{8}$$

Hence (7) and (3) imply that

$$\ln S(t) - \ln S(0) \leq \bar{\alpha}t - \theta\int_0^t S(s)\mathrm{d}s + \Lambda(t). \tag{9}$$

As a result,

$$t^{-1}\left\{\ln S(t) - \ln S(0)\right\} \leq \bar{\alpha} + t^{-1}\Lambda(t).$$

According to (8)

$$\limsup_{t\to+\infty} t^{-1}\ln S(t) \leq \bar{\alpha} < 0.$$

Thereby, $\lim_{t\to+\infty} S(t) = 0$, a.s.. ■

Theorem 3. *If $\bar{\alpha} \geq 0$, then*

$$\limsup_{t\to+\infty} t^{-1}\int_0^t S(s)\mathrm{d}s \leq \frac{\bar{\alpha}}{\theta}, \quad a.s.. \tag{10}$$

Particularly, if $\bar{\alpha} = 0$, then $\lim_{t\to+\infty} t^{-1}\int_0^t S(s)\mathrm{d}s = 0$, namely, the species is non-persistent in the mean.

Proof: For $\forall\epsilon > 0$, there is an $\tilde{T} > 0$ such that for $t > \tilde{T}$,

$$t^{-1}\left[\ln S(0) + \bar{\alpha} + \Lambda(t)\right] \leq \bar{\alpha} + \epsilon.$$

Set $\xi = \bar{\alpha} + \epsilon$. According to (9), for $t > \tilde{T}$,

$$\begin{aligned} \ln S(t) &\leq \ln S(0) + \bar{\alpha}t - \theta\int_0^t S(s)\mathrm{d}s + \Lambda(t) \\ &\leq \xi t - \theta\int_0^t S(s)\mathrm{d}s. \end{aligned}$$

Set $\pi(t) = \int_0^t S(s)\mathrm{d}s$, hence

$$e^{\theta\pi(t)}(\mathrm{d}\alpha/\mathrm{d}t) \leq e^{\xi t}, \quad t \geq \tilde{T}.$$

Therefore,

$$e^{\theta\pi(t)} \leq e^{\theta\pi(\tilde{T})} + \theta\xi^{-1}e^{\xi t} - \theta\xi^{-1}e^{\xi\tilde{T}}.$$

Taking logarithm gives

$$\pi(t) \leq \theta^{-1}\ln\left\{\theta\xi^{-1}e^{\xi t} + e^{\theta\pi(\tilde{T})} - \theta\xi^{-1}e^{\epsilon\tilde{T}}\right\}.$$

Consequently,

$$\begin{aligned} &\limsup_{t\to+\infty} t^{-1}\int_0^t S(s)\mathrm{d}s \leq \theta^{-1} \\ &\times \limsup_{t\to+\infty}\left\{t^{-1}\ln\left\{\theta\xi^{-1}e^{\xi t} + e^{\theta\pi(\tilde{T})} - \theta\xi^{-1}e^{\xi\tilde{T}}\right\}\right\}. \end{aligned}$$

By L'Hospital's rule, one has

$$\limsup_{t\to+\infty} t^{-1}\int_0^t S(s)\mathrm{d}s \leq \frac{\xi}{\theta} = \frac{\bar{\alpha} + \epsilon}{\theta}.$$

An application of the arbitrariness of ϵ gives (10). ■

Theorem 4. *If $\bar{\alpha} > 0$, then $\limsup_{t\to+\infty} S(t) > 0$, namely, the species is weakly persistent.*

Proof: Set $J = \{\omega : \lim_{t\to+\infty} S(t, \omega) = 0\}$. If $\mathrm{P}\{J\} > 0$, then for arbitrary $\omega \in J$, $\lim_{t\to+\infty} S(t, \omega) = 0$. That is to say,

$$\limsup_{t\to+\infty} t^{-1}[\ln S(t, \omega) - \ln S(0)] < 0,$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t \left[\theta S(s)+\frac{\sigma S(s)}{\varpi^2+S^2(s)}\right]\mathrm{d}s=0.$$

Then (7) and (8) indicate that

$$0>\limsup_{t\to+\infty} t^{-1}\ln S(t,\omega)=\bar{\alpha}>0.$$

One then derive a contradiction. ∎

Theorem 5. *If $\bar{\alpha}>0$, then for any $\epsilon\in(0,1)$, there are two constants $g_1=g_1(\epsilon)>0,\ g_2=g_2(\epsilon)>0$ such that*

$$\liminf_{t\to+\infty}\mathrm{P}\{S(t)\geq g_1\}\geq 1-\epsilon,$$

$$\liminf_{t\to+\infty}\mathrm{P}\{S(t)\leq g_2\}\geq 1-\epsilon.$$

namely, the species is stochastically permanent.

Proof: Set

$$U_2(S)=1/S^2,\ S>0.$$

By Itô's formula,

$$\begin{aligned}
&\mathrm{d}U_2(S)\\
&=2U_2(S)\left[\theta S+\frac{\sigma S}{\varpi^2+S^2}-b\right]\mathrm{d}t\\
&\quad+3\sum_{i=1}^n\lambda_i^2U_2(S)\mathrm{d}t-2\sum_{i=1}^n\lambda_iU_2(S)\mathrm{d}\psi_i(t)\\
&=2U_2(S)\left\{\theta S-b+1.5\sum_{i=1}^n\lambda_i^2+\frac{\sigma S}{\varpi^2+S^2}\right\}\mathrm{d}t\\
&\quad-2\sum_{i=1}^n\lambda_iU_2(S)\mathrm{d}\psi_i(t).
\end{aligned}$$

Let $\mu\in(0,1)$ be a constant satisfying

$$\mu<\bar{\alpha}/\lambda_1^2. \tag{11}$$

Set

$$U_3(S)=(1+U_2(S))^{\mu}.$$

By Itô's formula,

$$\mathbb{E}U_3(S(t))=U_3(S(0))+\mathbb{E}\int_0^t\mathcal{L}U_3(S(s))\mathrm{d}s,$$

where

$$\begin{aligned}
&\mathcal{L}U_3(S)\\
&=2\mu(1+U_2(S))^{\mu-2}\Big\{(U_2(S)+U_2^2(S))\\
&\quad\times\left[\theta S+\frac{\sigma S}{\varpi^2+S^2}-b+1.5\sum_{i=1}^n\lambda_i^2\right]\\
&\quad+(\mu-1)U_2^2(S)\textstyle\sum_{i=1}^n\lambda_i^2\Big\}\\
&=2\mu(1+U_2(S))^{\mu-2}\Big\{\Big[-b+\frac{\sum_{i=1}^n\lambda_i^2}{2}\\
&\quad+\mu\sum_{i=1}^n\lambda_i^2\Big]U_2^2(S)+\theta U_2^{1.5}(S)\\
&\quad+\frac{\sigma}{\varpi^2+S^2}U_2^{1.5}(S)+\Big[-b+1.5\sum_{i=1}^n\lambda_i^2\\
&\quad+\frac{\sigma S}{\varpi^2+S^2}\Big]U_2(S)+\theta U_2^{0.5}(S)\Big\}\\
&\leq 2\mu(1+U_2(S))^{\mu-2}\Big\{\Big[-\bar{\alpha}\\
&\quad+\mu\sum_{i=1}^n\lambda_i^2\Big]U_2^2(S)+\left(\theta+\frac{\sigma}{\varpi^2}\right)U_2^{1.5}(S)\\
&\quad+\left[1.5\sum_{i=1}^n\lambda_i^2+\frac{\sigma}{2\varpi}\right]U_2(S)+\theta U_2^{0.5}(S)\Big\}\\
&=2\mu(1+U_2(S))^{\mu-2}\Big\{-\Big[\bar{\alpha}\\
&\quad-\mu\sum_{i=1}^n\lambda_i^2\Big]U_2^2(a)+\left(\theta+\frac{\sigma}{\varpi^2}\right)U_2^{1.5}(S)\\
&\quad+\left[1.5\sum_{i=1}^n\lambda_i^2+\frac{\sigma}{2\varpi}\right]U_2(S)+\theta U_2^{0.5}(S)\Big\}.
\end{aligned}$$

Let $\vartheta>0$ be a constant obeying

$$\bar{\alpha}-\mu\sum_{i=1}^n\lambda_i^2-\frac{\vartheta}{2\mu}>0. \tag{12}$$

Set

$$U_4(S)=e^{\vartheta t}U_3(S).$$

By Itô's formula,

$$\mathbb{E}U_4(S(t))=U_3(S(0))+\mathbb{E}\int_0^t\mathcal{L}U_4(S(s))\mathrm{d}s,$$

where

$$\begin{aligned}
&\mathcal{L}U_4(S)\\
&=\vartheta e^{\vartheta t}(1+U_2(S))^{\mu}+e^{\vartheta t}\mathcal{L}U_3(S)\\
&\leq 2e^{\vartheta t}\mu(1+U_2(S))^{\mu-2}\\
&\quad\times\left\{-\left[\bar{\alpha}-\mu\sum_{i=1}^n\lambda_i^2-\frac{\vartheta}{2\mu}-\epsilon\right]U_2^2(S)\right.\\
&\quad+\left(\theta+\frac{\sigma}{\varpi^2}\right)U_2^{1.5}(S)\\
&\quad+\left[1.5\sum_{i=1}^n\lambda_i^2+\frac{\sigma}{2\varpi}+\frac{\vartheta}{\mu}\right]U_2(S)\\
&\quad\left.+\theta U_2^{0.5}(S)+\frac{\vartheta}{2\mu}\right\}\\
&=: e^{\vartheta t}g(S).
\end{aligned}$$

In view of (12),

$$\tilde{g}:=\sup_{S>0}g(S)<+\infty.$$

Thus,

$$\mathbb{E}[e^{\vartheta t}(1+U_2(S))^\mu] \le (1+S^{-2}(0))^\mu + \vartheta^{-1}\tilde{g}(e^{\vartheta t}-1).$$

That is to say,

$$\begin{aligned}&\limsup_{t\to+\infty}\mathbb{E}[S^{-2\mu}(t)] = \limsup_{t\to+\infty}\mathbb{E}[U_2^\mu(S(t))]\\ &\le \limsup_{t\to+\infty}\mathbb{E}[(1+U_2(S(t)))^\mu] \le \tilde{g}.\end{aligned} \tag{13}$$

For any $\epsilon > 0$, define $g_1 = (\epsilon/\tilde{g})^{\frac{1}{2\mu}}$. By Chebyshev's inequality,

$$\begin{aligned}&\mathrm{P}\{S(t) < g_1\} = \mathrm{P}\{S^{-2\mu}(t) > g_1^{-2\mu}\}\\ &\le \frac{\mathbb{E}[S^{-2\mu}(t)]}{g_1^{-2\mu}} = g_1^{2\mu}\mathbb{E}[S^{-2\mu}(t)].\end{aligned}$$

Hence,

$$\limsup_{t\to+\infty}\mathrm{P}\{S(t) < g_1\} \le \epsilon.$$

As a result,

$$\liminf_{t\to+\infty}\mathrm{P}\{S(t) \ge g_1\} \ge 1-\epsilon.$$

In order to testify

$$\liminf_{t\to+\infty}\mathrm{P}\{S(t) \le g_2\} \ge 1-\epsilon, \tag{14}$$

set

$$U_5(S) = S^f,\ S>0,\ f>0.$$

By Itô's formula,

$$\begin{aligned}&\mathrm{d}(e^tU_5(S))\\ &= f\sum_{i=1}^n \lambda_i e^t S^f \mathrm{d}\psi_i(t) + e^t\Big\{S^f\\ &+fS^f\Big[b-\theta S-\frac{\sigma S}{\varpi^2+S^2}+\frac{f-1}{2}\sum_{i=1}^n\lambda_i^2\Big]\Big\}\mathrm{d}t\\ &\le e^tg_3\mathrm{d}t + fe^tS^f\sum_{i=1}^n\lambda_i\mathrm{d}\psi_i(t),\end{aligned}$$

where $g_3 > 0$ is a constant. Thus,

$$\limsup_{t\to+\infty}\mathbb{E}[S^f(t)] \le g_3. \tag{15}$$

By Chebyshev's inequality, one has (14). ■

IV. Upper- and lower-growth rates

Theorem 6. *For model (3), one has*

$$\limsup_{t\to+\infty}\frac{\ln S(t)}{\ln t} \le 1, \quad a.s.. \tag{16}$$

Proof: By Itô's formula,

$$\begin{aligned}\mathrm{d}[e^t\ln S] &= e^t\Big[\ln S + b - \frac{\sum_{i=1}^n\lambda_i^2}{2} - \theta S\\ &-\frac{\sigma S}{\varpi^2+S^2}\Big]\mathrm{d}t + \sum_{i=1}^n\lambda_i e^t\mathrm{d}\psi_i(t).\end{aligned}$$

Thus,

$$\begin{aligned}e^t\ln S(t)/S(0) &= \int_0^t e^s\Big[\ln S(s) + b - \frac{\sum_{i=1}^n\lambda_i^2}{2}\\ &-\theta S(s) - \frac{\sigma S(s)}{\varpi^2+S^2(s)}\Big]\mathrm{d}s + \Lambda_2(t),\end{aligned} \tag{17}$$

where

$$\Lambda_2(t) = \int_0^t\sum_{i=1}^n\lambda_i e^s\mathrm{d}\psi_i(s).$$

That is to say,

$$\langle\Lambda_2(t),\Lambda_2(t)\rangle = \int_0^t e^{2s}\sum_{i=1}^n\lambda_i^2\mathrm{d}s.$$

By the exponential martingale inequality, for any $\tau > 1$ and $\zeta > 0$,

$$\begin{aligned}&\mathrm{P}\Big\{\sup_{0\le t\le\zeta m}\Big[\Lambda_2(t) - \frac{e^{-\zeta m}}{2}\langle\Lambda_2(t),\Lambda_2(t)\rangle\Big]\\ &> \tau e^{\zeta m}\ln m\Big\} \le m^{-\tau}.\end{aligned}$$

According to Borel-Cantelli's lemma, for almost all $\omega\in\Omega$, there is a m_2 such that for any $m \ge m_2$,

$$\Lambda_2(t) \le \frac{e^{-\zeta m}}{2}\langle\Lambda_2(t),\Lambda_2(t)\rangle + e^{\zeta m}\ln m, \quad 0\le t\le\zeta m.$$

Hence for $m\ge m_2,\ 0\le t\le\zeta m$,

$$\Lambda_2(t) \le \frac{e^{-\zeta m}}{2}\int_0^t e^{2s}\sum_{i=1}^n\lambda_i^2\mathrm{d}s + \tau e^{\zeta m}\ln m.$$

By (17), for $m\ge m_2,\ 0\le t\le\zeta m$,

$$\begin{aligned}&e^t\ln S(t) - \ln S(0)\\ &\le \tau e^{\zeta m}\ln m + \int_0^t e^s\Big[\ln S + b - \frac{\sum_{i=1}^n\lambda_i^2}{2} - \theta S\\ &-\frac{\sigma S(s)}{\varpi^2+S^2(s)}\Big]\mathrm{d}s + \frac{e^{-\zeta m}}{2}\sum_{i=1}^n\lambda_i^2\int_0^t e^{2s}\mathrm{d}s\\ &= \int_0^t e^s\Big[\ln S + b - \theta S - \frac{1-e^{s-\zeta m}}{2}\sum_{i=1}^n\lambda_i^2\Big]\mathrm{d}s\\ &\ + \tau e^{\zeta m}\ln m\\ &\le \int_0^t e^s\Big[\ln S(s) + b - \theta S(s)\Big]\mathrm{d}s + \tau e^{\zeta m}\ln m\\ &\le g_4(e^t-1) + \tau e^{\zeta m}\ln m,\end{aligned}$$

where $g_4 > 0$ is a constant. Then for $0 < \zeta(m-1) \le t \le \zeta m$ and $m \ge m_2$,

$$\begin{aligned}\frac{\ln S(t)}{\ln t} &\le \frac{e^{-t}\ln S(0)}{\ln t} + \frac{g_4(1-e^{-t})}{\ln t}\\ &+\frac{\tau e^{-\zeta(m-1)}e^{\zeta m}\ln m}{\ln(\zeta(m-1))}.\end{aligned}$$

As a result,

$$\limsup_{t\to+\infty}\frac{\ln S(t)}{\ln t} \le \tau e^\zeta.$$

Letting $\tau\to 1$ and $\zeta\to 0$ yields (16). ■

Theorem 6 explores the upper-growth rate of $S(t)$, now we test the lower-growth rate of $S(t)$.

Theorem 7. *If $\bar{\alpha} > 0$, then*

$$\liminf_{t\to+\infty}\frac{\ln S(t)}{\ln t} \ge -\frac{1}{2\mu}. \tag{18}$$

Proof: By (13), there is an $g_5 > 0$ satisfying

$$\mathbb{E}\Big[\Big(1+U_2(S(t))\Big)^\mu\Big] \le g_5, \quad t\ge 0. \tag{19}$$

By Itô formula,

$$\begin{aligned}
&\mathrm{d}[(1+U_2(S))^\mu] \\
&= 2\mu(1+U_2(S))^{\mu-2}\bigg\{\bigg[\frac{\sum_{i=1}^n \lambda_i^2}{2} - b + \frac{\sigma S}{\varpi^2+S^2} \\
&\quad + \mu\sum_{i=1}^n \lambda_i^2\bigg]U_2^2(S) + \theta U_2^{1.5}(S) + \bigg[-b \\
&\quad + 1.5\sum_{i=1}^n \lambda_i^2 + \frac{\sigma S}{\varpi^2+S^2}\bigg]U_2(S) + \theta U_2^{0.5}(S)\bigg\} \\
&\quad - 2\mu(1+U_2(S))^{\mu-1}\sum_{i=1}^n \lambda_i U_2(S)\mathrm{d}\psi_i(t) \\
&\le 2\mu(1+U_2(S))^{\mu-2}\Big\{\varrho_1 U_2^2(S) + \varrho_2 U_2^{1.5}(S) \\
&\quad + \varrho_3 U_2(S) + \varrho_2 U_2^{0.5}(S)\Big\} \\
&\quad - 2\mu(1+U_2(S))^{\mu-1}\sum_{i=1}^n \lambda_i U_2(S)\mathrm{d}\psi_i(t),
\end{aligned}$$

where

$$\varrho_1 = -b + \left(\mu + \frac{1}{2}\right)\sum_{i=1}^n \lambda_i^2 + \frac{\sigma}{2\varpi},\ \varrho_2 = \theta,$$

$$\varrho_3 = -b + 1.5\sum_{i=1}^n \lambda_i^2 + \frac{\sigma}{2\varpi}.$$

Let g_6 be a positive constant obeying

$$\begin{aligned}
&2\mu(\varrho_1 U_2^2(S) + \varrho_2 U_2^{1.5}(S) + \varrho_3 U_2(S) + \varrho_2 U_2^{0.5}(S)) \\
&\le g_6(1+U_2(S))^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{d}((1+U_2(S))^\mu) \le g_6(1+U_2(S))^\mu \mathrm{d}t \\
&-2\mu(1+U_2(S))^{\mu-1}\sum_{i=1}^n \lambda_i U_2(S)\mathrm{d}\psi_i(t).
\end{aligned} \tag{20}$$

Let κ be a positive constant obeying

$$g_6\kappa + 12\mu\kappa^{0.5}\sum_{i=1}^n \lambda_i < \frac{1}{2}. \tag{21}$$

Let $L = 1, 2....$ In view of (20),

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}(1+U_2(S(t))^\mu\bigg) \\
&\le \mathbb{E}\Big(1+U_2(S((L-1)\kappa))\Big)^\mu + g_6 \\
&\times\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\bigg|\int_{(L-1)\kappa}^t \Big(1+U_2(S(s))\Big)^\mu \mathrm{d}s\bigg|\bigg) \\
&+2\mu\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\bigg|\int_{(L-1)\kappa}^t \Big(1+U_2(S(s))\Big)^{\mu-1} \\
&\times\bigg[\sum_{i=1}^n \lambda_i U_2(S(s))\mathrm{d}\psi_i(s)\bigg]\bigg|\bigg).
\end{aligned} \tag{22}$$

By Burkholder-Davis-Gundy's inequality

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\bigg|\int_{(L-1)\kappa}^t \Big(1+U_2(S(s))\Big)^{\mu-1} \\
&\quad\times\bigg[\sum_{i=1}^n \lambda_i U_2(S(s))\mathrm{d}\psi_i(s)\bigg]\bigg|\bigg) \\
&\le \mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\bigg|\int_{(L-1)\kappa}^t \Big(1+U_2(S(s))\Big)^{\mu-1} \\
&\quad\times\bigg[\sum_{i=1}^n \lambda_i U_2(S(s))\bigg]\mathrm{d}\psi_i(s)\bigg|\bigg) \\
&\le 6\mathbb{E}\bigg(\int_{(L-1)\kappa}^{L\kappa}\Big(1+U_2(S(s))\Big)^{2\mu-2} \\
&\quad\times\bigg[\sum_{i=1}^n \lambda_i U_2(S(s))\bigg]^2\mathrm{d}s\bigg)^{0.5} \\
&\le 6\kappa^{0.5}\sum_{i=1}^n \lambda_i \times \mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\Big(1+U_2(S(t))\Big)^\mu\bigg).
\end{aligned} \tag{23}$$

In addition,

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\bigg|\int_{(L-1)\kappa}^t \Big(1+U_2(S(s))\Big)^\mu \mathrm{d}s\bigg|\bigg) \\
&\le \mathbb{E}\bigg(\int_{(L-1)\kappa}^{L\kappa}\bigg|\Big(1+U_2(S(s))\Big)^\mu\bigg|\mathrm{d}s\bigg) \\
&\le \kappa\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\Big(1+U_2(S(t))\Big)^\mu\bigg).
\end{aligned} \tag{24}$$

According to (22), (23) and (24),

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}(1+U_2(S(t))^\mu\bigg) \\
&\quad\le \mathbb{E}\Big(1+U_2\Big(S((L-1)\kappa)\Big)\Big)^\mu \\
&\quad\quad + \bigg[g_6\kappa + 12\mu\kappa^{0.5}\sum_{i=1}^n \lambda_i\bigg] \\
&\quad\times\ \mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}\Big(1+U_2(S(t))\Big)^\mu\bigg).
\end{aligned}$$

We then deduce from (19) and (21) that

$$\mathbb{E}\bigg(\sup_{(L-1)\kappa\le t\le L\kappa}(1+U_2(S(t))^\mu\bigg) \le 2g_5.$$

For any $\epsilon > 0$, it follows from Chebyshev's inequality that

$$\mathrm{P}\bigg\{\sup_{(L-1)\kappa\le t\le L\kappa}\Big(1+U_2(S)\Big)^\mu > (L\kappa)^{1+\epsilon}\bigg\} \le \frac{2g_5}{(L\kappa)^{1+\epsilon}}.$$

Then Borel-Cantelli's lemma means that for almost all $\omega \in \Omega$, there is an integer N_0 such that for any $N \ge N_0$ and $(L-1)\kappa \le t \le L\kappa$,

$$\frac{\ln(1+U_2(S(t))^\mu}{\ln t} \le \frac{(1+\epsilon)\ln(L\kappa)}{\ln((L-1)\kappa)}.$$

Thus,

$$\limsup_{t\to+\infty}\frac{\ln(1+U_2(S(t))^\mu}{\ln t} \le 1+\epsilon.$$

Letting $\epsilon \to 0$ gives

$$\limsup_{t\to+\infty}\frac{\ln(1+U_2(S(t))^\mu}{\ln t} \le 1.$$

Hence,

$$\limsup_{t\to+\infty}\frac{\ln(S^{-2\mu}(t))}{\ln t}\le 1,$$

which means (18). ■

V. GLOBAL ASYMPTOTIC STABILITY (GAS)

Definition 1. *Eq. (3) is called GAS if*

$$\lim_{t\to+\infty}|S(s_1;t)-S(s_2;t)|=0,$$

where $S(s_1;t)$ and $S(s_2;t)$ are two arbitrary solutions of Eq. (3) with initial data $S(0)=s_1>0$ and $S(0)=s_2>0$, respectively.

Lemma 1. *([6]) Let $X(t)$ be an $n-$dimensional stochastic process which obeys*

$$\mathbb{E}|X(t)-X(s)|^{\alpha_1}\le c|t-s|^{1+\alpha_2}, 0\le s,t<\infty$$

for some constants $\alpha_1>0,\alpha_2>0$ and $c>0$. Then almost each sample path of $X(t)$ is locally uniformly Hölder continuous with exponent $\theta\in(0,\alpha_2/\alpha_1)$.

Lemma 2. *Almost each sample path of $S(t)$ is uniformly continuous.*

Proof: By (15), for any $f>0$, there is a $G_1(f)$ such that

$$\mathbb{E}|S(t)|^f\le G_1(f).$$

Rewritten Eq. (3) gives

$$S(t)=S(0)+\int_0^t S\left[b-\theta S(t)-\frac{\sigma S(t)}{\varpi+S^2(t)}\right]\mathrm{d}s+\sum_{i=1}^n\lambda_i\int_0^t S(s)\mathrm{d}\psi_i(s).$$

One can see that

$$\begin{aligned}
&\mathbb{E}\left|S\left[b-\theta S-\frac{\sigma S(t)}{\varpi+S^2}\right]\right|^f\\
&=\mathbb{E}\left[|S|^f\left|b-\theta S-\frac{\sigma S(t)}{\varpi+S^2}\right|^f\right]\\
&\le 0.5\mathbb{E}|S|^{2f}+0.5\mathbb{E}\left|b-\theta S-\frac{\sigma S(t)}{\varpi+S^2}\right|^{2f}\\
&\le 0.5\left\{G_1(2f)+3^{2f-1}\left[|b|^{2f}+\theta^{2f}\mathbb{E}|S|^{2f}+\frac{\sigma^{2f}}{\varpi^f}\right]\right\}\\
&\le 0.5\left\{G_1(2f)+3^{2f-1}\left[|b|^{2f}+\theta^{2f}G_1(2f)+\frac{\sigma^{2f}}{\varpi^f}\right]\right\}\\
&=G_2(f).
\end{aligned}$$

Additionally, by the moment inequality for stochastic integrals, it follows that for $0\le t_1\le t_2$ and $f>2$,

$$\begin{aligned}
&\mathbb{E}\left|\int_{t_1}^{t_2}\sum_{i=1}^n\lambda_i S(s)\mathrm{d}\psi_i(s)\right|^f\\
&\le n^{f-1}\sum_{i=1}^n\mathbb{E}\left|\int_{t_1}^{t_2}\lambda_i S(s)\mathrm{d}\psi_i(s)\right|^f\\
&\le n^{f-1}\sum_{i=1}^n\lambda_i^{2f}\left[\frac{f^2-f}{2}\right]^{f/2}(t_2-t_1)^{\frac{f-2}{2}}\int_{t_1}^{t_2}\mathbb{E}|S|^f\mathrm{d}s\\
&\le n^{f-1}\sum_{i=1}^n[\lambda_i^2]^f\left[\frac{f(f-1)}{2}\right]^{f/2}(t_2-t_1)^{\frac{p}{2}}G_1(f).
\end{aligned}$$

As a result, for $0<t_1<t_2<\infty$, $t_2-t_1\le 1$, $1/f+1/\tilde{f}=1$, we have

$$\begin{aligned}
&\mathbb{E}(|S(t_2)-S(t_1)|^f)=\mathbb{E}\left|\int_{t_1}^{t_2}S(s)\left[b-\theta S(s)\right.\right.\\
&\left.\left.-\frac{\sigma S(s)}{\varpi+S^2(s)}\right]\mathrm{d}s+\int_{t_1}^{t_2}S(s)\sum_{i=1}^n\lambda_i\mathrm{d}\psi_i(s)\right|^f\\
&\le 2^{f-1}\mathbb{E}\left|\int_{t_1}^{t_2}S(s)\left[b-\theta S(s)-\frac{\sigma S(s)}{\varpi+S^2(s)}\right]\mathrm{d}s\right|^f\\
&+2^{f-1}\mathbb{E}\left|\int_{t_1}^{t_2}S(s)\sum_{i=1}^n\lambda_i\mathrm{d}\psi_i(s)\right|^f\\
&\le 2^{f-1}(t_2-t_1)^{f/\tilde{f}}\int_{t_1}^{t_2}\mathbb{E}\left|S\left[b-\theta S-\frac{\sigma S}{\varpi+S^2}\right]\right|^p\mathrm{d}s\\
&+2^{f-1}n^{f-1}\sum_{i=1}^n\lambda_i^{2f}\left[\frac{f^2-f}{2}\right]^{f/2}(t_2-t_1)^{f/2}G_1(f)\\
&\le 2^{f-1}(t_2-t_1)^{f/\tilde{f}+1}G_2(f)\\
&+2^{f-1}n^{f-1}\sum_{i=1}^n\lambda_i^{2f}\left[\frac{f^2-f}{2}\right]^{f/2}(t_2-t_1)^{f/2}G_1(f)\\
&\le 2^{f-1}(t_2-t_1)^{f/2}\left[(t_2-t_1)^{f/2}+(\tfrac{f^2-f}{2})^{f/2}\right]G_3(f)\\
&\le 2^{f-1}(t_2-t_1)^{f/2}\left[(1+(\tfrac{f^2-f}{2})^{f/2}\right]G_3(f),
\end{aligned}$$

where

$$G_3(f)=\max\{G_2(f),n^{f-1}\sum_{i=1}^n[\lambda_i^2]^fG_1(f)\}.$$

According to Lemma 1, almost each sample path of $S(t)$ is locally uniformly Hölder-continuous with exponent $\theta\in(0,\frac{f-2}{2f})$. ■

Theorem 8. *If*

$$\theta>\sigma/\varpi, \tag{25}$$

then Eq. (3) is GAS.

Proof: Define

$$W(t)=|\ln S(s_1;t)-\ln S(s_2;t)|,$$

Computation $\mathrm{d}^+W(t)$, we have

$$\begin{aligned}
&\mathrm{d}^+W(t) = \mathrm{sgn}(S(s_1;t) - S(s_2;t)) \\
&\times\bigg\{\bigg[\frac{\mathrm{d}S(s_1;t)}{S(s_1;t)} - \frac{(\mathrm{d}S(s_1;t))^2}{2S^2(s_1;t))}\bigg] \\
&-\bigg[\frac{\mathrm{d}S(s_2;t)}{S(s_2;t)} - \frac{(\mathrm{d}S(s_2;t))^2}{2S^2(s_2;t)}\bigg]\bigg\} \\
&= \mathrm{sgn}(S(s_1) - S(s_2))\bigg\{-\theta[S(s_1) - S(s_2)] \\
&-\sigma\frac{(\varpi - S(s_1;t)S(s_2;t))(S(s_1;t) - S(s_2;t))}{(\varpi + S^2(s_1;t))(\varpi + S^2(s_2;t))}\bigg\}\mathrm{d}t \\
&\leq \bigg\{-\theta|S(s_1;t) - S(s_2;t)| \\
&+\sigma\frac{\varpi + S(s_1)S(s_2)}{(\varpi + S^2(s_1))(\varpi + S^2(s_2))}|S(s_1) - S(s_2)|\bigg\}\mathrm{d}t \\
&= \bigg\{-\theta \\
&+\frac{\sigma(\varpi + S(s_1)S(s_2))}{\varpi^2 + \varpi(S^2(s_1) + S^2(s_2)) + S^2(s_1)S^2(s_2;t)}\bigg\} \\
&\quad\times |S(s_1;t) - S(s_2;t)|\mathrm{d}t \\
&\leq \bigg\{-\theta + \frac{\sigma(\varpi + S(s_1;t)S(s_2;t))}{\varpi^2 + 2\varpi S(s_1;t)S(s_2;t)}\bigg\} \\
&\quad\times |S(s_1;t) - S(s_2;t)|\mathrm{d}t \\
&\leq \bigg\{-\theta + \frac{\sigma(\varpi + S(s_1;t)S(s_2;t))}{\varpi^2 + \varpi S(s_1;t)S(s_2;t)}\bigg\} \\
&\quad\times |S(s_1;t) - S(s_2;t)|\mathrm{d}t \\
&= \bigg\{-\theta + \frac{\sigma}{\varpi}\bigg\}|S(s_1;t) - S(s_2;t)|\mathrm{d}t.
\end{aligned}$$

As a result,

$$W(t) \leq W(0) - \int_0^t \bigg[\theta - \frac{\sigma}{\varpi}\bigg]\big|S(s_1;\tau) - S(s_2;\tau)\big|\mathrm{d}\tau.$$

Therefore,

$$W(t) + \int_0^t \bigg[\theta - \frac{\sigma}{\varpi}\bigg]\big|S(s_1;\tau) - S(s_2;\tau)\big|\mathrm{d}\tau \leq W(0) < \infty.$$

Hence by $V(t) \geq 0$ and (25),

$$|S(s_1;t) - S(s_2;t)| \in L^1[0,\infty)$$

According to Lemma 2 and Barbalat's result [3], we obtain the required assertion. ■

VI. Explicit density function of the invariant measure

The explicit density function of the invariant measure can test the growth of the budworm more accurately. Thus in this part, we test this problem.

Theorem 9. *If $\bar{\alpha} > 0$, then the density function of the invariant measure is*

$$\rho(x) = \frac{\bar{\alpha}(x)}{\int_0^{+\infty}\bar{\alpha}(\tau)\mathrm{d}\tau}, \tag{26}$$

where

$$\begin{aligned}
\bar{\alpha}(x) &= x^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\times\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\},
\end{aligned}$$

$$\overline{\lambda^2} = \sum_{i=1}^n \lambda_i^2.$$

Proof: Define

$$\phi_1(\tau) = \tau\bigg(b - \theta\tau - \frac{\sigma\tau}{\varpi + \tau^2}\bigg),\ \phi_2^2(\tau) = \overline{\lambda^2}\tau^2,$$

$$\beta(x) = \exp\bigg\{-\int_a^x \frac{2\phi_1(\tau)}{\phi_2^2(\tau)}\mathrm{d}\tau\bigg\},$$

where a is an arbitrary positive constant. It then follows that

$$\begin{aligned}
\beta(x) &= \exp\bigg\{-\frac{2}{\overline{\lambda^2}}\int_a^x\bigg[\frac{b}{\tau} - \theta - \frac{\sigma}{\varpi + \tau^2}\bigg]\mathrm{d}\tau\bigg\} \\
&= \exp\bigg\{-\frac{2}{\overline{\lambda^2}}\bigg[b(\ln x/a) - \theta(x - a) \\
&\quad - \frac{\sigma}{\sqrt{\varpi}}\bigg(\arctan\frac{x}{\sqrt{\varpi}} - \arctan\frac{a}{\sqrt{\varpi}}\bigg)\bigg]\bigg\} \\
&= \exp\bigg\{\frac{2}{\overline{\lambda^2}}\bigg[b\ln a - \theta a - \frac{\sigma}{\sqrt{\varpi}}\arctan\frac{a}{\sqrt{\varpi}}\bigg]\bigg\} \\
&\quad\times\exp\bigg\{-\frac{2}{\overline{\lambda^2}}\bigg[b\ln x - \theta x - \frac{\sigma}{\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg]\bigg\} \\
&= \Psi_1 x^{-\frac{2b}{\lambda^2}} \times \exp\bigg\{\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\times\exp\bigg\{\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\} \\
&= \Psi_1 x^{\frac{\overline{\lambda^2}-2b}{\lambda^2}-1} \times \exp\bigg\{\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\times\exp\bigg\{\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\},
\end{aligned}$$

where

$$\Psi_1 = \exp\bigg\{\frac{2}{\overline{\lambda^2}}\bigg[b\ln a - \theta a - \frac{\sigma}{\sqrt{\varpi}}\arctan\frac{a}{\sqrt{\varpi}}\bigg]\bigg\}.$$

Now define

$$\bar{\alpha}(x) = \frac{1}{\phi_2^2(x)\beta(x)}.$$

For sufficiently small $0 < c < 1/\varpi$, one obtains

$$\begin{aligned}
&\int_0^{+\infty}\bar{\alpha}(x)\mathrm{d}x \\
&= \Psi_1^{-1}\lambda^{-2}\int_0^{+\infty} x^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\times\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\}\mathrm{d}x \\
&= \Psi_1^{-1}\lambda^{-2}\int_0^c x^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\}\mathrm{d}x \\
&\quad+\Psi_1^{-1}\lambda^{-2}\int_c^{1/\varpi} y^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\}\mathrm{d}x \\
&\quad+\Psi_1^{-1}\lambda^{-2}\int_{1/\varpi}^{+\infty} y^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\}\mathrm{d}x.
\end{aligned} \tag{27}$$

It then follows from $2b - \overline{\lambda^2} > 0$ that

$$\begin{aligned}
&\int_0^c x^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\exp\bigg\{-\frac{2\theta}{\overline{\lambda^2}}x\bigg\} \\
&\quad\times\exp\bigg\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\bigg\}\mathrm{d}x \\
&\leq \int_0^c x^{\frac{2b-\overline{\lambda^2}}{\lambda^2}-1}\mathrm{d}x \leq \Psi_2,
\end{aligned} \tag{28}$$

where $\Psi_2 > 0$ is a constant. Due to the fact that

$$x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\Big\{-\frac{2\theta}{\overline{\lambda^2}}x\Big\}\exp\Big\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\Big\}$$

is continuous on $[c, 1/\varpi]$, as a result,

$$\begin{aligned}\int_c^{1/\varpi} & x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\Big\{-\frac{2\theta}{\overline{\lambda^2}}x\Big\} \\ & \times\exp\Big\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\Big\}\mathrm{d}x \le \Psi_3,\end{aligned} \tag{29}$$

where $\Psi_3 > 0$ is a constant. Moreover,

$$\begin{aligned}\int_{1/\varpi}^{+\infty} & x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\Big\{-\frac{2\theta}{\overline{\lambda^2}}x\Big\} \\ & \times\exp\Big\{-\frac{2\sigma}{\overline{\lambda^2}\sqrt{\varpi}}\arctan\frac{x}{\sqrt{\varpi}}\Big\}\mathrm{d}x \\ \le & \int_{1/\varpi}^{+\infty} x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\Big\{-\frac{2\theta}{\overline{\lambda^2}}x\Big\}\mathrm{d}x \\ = & \Big(\frac{\overline{\lambda^2}}{2\theta}\Big)^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}}\int_{2\theta/(\overline{\lambda^2}\varpi)}^{+\infty} x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\big\{-x\big\}\mathrm{d}x \\ \le & \Big(\frac{\overline{\lambda^2}}{2\theta}\Big)^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}}\int_0^{+\infty} x^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}-1}\exp\big\{-x\big\}\mathrm{d}x \\ = & \Big(\frac{\overline{\lambda^2}}{2\theta}\Big)^{\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}}\Gamma\Big(\frac{2b-\overline{\lambda^2}}{\overline{\lambda^2}}\Big) := \Psi_4,\end{aligned} \tag{30}$$

where $\Gamma(\cdot)$ is the Gamma Function ([13]). When (28), (29) and (30) are used in (27), we get

$$\int_0^{+\infty}\bar{\alpha}(x)\mathrm{d}x < +\infty.$$

Define

$$\rho(x) = \frac{\bar{\alpha}(x)}{\int_0^{+\infty}\bar{\alpha}(\tau)\mathrm{d}\tau}.$$

One can see that $\rho(x)$ solves the following forward Kolmogorov equation of Eq. (3) in steady-state

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big(\rho(x)\phi_2^2(x)\big) - 2\frac{\mathrm{d}}{\mathrm{d}x}\big(\rho(x)\phi_1(x)\big) = 0. \tag{31}$$

Therefore, $\rho(x)$ is the density function of the stationary distribution of Eq. (3). ■

VII. Generalizations

In the previous sections, we have probed some dynamical properties of model (3). As a matter of fact, some theoretical findings can be extended. Consider the following stochastic hybrid model

$$\begin{aligned}\mathrm{d}S &= S\Big(b(\varphi) - \theta(\varphi)S - \frac{\sigma(\varphi)S}{\varpi(\varphi)+S^2}\Big)\mathrm{d}t \\ &+ \sum_{i=1}^n \lambda_i(\varphi)S\mathrm{d}\psi_i(t),\end{aligned} \tag{32}$$

where $\varphi = \varphi(t)$ is a continuous-time finite-state Markov chain which is independent with $\psi_i(t)$. Let $\mathcal{L} = \{1, 2, ..., L\}$ represent the finite-state space of $\varphi(t)$, then the mechanism of the model portrayed by Eq. (32) could be illustrated as follows. Hypothesize that in the beginning, $\psi_i(0) = j \in \mathcal{L}$, then Eq. (32) follows

$$\begin{aligned}\mathrm{d}S &= S\Big(b(j) - \theta(j)S - \frac{\sigma(j)S}{\varpi(j)+S^2}\Big)\mathrm{d}t \\ &+ \sum_{i=1}^n \lambda_i(j)S\mathrm{d}\psi_i(t),\end{aligned}$$

until $\varphi(t)$ jumps to a new state, say, $k \in \mathcal{L}$, then Eq. (32) follows

$$\begin{aligned}\mathrm{d}S &= S\Big(b(k) - \theta(k)S - \frac{\sigma(k)S}{\varpi(k)+S^2}\Big)\mathrm{d}t \\ &+ \sum_{i=1}^n \lambda_i(k)S\mathrm{d}\psi_i(t),\end{aligned}$$

until $\varphi(t)$ jumps again.

For Eq. (32), we have the following results. To begin with, similar to the proof of Theorem 1, one can testify that

Theorem 10. *For any $(S(0), \varphi(0)) \in (0, +\infty)\times\mathcal{L}$, Eq. (32) has a unique global solution $(S(t), \varphi(t)) \in (0, +\infty)\times\mathcal{L}$ a.s.*

Theorem 11. *If $\widetilde{\alpha} < 0$, then the species portrayed by Eq. (32) goes to extinction, where*

$$\widetilde{\alpha} = \sum_{j=1}^L \pi_j\alpha(j),\ \alpha(\varphi) = b(\varphi) - \frac{1}{2}\sum_{i=1}^n\lambda_i^2(\varphi),$$

and π is the unique stationary probability distribution of $\psi(t)$.

Proof: By Itô's formula,

$$\begin{aligned}\mathrm{d}\ln S &= \Big[\alpha(\varphi) - \theta(\varphi)S - \frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}\Big]\mathrm{d}t \\ &+ \sum_{i=1}^n\lambda_i(\varphi)\mathrm{d}\psi_i(t).\end{aligned}$$

As a result,

$$\begin{aligned}\ln S(t)/S(0) &= \int_0^t\alpha(\varphi(s))\mathrm{d}s \\ -\int_0^t &\Big[\theta(\varphi(s))S(s) + \frac{\sigma(\varphi(s))S(s)}{\varpi^2(\varphi(s))+S^2(s)}\Big]\mathrm{d}s + \widetilde{\Lambda}(t),\end{aligned} \tag{33}$$

where

$$\widetilde{\Lambda}(t) = \sum_{i=1}^n\int_0^t\lambda_i(\varphi(s))\mathrm{d}\psi_i(s).$$

Notice that

$$\lim_{t\to+\infty}t^{-1}\widetilde{\Lambda}(t) = 0, \quad a.s.. \tag{34}$$

We then deduce from (33) and (3) that

$$\begin{aligned}&\ln S(t) - \ln S(0) \\ &\le \int_0^t\alpha(\varphi(s))\mathrm{d}s - \int_0^t\theta(\varphi(s))S(s)\mathrm{d}s + \widetilde{\Lambda}(t) \\ &\le \int_0^t\alpha(\varphi(s))\mathrm{d}s - \check{\theta}\int_0^t S(s)\mathrm{d}s + \widetilde{\Lambda}(t),\end{aligned} \tag{35}$$

where $\check{\theta} = \min_{j\in\mathcal{L}}\{\theta(j)\}$. It follows that,

$$t^{-1}\Big\{\ln S(t) - \ln S(0)\Big\} \le t^{-1}\int_0^t\alpha(\varphi)\mathrm{d}s + t^{-1}\widetilde{\Lambda}(t).$$

According to (34) and

$$\lim_{t\to+\infty}t^{-1}\int_0^t\alpha(\varphi(s))\mathrm{d}s = \widetilde{\alpha},$$

we get
$$\limsup_{t\to+\infty} t^{-1}\ln S(t) \le \widetilde{\alpha} < 0.$$
As a result, $\lim\limits_{t\to+\infty} S(t)=0$, a.s.. ∎

Theorem 12. *If $\widetilde{\alpha}\ge 0$, then*
$$\limsup_{t\to+\infty} t^{-1}\int_0^t S(s)\mathrm{d}s \le \frac{\widetilde{\alpha}}{\breve{\theta}},\quad a.s.. \tag{36}$$
Particularly, if $\widetilde{\alpha}=0$, then the species portrayed by Eq. (32) is non-persistent in the mean.

Proof: For $\forall\epsilon>0$, there is an $\tilde{T}>0$ such that for $t>\tilde{T}$,
$$t^{-1}\left[\ln S(0)+\int_0^t \alpha(\varphi(s))\mathrm{d}s+\widetilde{\Lambda}(t)\right]\le\widetilde{\alpha}+\epsilon.$$
Set $\widetilde{\xi}=\widetilde{\alpha}+\epsilon$. In view of (35), for $t>\tilde{T}$,
$$\begin{aligned}\ln S(t) &\le \ln S(0)+\widetilde{\alpha}t-\breve{\theta}\int_0^t S(s)\mathrm{d}s+\widetilde{\Lambda}(t)\\ &\le \widetilde{\xi}t-\breve{\theta}\int_0^t S(s)\mathrm{d}s.\end{aligned}$$
The following proof is similar to that of Theorem 3 and hence is omitted. ∎

Theorem 13. *If $\widetilde{\alpha}>0$, then the species portrayed by Eq. (32) is weakly persistent.*

Proof: Set $\tilde{J}=\{\omega:\lim\limits_{t\to+\infty}S(t,\omega)=0\}$. If $\mathrm{P}\{\tilde{J}\}>0$, then for any $\omega\in\tilde{J}$, $\lim\limits_{t\to+\infty}S(t,\omega)=0$. That is to say,
$$\limsup_{t\to+\infty} t^{-1}[\ln S(t,\omega)-\ln S(0)]<0,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t\left[\theta(\varphi)S(s)+\frac{\sigma(\varphi(s))S(s)}{\varpi^2(\varphi(s))+S^2(s)}\right]\mathrm{d}s=0.$$
We then deduce from (33) and (34) that
$$0>\limsup_{t\to+\infty}t^{-1}\ln S(t,\omega)=\widetilde{\alpha}>0.$$
This is a contradiction. ∎

Theorem 14. *If $\breve{\alpha}>0$, then the species portrayed by Eq. (32) is stochastically permanent.*

Proof: Set
$$\tilde{U}_2(S)=1/S^2,\ S>0.$$
By Itô's formula,
$$\begin{aligned}\mathrm{d}\tilde{U}_2(S)&=2\tilde{U}_2(S)\left[\theta(\varphi)S+\frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}-b(\varphi)\right]\mathrm{d}t\\ &+3\sum_{i=1}^n\lambda_i^2(\varphi)\tilde{U}_2(S)\mathrm{d}t-2\sum_{i=1}^n\lambda_i(\varphi)\tilde{U}_2(S)\mathrm{d}\psi_i(t)\\ &=2\tilde{U}_2(S)\Bigg\{\theta(\varphi)S-b(\varphi)+1.5\sum_{i=1}^n\lambda_i^2(\varphi)\\ &+\frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}\Bigg\}\mathrm{d}t-2\sum_{i=1}^n\lambda_i(\varphi)\tilde{U}_2(S)\mathrm{d}\psi_i(t).\end{aligned}$$
Let $\mu\in(0,1)$ be a constant satisfying
$$\mu<\breve{\alpha}/\sum_{i=1}^n\hat{\lambda}_i^2. \tag{37}$$
Set
$$\tilde{U}_3(S)=(1+\tilde{U}_2(S))^\mu.$$
According to Itô's formula,
$$\mathbb{E}\tilde{U}_3(S(t))=\tilde{U}_3(S(0))+\mathbb{E}\int_0^t\mathcal{L}\tilde{U}_3(S(s))\mathrm{d}s,$$
where
$$\begin{aligned}&\mathcal{L}\tilde{U}_3(S)\\ &=2\mu(1+\tilde{U}_2(S))^{\mu-2}\Bigg\{(\tilde{U}_2(S)+\tilde{U}_2^2(S))\\ &\times\left[\theta(\varphi)S+\frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}-b(\varphi)+1.5\sum_{i=1}^n\lambda_i^2(\varphi)\right]\\ &+(\mu-1)\tilde{U}_2^2(S)\textstyle\sum_{i=1}^n\lambda_i^2(\varphi)\Bigg\}\\ &=2\mu(1+\tilde{U}_2(S))^{\mu-2}\Bigg\{\Bigg[-b(\varphi)+\frac{\sum_{i=1}^n\lambda_i^2(\varphi)}{2}\\ &+\mu\sum_{i=1}^n\lambda_i^2(\varphi)\Bigg]\tilde{U}_2^2(S)+\theta\tilde{U}_2^{1.5}(S)\\ &+\frac{\sigma(\varphi)}{\varpi^2(\varphi)+S^2}\tilde{U}_2^{1.5}(S)+\Bigg[-b(\varphi)+1.5\sum_{i=1}^n\lambda_i^2(\varphi)\\ &+\frac{\sigma S}{\varpi^2(\varphi)+S^2}\Bigg]\tilde{U}_2(S)+\theta(\varphi)\tilde{U}_2^{0.5}(S)\Bigg\}\\ &\le 2\mu(1+\tilde{U}_2(S))^{\mu-2}\Bigg\{\Bigg[-\breve{\alpha}\\ &\quad+\mu\sum_{i=1}^n\hat{\lambda}_i^2\Bigg]\tilde{U}_2^2(S)+\left(\hat{\theta}+\frac{\hat{\sigma}}{\breve{\varpi}^2}\right)\tilde{U}_2^{1.5}(S)\\ &\quad+\left[1.5\sum_{i=1}^n\hat{\lambda}_i^2+\frac{\hat{\sigma}}{2\breve{\varpi}}\right]\tilde{U}_2(S)+\hat{\theta}\tilde{U}_2^{0.5}(S)\Bigg\}.\end{aligned}$$
Let $\vartheta>0$ be a constant obeying
$$\breve{\alpha}-\mu\sum_{i=1}^n\hat{\lambda}_i^2-\frac{\vartheta}{2\mu}>0. \tag{38}$$
Set
$$\tilde{U}_4(S)=e^{\vartheta t}\tilde{U}_3(S).$$
In view of Itô's formula,
$$\mathbb{E}\tilde{U}_4(S(t))=\tilde{U}_3(S(0))+\mathbb{E}\int_0^t\mathcal{L}\tilde{U}_4(S(s))\mathrm{d}s,$$
where
$$\begin{aligned}\mathcal{L}\tilde{U}_4(S)&=\vartheta e^{\vartheta t}(1+\tilde{U}_2(S))^\mu+e^{\vartheta t}\mathcal{L}\tilde{U}_3(S)\\ &\le 2e^{\vartheta t}\mu(1+\tilde{U}_2(S))^{\mu-2}\\ &\quad\times\Bigg\{-\left[\breve{\alpha}-\mu\sum_{i=1}^n\hat{\lambda}_i^2-\frac{\vartheta}{2\mu}-\epsilon\right]\tilde{U}_2^2(S)\\ &\quad+\left(\hat{\theta}+\frac{\hat{\sigma}}{\breve{\varpi}^2}\right)\tilde{U}_2^{1.5}(S)\\ &\quad+\left[1.5\sum_{i=1}^n\hat{\lambda}_i^2+\frac{\hat{\sigma}}{2\breve{\varpi}}+\frac{\vartheta}{\mu}\right]\tilde{U}_2(S)\\ &\quad+\hat{\theta}\tilde{U}_2^{0.5}(S)+\frac{\vartheta}{2\mu}\Bigg\}\\ &=:e^{\vartheta t}h(S).\end{aligned}$$
By virtue of (38),
$$\tilde{h}:=\sup_{S>0}h(S)<+\infty.$$
Thus,
$$\mathbb{E}[e^{\vartheta t}(1+\tilde{U}_2(S))^\mu]\le(1+S^{-2}(0))^\mu+\vartheta^{-1}\tilde{h}(e^{\vartheta t}-1).$$

Consequently,

$$\begin{aligned}&\limsup_{t\to+\infty}\mathbb{E}[S^{-2\mu}(t)]=\limsup_{t\to+\infty}\mathbb{E}[\tilde U_2^{\mu}(S(t))]\\&\le\limsup_{t\to+\infty}\mathbb{E}[(1+\tilde U_2(S(t)))^{\mu}]\le\tilde h.\end{aligned}$$

For any $\epsilon>0$, define $h_1=\left(\epsilon/\tilde h\right)^{\frac{1}{2\mu}}$. In view of Chebyshev's inequality,

$$\begin{aligned}&\mathrm{P}\left\{S(t)<h_1\right\}=\mathrm{P}\left\{S^{-2\mu}(t)>h_1^{-2\mu}\right\}\\&\le\frac{\mathbb{E}[S^{-2\mu}(t)]}{h_1^{-2\mu}}=h_1^{2\mu}\mathbb{E}[S^{-2\mu}(t)].\end{aligned}$$

It follows that,

$$\limsup_{t\to+\infty}\mathrm{P}\left\{S(t)<h_1\right\}\le\epsilon.$$

As a result,

$$\liminf_{t\to+\infty}\mathrm{P}\left\{S(t)\ge h_1\right\}\ge1-\epsilon.$$

No we testify that

$$\liminf_{t\to+\infty}\mathrm{P}\left\{S(t)\le g_2\right\}\ge1-\epsilon.\tag{39}$$

Let

$$\tilde U_5(S)=S^f,\ S>0,\ f>0.$$

According to Itô's formula,

$$\begin{aligned}&\mathrm{d}(e^t\tilde U_5(S))\\&=e^t\left\{S^f+fS^f\left[b(\varphi)-\theta(\varphi)S-\frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}\right.\right.\\&\quad\left.\left.+\frac{f-1}{2}\sum_{i=1}^n\lambda_i^2(\varphi)\right]\right\}\mathrm{d}t\\&\quad+f\sum_{i=1}^n\lambda_i(\varphi)e^tS^f\mathrm{d}\psi_i(t)\\&\le e^t\tilde g_3\mathrm{d}t+fe^tS^f\sum_{i=1}^n\lambda_i(\varphi)\mathrm{d}\psi_i(t),\end{aligned}$$

where $\tilde g_3>0$ is a constant. As a result,

$$\limsup_{t\to+\infty}\mathbb{E}[S^f(t)]\le\tilde g_3.$$

In view of Chebyshev's inequality, one gets (39). ■

Theorem 15. *For model (32), one has*

$$\limsup_{t\to+\infty}\frac{\ln S(t)}{\ln t}\le1,\quad a.s..\tag{40}$$

Proof: We deduce from Itô's formula that

$$\begin{aligned}\mathrm{d}[e^t\ln S]&=e^t\left[\ln S+b(\varphi)-\frac{\sum_{i=1}^n\lambda_i^2(\varphi)}{2}-\theta(\varphi)S\right.\\&\left.-\frac{\sigma(\varphi)S}{\varpi^2(\varphi)+S^2}\right]\mathrm{d}t+\sum_{i=1}^n\lambda_i(\varphi)e^t\mathrm{d}\psi_i(t).\end{aligned}$$

Thus,

$$\begin{aligned}&e^t\ln S(t)-\ln S(0)\\&=\int_0^te^s\left[\ln S(s)+b(\varphi(s))-\frac{\sum_{i=1}^n\lambda_i^2(\varphi(s))}{2}\right.\\&\left.-\theta(\varphi(s))S(s)-\frac{\sigma(\varphi(s))S(s)}{\varpi^2(\varphi(s))+S^2(s)}\right]\mathrm{d}s+\widetilde\Lambda_2(t),\end{aligned}\tag{41}$$

where

$$\widetilde\Lambda_2(t)=\int_0^t\sum_{i=1}^n\lambda_i(\varphi(s))e^s\mathrm{d}\psi_i(s).$$

Therefore,

$$\langle\widetilde\Lambda_2(t),\widetilde\Lambda_2(t)\rangle=\int_0^te^{2s}\sum_{i=1}^n\lambda_i^2(\varphi(s))\mathrm{d}s.$$

By the exponential martingale inequality, for any $\tau>1$ and $\zeta>0$,

$$\begin{aligned}&\mathrm{P}\left\{\sup_{0\le t\le\zeta m}\left[\widetilde\Lambda_2(t)-\frac{e^{-\zeta m}}{2}\langle\widetilde\Lambda_2(t),\widetilde\Lambda_2(t)\rangle\right]>\tau e^{\zeta m}\ln m\right\}\\&\le m^{-\tau}.\end{aligned}$$

It then follows from Borel-Cantelli's lemma that for almost all $\omega\in\Omega$, there is a m_2 such that for any $m\ge m_2$,

$$\widetilde\Lambda_2(t)\le\frac{e^{-\zeta m}}{2}\langle\widetilde\Lambda_2(t),\widetilde\Lambda_2(t)\rangle+e^{\zeta m}\ln m,\ 0\le t\le\zeta m.$$

Hence for $m\ge m_2,\ 0\le t\le\zeta m$,

$$\widetilde\Lambda_2(t)\le\frac{e^{-\zeta m}}{2}\int_0^te^{2s}\sum_{i=1}^n\lambda_i^2\mathrm{d}s+\tau e^{\zeta m}\ln m.$$

By (41), for $m\ge m_2,\ 0\le t\le\zeta m$,

$$\begin{aligned}&e^t\ln S(t)-\ln S(0)\\&\le\int_0^te^s\left[\ln S(s)+b(\varphi(s))-\frac{\sum_{i=1}^n\lambda_i^2(\varphi(s))}{2}\right.\\&\left.-\theta(\varphi(s))S(s)-\frac{\sigma(\varphi(s))S(s)}{\varpi^2(\varphi(s))+S^2(s)}\right]\mathrm{d}s\\&+\frac{e^{-\zeta m}}{2}\sum_{i=1}^n\int_0^t\lambda_i^2(\varphi(s))e^{2s}\mathrm{d}s+\tau e^{\zeta m}\ln m\\&=\int_0^te^s\left[\ln S(s)+b(\varphi(s))-\theta(\varphi(s))S(s)\right.\\&\left.-\frac{1-e^{s-\zeta m}}{2}\sum_{i=1}^n\lambda_i^2(\varphi(s))\right]\mathrm{d}s+\tau e^{\zeta m}\ln m\\&\le\int_0^te^s\left[\ln S(s)+\hat b-\check\theta S(s)\right]\mathrm{d}s+\tau e^{\zeta m}\ln m\\&\le\widetilde g_4(e^t-1)+\tau e^{\zeta m}\ln m,\end{aligned}$$

where $\widetilde g_4>0$ is a constant. Then for $0<\zeta(m-1)\le t\le\zeta m$ and $m\ge m_2$,

$$\frac{\ln S(t)}{\ln t}\le\frac{e^{-t}\ln S(0)}{\ln t}+\frac{\widetilde g_4(1-e^{-t})}{\ln t}+\frac{\tau e^{-\zeta(m-1)}e^{\zeta m}\ln m}{\ln(\zeta(m-1))}.$$

It follows that,

$$\limsup_{t\to+\infty}\frac{\ln S(t)}{\ln t}\le\tau e^{\zeta}.$$

Letting $\tau\to1$ and $\zeta\to0$ gives (40). ■

VIII. Applications to Spruce Budworm

Now we use the above findings to explore the growth of spruce budworm (*Choristoneura fumiferana* Clemens) in eastern North America. In accordance to [7], [9], $r=1.6$, $\theta=6\times10^{-7}$, $\sigma=3\times10^4$, $\varpi=8\times10^8$, $\overline{\lambda^2}=2.6$. Thus $\alpha=0.3>0$. In view of Theorem 5 and Theorem 9, the species is permanent and has the following explicit density function

$$\rho(x)\frac{\bar\alpha(x)}{\int_0^{+\infty}\bar\alpha(\tau)\mathrm{d}\tau},$$

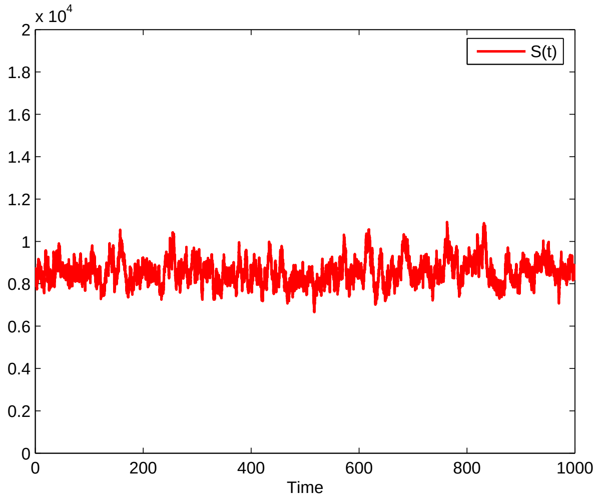

Fig. 1: A sample path of Eq.(3) at $t = 3000$ with $r = 1.6$, $\theta = 6 \times 10^{-7}$, $\sigma = 3 \times 10^4$, $\varpi = 8 \times 10^8$, $\overline{\lambda^2} = 2.6$, step size $\Delta t = 0.01$.

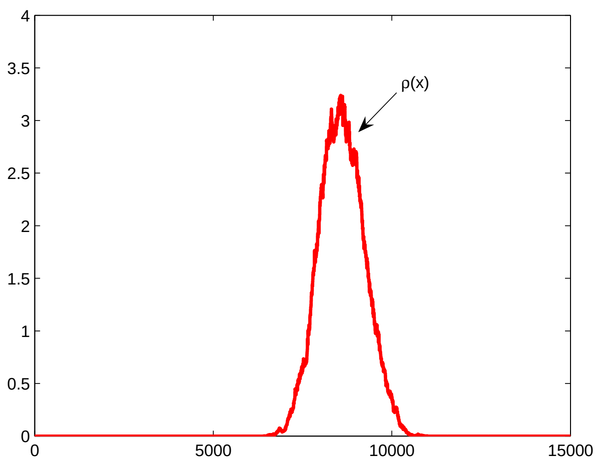

Fig. 2: Density function of Eq.(3) with parameters given in Fig.1.

where

$$\begin{aligned}\bar{\alpha}(x) &= x^{-0.77} \exp\left\{ - 4.62 \times 10^{-7} x \right\} \\ &\quad \times \exp\left\{ - 0.816 \arctan \frac{x}{2.828 \times 10^4} \right\}.\end{aligned}$$

See Fig.1 (a sample path of $S(t)$) and Fig.2 (the density function of $S(t)$ at $t = 3000$).

Remark 1. *In this report, we only consider the effects of stochastic perturbations, it is of interest to consider the effect of time delay ([4], [16]). In addition, one may use the fuzzy approach ([19], [20]) to depict the fluctuations of the parameters. Moreover, this report tests the differential models, one may consider the discrete models ([17], [18]).*

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