Finite-Time Stabilization of General Stochastic Nonlinear Systems with Application to a Liquid-Level System

Wenhui Zhang, Yanling Shang, Quan Sun and Fangzheng Gao

***Abstract*—The problem of finite-time stabilization (FTS) is addressed for a class of p-normal stochastic nonlinear systems. The considered systems are general in sense that they have the powers of both odd and even rational numbers. By skillfully combining sign function with adding a power integrator (AAPI) technique, a continuous state feedback controller which achieve that the closed-loop system (CLS) is finite-time stable in probability (FTS in probability) is given. The application to a liquid-level system are given to confirmed its effectiveness.**

***Index Terms*—p-normal stochastic nonlinear systems, adding a power integrator(AAPI), finite-time stabilization(FTS)**

I. Introduction

As we known, stochastic model has played a major role in the engineering field because stochastic noisefrequently and inevitably produces in various realistic dynamic systems [1]. Since backstepping technique was extended to stochastic settings [2], the increasing efforts toward controller design of stochastic nonlinear systems (SNSs) has been made, see, e.g., [3-6]. However, the above-mentioned results only addressed the asymptotic convergence of system trajectories.

However, in practical application, the finite-time control laws are more desirable since finite-time stable systems enjoy the good properties of faster convergence, stronger robustness and better disturbance attenuation. In the deterministic setting, a Lyapunov stability theorem of finite-time stability was proposed in [7], which offered a basic control design tool for nonlinear control systems, and stimulated a lot of good results on the FTS of nonlinear systems with different structures [8-12]. Recently, on based of the theorem of stochastic finite-time stability achieved in [13], some results on FTS of SNSs have also been developed, refer to [14-17] and references therein. However, note that the considered systems in the above-mentioned literature all require that the powers are odd rational numbers. Although this restriction is very common, there are may practical systems that indeed do not fulfill such assumption, for instance, some powers of the liquid-level system given in [18] is even rational number. Therefore, the interesting questions naturally arises: *for a SNS, if some (or all) power orders are even rational numbers, is it possible to achieve its FTS? If possible, under what conditions and and how can we design such controller?* To our best knowledge, these questions have not been well-addressed in the literature.

Motivated by the above observations, in this paper we devote ourself to solving the problem of FTS for a class of stochastic nonlinear systems with the powers of both odd and even rational numbers using state feedback. The significant contributions of this paper can be summarized as follows. (i) Fully taking into consideration of practical system requirements, the FTS problem of SNSs that have the powers of both odd and even rational numbers is considered. (ii) A weaker sufficient condition on characterizing the powers and the nonlinear functions of SNSs is derived. (iii) By successfully overcoming some essential difficulties such as the weaker assumptions on the power order and the system growth, and the construction of Lyapunov function, a novel method to FTS of SNSs by state feedback is given, rendering a much more general results than the existing ones. (iv) As an application of the presented theoretical result, the problem of FTS of liquid-level system is solved by using the proposed method.

Notations. Throughout this paper, the notations used are fairly standard. Specifically, $\mathbb{R}^{+}_{odd}$:={$q \in \mathbb{R}^{+}$: q is a ratio of odd integers}, $\mathbb{R}^{\geq 1}_{odd}$:= {$q \in \mathbb{R}^{+}_{odd}$: $q \geq 1$ is a ratio of odd integers}. $\mathbb{N}^{\geq 1}_{odd}$:={$q \geq 1$ is an odd integer }.

II. Problem formulation and preliminaries

Consider the following class of p-normal SNSs:

$$\begin{cases} dx_1 = (h_1(t)[x_2]^{p_1} + f_1(\bar{x}_1))dt + g_1^T(\bar{x}_1)dw, \\ dx_2 = (h_2(t)[x_3]^{p_2} + f_2(\bar{x}_2))dt + g_2^T(\bar{x}_2)dw, \\ \quad \vdots \\ dx_{n-1} = (h_{n-1}(t)[x_n]^{p_{n-1}} + f_{n-1}(\bar{x}_{n-1}))dt \\ \qquad + g_i^T(\bar{x}_i)dw, \\ dx_n = (h_n(t)[u]^{p_n} + f_n(\bar{x}_n))dt + g_n^T(\bar{x}_n)dw, \end{cases} \tag{1}$$

where $x = (x_1, \cdots, x_i)^T \in \mathbb{R}^i$, $u \in \mathbb{R}^n$ are the system state and the control input, respectively. $p_i \in \mathbb{R}^+ \setminus \{0\}$, $i = 1, \cdots, n$, are the power orders; w is an q-dimensional independent standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration, and P

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being a probability measure. The drift term $f_i : \mathbb{R}^i \to \mathbb{R}$ and the diffusion term $g_i : \mathbb{R}^i \to \mathbb{R}^q$ are continuous functions with $f_i(0) = 0$ and $g_i(0) = 0$, $i = 1, \cdots, n$.

The aim of this paper is to present a state feedback controller which stabilizes system (1) within finite time under the following wild assumption.

Assumption 1. There are positive constants h_{i1} and h_{i2}, $i = 1, \ldots, n$ such that

$$h_{i1} \leq h_i(t) \leq h_{i2}. \tag{2}$$

Assumption 2. For $i = 1, \cdots, n$, there are smooth functions $\varphi_i(\bar{x}_i) \geq 0$, $\phi_i(\bar{x}_i) \geq 0$ and constants $\tau \in (-1, 0)$, $\lambda_{i,j,1} \geq 0$, $\lambda_{i,j,2} \geq 0$ such that

$$|f_i(\bar{x}_i)| \leq \varphi_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\lambda_{i,j,1} + \frac{r_i + \tau}{r_j}}, \tag{3}$$

and

$$|g_i(\bar{x}_i)| \leq \phi_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\lambda_{i,j,2} + \frac{2r_i + \tau}{2r_j}}, \tag{4}$$

where r_i's are defined as

$$r_1 = 1, \ \ r_{i+1} = \frac{r_i + \tau}{p_i} > 0, \ \ i = 1, \cdots, n. \tag{5}$$

Remark 1. As a Hölder version of the system growth condition, Assumption 2 clearly relaxes and includes the Lipschitz growth condition used in [14-17] as special cases. Furthermore, from Assumption we have there are smooth functions $\bar{\varphi}_i(\bar{x}_i) \geq 0$, $\bar{\phi}_i(\bar{x}_i) \geq 0$ such that

$$|f_i(\bar{x}_i)| \leq \bar{\varphi}_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\frac{r_i + \tau}{r_j}}, \tag{6}$$

and

$$|g_i(\bar{x}_i)| \leq \bar{\phi}_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\frac{2r_i + \tau}{2r_j}}. \tag{7}$$

Consider the following SNSs

$$dx = f(x)dt + g(x)dw, \ \ x(0) = x_0 \in \mathbb{R}^n, \tag{8}$$

where $f(x)$ and $g(x)$ are continuous in x and satisfy $f(0) = 0$, $g(0) = 0$ for all $t \geq 0$.

Definition 1[13]**.** The trivial solution of system (8) is said to be finite-time stable in probability if the stochastic system admits a solution for any initial data $x_0 \in R^n$, denoted by $x(t, x_0)$, and the following statements hold:

(i) Finite-time attractive in probability: For every initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t \mid x(t, x_0) = 0\}$ called stochastic settling time, is finite almost surely, that is, $P\{\tau_{x_0} < \infty\} = 1$.

(ii) Stable in probability: For every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that $P\{|x(t, x_0)| < r, \forall t \geq 0\} \geq 1 - \varepsilon$, whenever $|x_0| < \delta$.

Remark 2. It should be noted that Definition 1 implies that the solution $x(t + \tau_{x_0}; x_0)$ is unique for $t \geq 0$, i.e., for any initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$, if the trivial solution of the stochastic system (1) is finite-time stable in probability and τ_{x_0} is the stochastic settling time, then $x(t; x_0) = 0$ with probability one for $t > \tau_{x_0}$.

Lemma 1[16]**.** For system (8), if there exists a C^2 function function $V : \mathbb{R}^n \to \mathbb{R}^+$, class $\mathcal{K}_\infty$ functions μ_1 and μ_2, real numbers $c > 0$ and $0 < \alpha < 1$, such that for all $t > 0$,

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|), \tag{9}$$

$$\mathcal{L}V(x)|_{(8)} \leq -cV^\alpha(x), \tag{10}$$

then the origin of (8) is finite-time stable in probability (FTS in probability).

Lemma 2[16]**.** For any $\zeta \in \mathbb{R}$, $\eta \in \mathbb{R}$, and $m \geq 1$, there hold: (i)$|\zeta + \eta|^m \leq 2^{m-1}|\zeta^m + \eta^m|$; (ii)$(|\zeta| + |\eta|)^{1/m} \leq |\zeta|^{1/m} + |\eta|^{1/m} \leq 2^{(m-1)/m}(|\zeta| + |\eta|)^{1/m}$.

Lemma 3[16]. For any positive real numbers c, d and any real-valued function $\vartheta(\zeta, \eta) > 0$, one has

$$|\zeta|^c|\eta|^d \leq \frac{c}{c+d}\vartheta(\zeta, \eta)|\zeta|^{c+d} + \frac{d}{c+d}\vartheta^{-c/d}(\zeta, \eta)|\eta|^{c+d}.$$

Lemma 4[16]**.** For any $\zeta, \eta \in \mathbb{R}$ and $0 < p \leq 1$, $a > 0$, one has

$$|\lceil\zeta\rceil^{ap} - \lceil\eta\rceil^{ap}| \leq 2^{1-p}|\lceil\zeta\rceil^a - \lceil\eta\rceil^a|^p.$$

Lemma 5[17]**.** For any positive real numbers $\eta_1, \eta_2, \cdots, \eta_n$ and $p > 0$, one has

$$(\sum_{i=1}^{n} \eta_i)^p \leq b(\sum_{i=1}^{n} \eta_i^p),$$

with $b = n^{p-1}$ if $p \geq 1$ and $b = 1$ if $0 < p < 1$.

Lemma 6[16]**.** $f(z) = \mathrm{sign}(z)|z|^a = [z]^a$ is C^2 for any $a \geq 2$, $z \in \mathbb{R}$. Furthermore, one has $f'(z) = a|z|^{a-1}$ and $f''(z) = a(a-1)[z]^{a-2}$.

III. Finite-time stabilization

In this section, a continuous state feedback controller for system (1) is constructed by employing the AAPI technique.

Step 1. Let $\rho \geq \max_{1 \leq i \leq n}\{2r_i\}$ and $l \geq \frac{5}{8}$ be positive numbers, and choose a Lyapunov function candidate as

$$V_1(x_1) = W_1(x_1) = \frac{r_1}{4l\rho}|x_1|^{\frac{4l\rho}{r_1}}. \tag{11}$$

By Assumption 1, Remark 1 and Lemmas 5–6, a simple computation gives

$$\begin{aligned}
\mathcal{L}V_1 &= h_1[\xi_1]^{\frac{4l\rho - r_1}{\rho}}[x_2]^{p_1} + [\xi_1]^{\frac{4l\rho - r_1}{\rho}} f_1 \\
&\quad + \tfrac{4l\rho - r_1}{2r_1}|\xi_1|^{\frac{(4l-1)\rho - r_1}{\rho}}|x_1|^{\frac{\rho - r_1}{r_1}} g_1^T g_1 \\
&\leq h_1[\xi_1]^{\frac{4l\rho - r_1}{\rho}}([x_2]^{p_1} - [x_2^*]^{p_1}) + h_1[\xi_1]^{\frac{4l\rho - r_1}{\rho}}[x_2^*]^{p_1} \\
&\quad + |\xi_1|^{\frac{4l\rho + \tau}{\rho}}\Big(\bar{\varphi}_1 + \frac{4l\rho - r_1}{2r_1}\bar{\phi}_1^2\Big),
\end{aligned} \tag{12}$$

with $\xi_1 = [x_1]^{\frac{\rho}{r_1}}$.

Obviously, the C^0 virtual controller

$$\begin{aligned}
x_2^* &= -\Big(\frac{2n + 2\bar{\varphi}_1 + 4l\rho - r_1}{2h_{11}}\bar{\phi}_1^2\Big)^{\frac{1}{p_1}}[\xi_1]^{\frac{r_1 + \tau}{p_1\rho}} \\
&:= -\beta_1^{\frac{r_2}{\rho}}(x_1)[\xi_1]^{\frac{r_2}{\rho}},
\end{aligned} \tag{13}$$

with $\beta_1 > 0$ being smooth, results in

$$\mathcal{L}V_1 \leq -n|\xi_1|^{\frac{4l\rho + \tau}{\rho}} + h_1[\xi_1]^{\frac{4l\rho - r_1}{\rho}}([x_2]^{p_1} - [x_2^*]^{p_1}). \tag{14}$$

Inductive Step. Suppose at step $i - 1$, there is a C^2 Lyapunov function V_{i-1}, which is positive definite and proper, and a set of C^0 virtual controllers $x_1^*, \cdots, x_i^*$ defined by

$$\begin{aligned} x_1^* &= 0, & \xi_1 &= [x_1]^{\frac{\rho}{r_1}} - [x_1^*]^{\frac{\rho}{r_1}}, \\ x_2^* &= -\beta_1^{\frac{r_2}{\rho}}(\bar{x}_1)[\xi_1]^{\frac{r_2}{\rho}}, & \xi_2 &= [x_2]^{\frac{\rho}{r_2}} - [x_2^*]^{\frac{\rho}{r_2}}, \\ &\vdots & &\vdots \\ x_i^* &= -\beta_{i-1}^{\frac{r_i}{\rho}}(\bar{x}_{i-1})[\xi_{i-1}]^{\frac{r_i}{\rho}}, & \xi_i &= [x_i]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}, \end{aligned} \tag{15}$$

with a set of smooth functions $\beta_j(\bar{x}_j) > 0$, $j = 1, \cdots, i-1$, such that

$$\begin{aligned} \mathcal{L}V_{i-1} \le & -(n-i+2)\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} \\ & + h_{i-1}[\xi_{i-1}]^{\frac{4l\rho-r_{i-1}}{\rho}}(x_i^{p_{i-1}} - x_i^{*p_{i-1}}). \end{aligned} \tag{16}$$

It is clear that (16) reduces to (14) when $i = 2$. In the following, we want to show that the similar conclusion still holds for step i. To this end, consider the ith Lyapunov function

$$\begin{aligned} V_i(\bar{x}_i) &= V_{i-1}(\bar{x}_{i-1}) + W_i(\bar{x}_i), \\ W_i(\bar{x}_i) &= \int_{x_i^*}^{x_i}\left[[s]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}\right]^{\frac{4l\rho-r_i}{\rho}} ds. \end{aligned} \tag{17}$$

Note the fact that $\rho/r_i \ge 2$ and $(4l-2)\rho \ge r_i$, the following proposition can be obtained by the similar procedure in [16] and [17].

Proposition 1. The Lyapunov function V_i defined by (17) is C^2, positive definite and proper, and moreover, for $j, k = 1, \cdots, i$, $j \ne k$, W_i satisfies

$$\begin{cases} \dfrac{\partial W_i}{\partial x_i} = [\xi_i]^{\frac{4l\rho-r_i}{\rho}}, \\ \dfrac{\partial^2 W_i}{\partial x_i^2} = \dfrac{4l\rho-r_i}{r_i}|\xi_i|^{\frac{(4l-1)\rho-r_i}{\rho}}|x_i|^{\frac{\rho-r_i}{r_i}}, \\ \dfrac{\partial^2 W_i}{\partial x_i \partial x_j} = \dfrac{\partial^2 W_i}{\partial x_j \partial x_i} = -\dfrac{4l\rho-r_i}{\rho}|\xi_i|^{\frac{(4l-1)\rho-r_i}{\rho}}\dfrac{\partial([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_j}, \\ \dfrac{\partial W_i}{\partial x_j} = -\dfrac{4l\rho-r_i}{\rho}\dfrac{\partial([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_j} \\ \times \displaystyle\int_{x_i^*}^{x_i}\left|[s]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}\right|^{\frac{(4l-1)\rho-r_i}{\rho}} ds, \\ \dfrac{\partial^2 W_i}{\partial x_j^2} = \dfrac{4l\rho-r_i}{\rho}\cdot\dfrac{(4l-1)\rho-r_i}{\rho}\Big(\dfrac{\partial([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_j}\Big)^2 \\ \times \displaystyle\int_{x_i^*}^{x_i}\left[[s]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}\right]^{\frac{(4l-2)\rho-r_i}{\rho}} ds \\ -\dfrac{4l\rho-r_i}{\rho}\dfrac{\partial^2([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_j^2} \times \displaystyle\int_{x_i^*}^{x_i}\left|[s]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}\right|^{\frac{(4l-1)\rho-r_i}{\rho}} ds, \\ \dfrac{\partial^2 W_i}{\partial x_j \partial x_k} = \dfrac{4l\rho-r_i}{\rho}\cdot\dfrac{(4l-1)\rho-r_i}{\rho}\dfrac{\partial([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_j}\dfrac{\partial([x_i^*]^{\frac{\rho}{r_i}})}{\partial x_k} \\ \times \displaystyle\int_{x_i^*}^{x_i}\left[[s]^{\frac{\rho}{r_i}} - [x_i^*]^{\frac{\rho}{r_i}}\right]^{\frac{(4l-2)\rho-r_i}{\rho}} ds. \end{cases} \tag{18}$$

Using Proposition 1, it is deduced from (16) that

$$\begin{aligned} \mathcal{L}V_i \le & -(n-i+2)\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} + h_i[\xi_i]^{\frac{4l\rho-r_i}{\rho}}x_{i+1}^{p_i} \\ & + h_{i-1}[\xi_{i-1}]^{\frac{4l\rho-r_{i-1}}{\rho}}(x_i^{p_{i-1}} - x_i^{*p_{i-1}}) + [\xi_i]^{\frac{4l\rho-r_i}{\rho}}f_i \\ & + \sum_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}(h_j x_{j+1}^{p_j} + f_j) + \frac{1}{2}\sum_{j,k=1}^{i}\Big|\frac{\partial^2 W_i}{\partial x_j \partial x_k}\Big||g_j^T g_k|. \end{aligned} \tag{19}$$

Next, to estimate the last four terms on the right-hand side of (19), the following propositions is introduced for system (1) based on Lemma 2–6 , whose proofs are similar those in [18] and thus omitted.

Proposition 2. There exists a positive constant c_{i1}, such that

$$\begin{aligned} & h_{i-1}[\xi_{i-1}]^{\frac{4l\rho-r_{i-1}}{\rho}}(x_i^{p_{i-1}} - x_i^{*p_{i-1}}) \\ & \le \frac{1}{4}|\xi_{i-1}|^{\frac{4l\rho+\tau}{\rho}} + |\xi_i|^{\frac{4l\rho+\tau}{\rho}}c_{i1}. \end{aligned} \tag{20}$$

Proposition 3. There exists a nonnegative smooth function c_{i2} such that

$$[\xi_i]^{\frac{3\rho-r_i}{\rho}}f_i \le \frac{1}{4}\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} + |\xi_i|^{\frac{4l\rho+\tau}{\rho}}c_{i2}. \tag{21}$$

Proposition 4. There exists a nonnegative smooth function $c_{i,3}$ such that

$$\sum_{j=1}^{i-1}\frac{\partial W_i}{\partial x_j}(x_{j+1}^{p_j} + f_j) \le \frac{1}{4}\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} + |\xi_i|^{\frac{4l\rho+\tau}{\rho}}c_{i3}. \tag{22}$$

Proposition 5. There exists a nonnegative smooth function $c_{i,4}$ such that

$$\frac{1}{2}\sum_{j,k=1}^{i}\Big|\frac{\partial^2 W_i}{\partial x_j \partial x_k}\Big||g_j^T g_k| \le \frac{1}{4}\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} + |\xi_i|^{\frac{4l\rho+\tau}{\rho}}c_{i4}. \tag{23}$$

Now, with the help of Propositions 2–5, we substitute (20)–(23) into (19) and obtain that

$$\begin{aligned} \mathcal{L}V_i \le & -(n-i+1)\sum_{j=1}^{i-1}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} \\ & + h_i[\xi_i]^{\frac{4l\rho-r_i}{\rho}}(x_{i+1}^{p_i} - x_{i+1}^{*p_i}) + h_i[\xi_i]^{\frac{4l\rho-r_i}{\rho}}x_{i+1}^{*p_i} \\ & + |\xi_i|^{\frac{4l\rho+\tau}{\rho}}(c_{i1} + c_{i2} + c_{i3} + c_{i4}). \end{aligned} \tag{24}$$

It is easy to see that the virtual controller

$$\begin{aligned} x_{i+1}^* &= -\left(\frac{n-i+1+\sum_{j=1}^{4}c_{ij}}{h_{i1}}\right)^{\frac{1}{p_i}}[\xi_i]^{\frac{r_i+\tau}{p_i\rho}} \\ &:= -\beta_i^{\frac{r_{i+1}}{\rho}}(\bar{x}_i)[\xi_i]^{\frac{r_{i+1}}{\rho}}, \end{aligned} \tag{25}$$

renders

$$\begin{aligned} \mathcal{L}V_i \le & -(n-i+1)\sum_{j=1}^{i}|\xi_j|^{\frac{4l\rho+\tau}{\rho}} \\ & + [\xi_i]^{\frac{4l\rho-r_i}{\rho}}(x_{i+1}^{p_i} - x_{i+1}^{*p_i}). \end{aligned} \tag{26}$$

This completes the inductive step

Hence, at the nth step, one concludes that there is a continuous state feedback controller of the form

$$u = x_{n+1}^* = -\beta_n^{\frac{r_{n+1}}{\rho}}(\bar{x}_n)[\xi_n]^{\frac{r_{n+1}}{\rho}}, \tag{27}$$

such that

$$\mathcal{L}V_n \le -\sum_{j=1}^{n} |\xi_j|^{\frac{4l\rho+\tau}{\rho}}, \tag{28}$$

where

$$\begin{aligned} V_n &= \sum_{j=1}^{n} W_j \\ &= \frac{r_1}{4l\rho}|x_1|^{\frac{4l\rho}{r_1}} + \sum_{j=2}^{n}\int_{x_j^*}^{x_j}\left[[s]^{\frac{\rho}{r_j}} - [x_j^*]^{\frac{\rho}{r_j}}\right]^{\frac{4l\rho-r_j}{\rho}} ds. \end{aligned} \tag{29}$$

The following theorem is given to summarizes the main result of the paper.

Theorem 1. For the SNS (1) with Assumptions 1 and 2, the continuous state feedback controller (27) such that the origin of the CLS is FTS in probability.

Proof. From Proposition 1, it is see that $V_n(x)$ is positive definite and proper. Then, there exist class $\mathcal{K}_\infty$ functions π_1 and π_2 such that

$$\pi_1(|x|) \le V_n(x) \le \pi_2(|x|). \tag{30}$$

Moreover, by using Lemma 4, it is easy to see that

$$\begin{aligned} V_n &= \sum_{j=1}^{n} W_j = W_1 + \sum_{j=2}^{n}\int_{x_j^*}^{x_j}\left[[s]^{\frac{\rho}{r_j}} - [x_j^*]^{\frac{\rho}{r_j}}\right]^{\frac{4l\rho-r_j}{\rho}} ds. \\ &\le \frac{r_1}{4l\rho}|\xi_1|^{4l} + \sum_{j=2}^{n}|\xi_j|^{\frac{4l\rho-r_j}{\rho}}|x_j - x_j^*| \\ &\le 2\sum_{j=1}^{n}|\xi_j|^{4l}. \end{aligned} \tag{31}$$

Let $\alpha = (4l\rho+\tau)/4l\rho$, which is positive and less than 1. With (31) and (28) in mind, by Lemma 3, we acquire that

$$\mathcal{L}V_n \le -V_n^{\alpha}/2^{\alpha}. \tag{32}$$

Therefore, according to Lemma 1, it can be obtained from (30) and (32) that the origin of the CLS (1) and (27) is FTS in probability. The proof is completed.

IV. Simulation Example

To verify the applicability of the proposed control scheme, we consider a liquid-level system shown in Fig. 1, whose parameters and variables are as:

- H_i liquid levels of tank i;
- H steady-state liquid levels of two tanks;
- A_i cross sections of tank i;
- α_1 cross sections of the inlet manual valves from tank 2 to tank 1 or from tank 1 to tank 2;
- α_2 cross sections of the right outlet manual valves of tank 2;
- Q inflow rate of this system;
- Q_1 inflow rate from tank 2 to tank 1;
- Q_2 outflow rate of this system;
- g gravitational acceleration.

According to the mass balance principle and the Bernoulli's law, one has

$$\begin{aligned} A_1\dot{H}_1 &= Q_1, \\ A_2\dot{H}_2 &= Q - Q_1 - Q_2, \\ Q_1 &= \begin{cases} \alpha_1\sqrt{2g|H_2-H_1|}, & H_2 \ge H_1, \\ -\alpha_1\sqrt{2g|H_2-H_1|}, & H_2 < H_1, \end{cases} \\ Q_2 &= \alpha_2\sqrt{2gH_2}. \end{aligned} \tag{33}$$

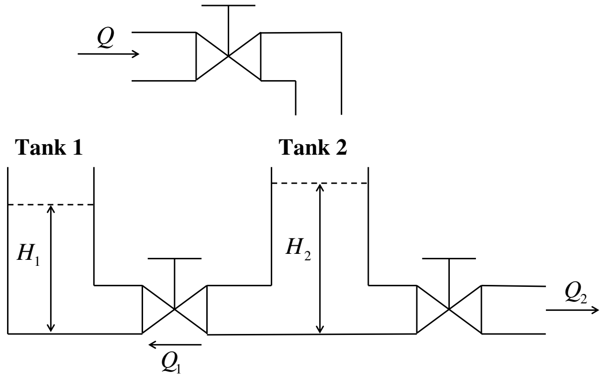

Fig. 1. Schematic diagram of the liquid-level system.

It is supposed that there exist white noises in the above system. Define the states $x_1 = H_1 - H$, $x_2 = H_2 - H_1$ and the control input $u = \frac{Q}{A_2} - \frac{\alpha_2\sqrt{2gH}}{A_2}$. System (33) can be written in the form

$$\begin{aligned} dx_1 &= h_1[x_2]^{\frac{1}{2}}dt + g_1(x_1)dw, \\ dx_2 &= ([u] + f_2(\bar{x}_2))dt + g_2(\bar{x}_2)dw, \end{aligned} \tag{34}$$

where $h_1 = \frac{\alpha_1\sqrt{2g}}{A_1}$ and $f_2(\bar{x}_2) = -\frac{A_1}{A_2}h_1[x_2]^{\frac{1}{2}} - \frac{\alpha_2\sqrt{2g}}{A_2}[x_1 + x_2 + H]^{\frac{1}{2}} + \frac{\alpha_2\sqrt{2g}}{A_2}[H]^{\frac{1}{2}}$, $g_1(x_1) = 0$ and $g_2(\bar{x}_2) = \frac{1}{10}x_1$. Note that the power of $[x_2]$ in the first equation of (34) is equal to $\frac{1}{2}$, which is an even rational number. This leads to the restriction on power order in [14-17] being not satisfied, and thus system (34) cannot be even finite-time stabilized using the presented design method. However, it is verified that Assumption 1 and 2 holds with $\tau_1 = \tau_2 = -\frac{1}{2}$, $r_1 = r_2 = 1$, $r_3 = \frac{1}{2}$, $\varphi_2 = \frac{\sqrt{2g}}{A_2}(\alpha_1 + \alpha_2)$ and $\phi_2 = \frac{1}{10}(1 + x_1^2)$. Therefore, the developed controller can be applicable. By choosing $\rho = 2$ and $l = \frac{5}{8}$, then one can design a FST controller for system (34) as

$$\begin{aligned} x_2^* &= -h_1^{-2}(1+l_1)^2[x_1] := -\beta_1^{\frac{1}{2}}[\xi_1]^{\frac{1}{2}}, \\ u &= -(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + l_2)[\xi_2]^{\frac{1}{4}}, \end{aligned} \tag{35}$$

where $\xi_2 = [x_2]^2 - [x_2^*]^2$, $c_{2,1} = \frac{1}{9}\cdot(\frac{32}{9})^8\cdot 2^{\frac{27}{4}}\cdot h_1^9$, $c_{2,2} = \frac{8}{9}\cdot(\frac{4}{9})^{\frac{1}{8}}\cdot(1+\beta_1^{\frac{1}{4}})^{\frac{9}{8}}\cdot\varphi_2^{\frac{9}{8}} + \varphi_2$, $c_{2,3} = \frac{7}{9}\cdot(\frac{64}{9})^{\frac{2}{7}}\cdot 2^{\frac{45}{14}}\cdot\beta_1^{\frac{9}{7}} + \frac{2}{3}\cdot(\frac{8}{3})^{\frac{1}{2}}\cdot 2^{\frac{15}{4}}\cdot\beta_1^{\frac{15}{8}}$ and $c_{2,4} = \frac{2}{9}\cdot(\frac{28}{9})^{\frac{7}{2}}\cdot 2^{\frac{27}{2}}\cdot\beta_1^9\cdot\phi_1^8$.

For simulation use, we pick up system parameters as: $g = 9.8$, $H = 1$, $A_1 = A_2 = \sqrt{2g} = 4.427$, $\alpha_1 = 0.2$ and $\alpha_2 = 0.1$. Furthermore, by setting the design parameters as: $l_1 = l_2 = 0.1$, Fig.2 is obtained to show the response of the ClS (34) and (35) with the initial conditions $x_1(0) = 0.3$ and $x_2(0) = -0.5$. It is clearly observed that the states of the CLS converge to the equilibrium in finite time, which demonstrates the effectiveness of the proposed method.

V. Conclusion

This paper has studied the problem of FTS for a class of SNSs in p-normal form with the powers of both odd and even rational numbers. By employing the AAPI technique, a recursive design approach for designing the state feedback finite-time controller is given to ensure the finite-time stability of the CLS. A practical application to finite-time control of a liquid-level system is provided to show the validity of the proposed method.

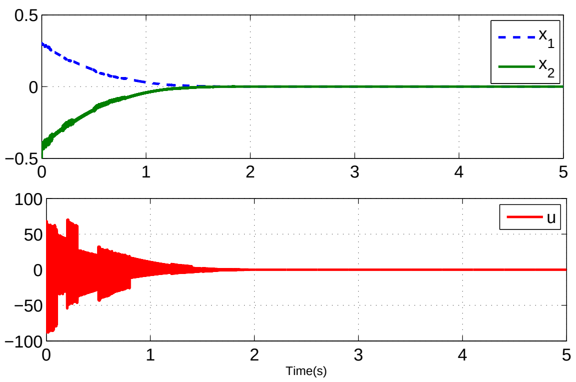

Fig. 2. The response of the CLS.

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