SP-Type Extragradient Iterative Methods for Solving Split Feasibility and Fixed Point Problems in Hilbert Spaces

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Abstract—**In this paper, we introduce the SP-type extragradient method with regularization for finding a common element of the solution set of the split feasibility and fixed point problems of pseudo-contractive mappings in real Hilbert spaces.**

Index Terms—**Split feasibility problem, SP-type extragradient method, Pseudo-contractive mapping, Fixed point problem.**

I. Introduction

We assume $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We denote the strong convergence and weak convergence by $\rightarrow$ and $\rightharpoonup$ respectively.

- A mapping $\mathcal{S}:\mathcal{C}\rightarrow\mathcal{C}$ is called a nonexpansive mapping if,
$$\|\mathcal{S}v-\mathcal{S}\vartheta\|\leq\|v-\vartheta\|,\quad \forall\, v,\vartheta\in\mathcal{C}. \tag{1}$$
- A mapping $\mathcal{T}:\mathcal{C}\rightarrow\mathcal{C}$ is called a pseudo-contractive if,
$$\langle\mathcal{T}v-\mathcal{T}\vartheta,v-\vartheta\rangle\leq\|v-\vartheta\|^2,\quad \forall\, v,\vartheta\in\mathcal{C}. \tag{2}$$
It is well-known that $\mathcal{T}$ is pseudo-contractive if and only if

$$\|\mathcal{T}v-\mathcal{T}\vartheta\|^2\leq\|v-\vartheta\|^2+\|(I-\mathcal{T})v-(I-\mathcal{T})\vartheta\|^2,\ \forall\, v,\vartheta\in\mathcal{C}. \tag{3}$$

The fixed point problem for the mapping $\mathcal{T}$ is the following: find $v\in\mathcal{C}$ such that $\mathcal{T}v=v$.
Denote by $\mathcal{F}(\mathcal{T})=\{v\in\mathcal{C}:\mathcal{T}v=v\}$ the set of solutions of the fixed point problem.
Phuengrattana and Suantai [1] introduced the SP iterative method as follows:

$$\begin{cases} q_n=(1-\gamma_n)v_n+\gamma_n\mathcal{T}v_n,\\ z_n=(1-\beta_n)z_n+\beta_n\mathcal{T}z_n,\\ v_{n+1}=(1-\alpha_n)y_n+\alpha_n\mathcal{T}q_n,\end{cases} \tag{4}$$

where $\alpha_n,\beta_n,\gamma_n\in(0,1)$ and $n\in\mathbb{N}$.

The above iterative methods (4) have been extensively studied by many authors (e.g. [2], [3], [4], [5]) for approximating fixed points of nonlinear mappings and solutions of nonlinear operator equations.

On the other hand, the split feasibility problems (SFP) have the following property:

$$\text{find}\quad v\in\mathcal{C}\quad\text{with}\quad\mathcal{A}v\in\mathcal{Q} \tag{5}$$

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where $\mathcal{A}$ is a bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Denote $\Upsilon_0=\{v\in\mathcal{C}:\mathcal{A}v\in\mathcal{Q}\}$ the set of solutions of the split feasibility problems (SFP) and $\Upsilon=\{v\in\mathcal{F}(\mathcal{T})\cap\mathcal{C}:\mathcal{A}v\in\mathcal{F}(\mathcal{S})\cap\mathcal{Q}\}$ the set of solutions of the split feasibility and fixed point problems where $\mathcal{T}:\mathcal{C}\rightarrow\mathcal{C}$ and $\mathcal{S}:\mathcal{Q}\rightarrow\mathcal{Q}$.

Censor and Segal [6] studied, in finite-dimensional spaces, for solving the problem (SCFPP) by using the following algorithm:

$$v_{n+1}=\mathcal{T}(v_n+\lambda\mathcal{A}^t(\mathcal{S}-\mathcal{I})\mathcal{A}v_n) \tag{6}$$

for each $n\geq 1$, where $\lambda\in(0,\frac{2}{\gamma})$ with γ being the largest eigenvalue of the matrix $\mathcal{A}^t\mathcal{A}$ ($\mathcal{A}^t$ is matrix transposition).

Moudafi [7] proved some weak convergence theorems in Hilbert spaces when two mappings $\mathcal{T}$ and $\mathcal{S}$ are quasi-nonexpansive mappings by the following relaxed algorithm:

$$v_{n+1}=(1-\alpha_n)q_n+\alpha_n\mathcal{T}z_n \tag{7}$$

for each $n\geq 1$, where $z_n=v_n+\lambda\mathcal{A}^*(\mathcal{S}-\mathcal{I})\mathcal{A}v_n$ for any $\alpha_n\in(0,1)$, $\beta\in(0,1)$ and $\lambda\in(0,\frac{1}{\beta\gamma})$ with γ being the spectral radius of operator $\mathcal{A}^*\mathcal{A}$.

Korpelevich [8] introduced the extra-gradient iterative method for solving a saddle point problems, many researchers have used and applied this iterative method for solving various problems (see e.g. [9], [10], [11], [12], [13], [14], [15], [16]).

For solving the split feasibility and fixed point problems, in 2012, Ceng et al. [17] proposed an iterative method by combining the extragradient iterative method with the idea of Nadezhkina and Takahashi [18] and proved that the sequences generated by their iterative method converge weakly to an element of the solutions of the split feasibility and fixed point problems.

Yao et al. [19] studied the split feasibility and fixed point problems by using the following iterative method for all $n\in\mathbb{N}$:

$$\begin{cases} v_0\in\mathcal{C}\ \text{chosen arbitrarily},\\ z_n=\mathcal{P}_{\mathcal{C}}(\gamma_n u+(1-\gamma_n)(v_n-\delta\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}v_n)),\\ v_{n+1}=(1-\alpha_n)q_n+\alpha_n\mathcal{T}((1-\beta_n)z_n+\beta_n\mathcal{T}z_n),\end{cases} \tag{8}$$

where α_n, β_n, $\gamma_n\in(0,1)$ and δ is a constant in $(0,\frac{1}{\|A\|^2})$.

Chen et al. [20] introduced an Ishikawa-type extragradient iterative method for pseudo-contractive mappings with Lipschitz assumption on $\mathcal{T}$.

Motivation and inspiration from the work of Ceng et al.

[17] and Yao et al. [19] as the following:

$$\begin{cases} v_0 \in \mathcal{C} \text{ chosen arbitrarily}, \\ q_n = \mathcal{P}_\mathcal{C}(v_n - \gamma_n \mathcal{A}^*(\mathcal{I} - \mathcal{S}\mathcal{P}_\mathcal{Q})\mathcal{A}v_n), \\ w_n = \mathcal{P}_\mathcal{C}(v_n - \gamma_n \mathcal{A}^*(\mathcal{I} - \mathcal{S}\mathcal{P}_\mathcal{Q})\mathcal{A}q_n), \\ z_n = (1-\beta_n)w_n + \beta_n \mathcal{T} w_n, \\ v_{n+1} = (1-\alpha_n)w_n + \alpha_n \mathcal{T} z_n, \end{cases} \quad (9)$$

for all $n \in \mathbb{N}$, where $\mathcal{S} : \mathcal{Q} \to \mathcal{Q}$ is a nonexpansive mapping, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint $\mathcal{A}^*$ and they proved that their sequences generated by their iterative methods converge weakly to solutions of the split feasibility and fixed point problems.

Here, we assume that the solution set of the split feasibility problems are nonempty. Let $f : \mathcal{H}_1 \to \mathbb{R}$ be a continuous differentiable function, the minimization problem:

$$\min_{v\in\mathcal{C}} f(v) := \frac{1}{2}\|\mathcal{A}v - \mathcal{P}_\mathcal{Q}\mathcal{A}v\|^2 \quad (10)$$

is ill-posed. Hence, Xu [22] considered the following Tikhonov regularized problem:

$$\min_{v\in\mathcal{C}} f^\rho(v) := \frac{1}{2}\|\mathcal{A}v - \mathcal{P}_\mathcal{Q}\mathcal{A}v\|^2 + \frac{1}{2}\rho\|v\|^2, \quad (11)$$

where $\rho > 0$ is the regularization parameter.

We can find the gradient that

$$\nabla f^\rho(v) = \nabla f(v) + \rho\mathcal{I} = \mathcal{A}^*(\mathcal{I} - \mathcal{P}_\mathcal{Q})\mathcal{A} + \rho\mathcal{I} \quad (12)$$

is $(\rho+\|\mathcal{A}\|^2)$-Lipschitz continuous and ρ-strongly monotone.

It is worth to emphasize that the traditional Tikhonov regularization is usually used to solve the ill-posed optimization problems. The advantage of a regularization method is its possible strong convergence to the minimum-norm solution of the optimization problems (see e.g. [23], [24], [25]).

Chen et al. [26] studied the split feasibility and fixed point problems by using the following iterative method:

$$\begin{cases} v_0 \in \mathcal{C} \text{ chosen arbitrarily}, \\ z_n = \mathcal{P}_\mathcal{C}(\mathcal{I} - \gamma_n \nabla f^{\rho_n})v_n, \\ v_{n+1} = \beta_n v_n + (1-\beta_n)\mathcal{S}\mathcal{P}_\mathcal{C}(v_n - \gamma_n \nabla f^{\rho_n}(z_n)), \end{cases} \quad (13)$$

for all $n \in \mathbb{N}$, where $\sum_{n=0}^{\infty} \rho_n < \infty$, $\{\gamma_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\|\mathcal{A}\|^2})$ and $\{\beta_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Then,both the sequences $\{v_n\}$ and $\{z_n\}$ converge weakly to an element $\hat{v} \in \Upsilon$.

Motivated and inspired by Chen et al. [26], Chen et al. [20], and Phuengrattana and Suantai [1], we introduce the iterative methods by using a combination of an extragradient method with regularization due to a generalized SP iterative method for solving the split feasibility and the fixed point problems of pseudo-contractive mappings with Lipschitz assumption on $\mathcal{C}$ and nonexpansive mappings on $\mathcal{Q}$.

II. Preliminaries

Let $\mathcal{C}$ be a closed convex subset of a Hilbert space $\mathcal{H}$. The mapping $\mathcal{P}_\mathcal{C} : \mathcal{H} \to \mathcal{C}$ is called the metric projection if $\mathcal{P}_\mathcal{C} v$ is the unique point in $\mathcal{C}$ with the property:

$$\|v - \mathcal{P}_\mathcal{C} v\| = \min\{\|v - \vartheta\| : \vartheta \in \mathcal{C}\} \text{ for all } v \in \mathcal{H}. \quad (14)$$

Proposition II.1 (see [27]). *For given $v \in \mathcal{H}$ and $\eta \in \mathcal{C}$:*

(i) $\eta = \mathcal{P}_\mathcal{C} v \Leftrightarrow \langle x - \eta, \vartheta - \eta \leq 0$ for all $\vartheta \in \mathcal{C}$;

(ii) $\eta = \mathcal{P}_\mathcal{C} v \Leftrightarrow \|v - \eta\|^2 \leq \|v - \vartheta\|^2 - \|\vartheta - \eta\|^2$ for all $\vartheta \in \mathcal{C}$;

(iii) $\langle v - \vartheta, \mathcal{P}_\mathcal{C} v - \mathcal{P}_\mathcal{C}\vartheta \geq \|\mathcal{P}_\mathcal{C} v - \mathcal{P}_\mathcal{C}\vartheta\|^2$ for all $\vartheta \in \mathcal{C}$

We also need other properties of nonlinear operators as the following:

(a) A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called an $\mathcal{L}$-Lipschitzian if there exists $\mathcal{L} > 0$ such that

$$\|\mathcal{T}v - \mathcal{T}\vartheta\| < \mathcal{L}\|v - \vartheta\|, \quad \text{for all } v, \vartheta \in \mathcal{H}, \quad (15)$$

if $\mathcal{L} = 1$, then $\mathcal{T}$ is called a nonexpansive;

(b) A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called a firmly nonexpansive if $2\mathcal{T} - \mathcal{I}$ is nonexpansive, or equivalently,

$$\langle v - \vartheta, \mathcal{T}v - \mathcal{T}\vartheta\rangle \geq \|\mathcal{T}v - \mathcal{T}\vartheta\|^2, \quad \text{for all } v, \vartheta \in \mathcal{H}, \quad (16)$$

alternatively, $\mathcal{T}$ is firmly nonexpansive if and only if $\mathcal{T}$ can be expressed as

$$\mathcal{T} = \frac{1}{2}(\mathcal{I} + \mathcal{S}),$$

where $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a nonexpansive;

Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a mapping. A mapping $\mathcal{I} - \mathcal{T}$ is said to be demiclosed at zero if for any sequence $\{v_n\} \subset \mathcal{H}$ with $v_n \rightharpoonup v$, and $v_n - \mathcal{T}v_n \to 0$, we have $v = \mathcal{T}v$.

(c) A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called a monotone if

$$\langle v - \vartheta, \mathcal{T}v - \mathcal{T}\vartheta\rangle \geq 0, \quad \text{for all } v, \vartheta \in \mathcal{H}; \quad (17)$$

(d) A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called a β-strongly monotone with $\beta > 0$, if

$$\langle v - \vartheta, \mathcal{T}v - \mathcal{T}\vartheta\rangle \geq \beta\|v - \vartheta\|^2, \quad \text{for all } v, \vartheta \in \mathcal{H}; \quad (18)$$

(e) A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is called a ν-inverse strongly monotone (ν-ism), with $\nu > 0$, if

$$\langle v - \vartheta, \mathcal{T}v - \mathcal{T}\vartheta\rangle \geq \nu\|\mathcal{T}v - \mathcal{T}\vartheta\|^2, \quad \text{for all } v, \vartheta \in \mathcal{H}; \quad (19)$$

It is well-known that the metric projection $\mathcal{P}_\mathcal{C} : \mathcal{H} \to \mathcal{C}$ is firmly nonexpansive, that is,

$$\begin{aligned} &\langle v - \vartheta, \mathcal{P}_\mathcal{C} v - \mathcal{P}_\mathcal{C}\vartheta\rangle \geq \|\mathcal{P}_\mathcal{C} v - \mathcal{P}_\mathcal{C}\vartheta\|^2 \\ &\Leftrightarrow \mathcal{P}_\mathcal{C} v - \mathcal{P}_\mathcal{C}\vartheta\|^2 \leq \|v - \vartheta\|^2 - \|(\mathcal{I} - \mathcal{P}_\mathcal{C})v - (\mathcal{I} - \mathcal{P}_\mathcal{C})\vartheta\|^2, \\ &\text{for all } v, \vartheta \in \mathcal{H}. \end{aligned} \quad (20)$$

Let $\mathcal{H}$ be a real Hilbert space. Then the following results hold:

(1) $\|v + \vartheta\|^2 = \|v\|^2 + 2\langle v, \vartheta\rangle + \|\vartheta\|^2 \quad \text{for all } v, \vartheta \in \mathcal{H}$;

(2) $\|v + \vartheta\|^2 \leq \|v\|^2 + 2\langle \vartheta, v + \vartheta\rangle \quad \text{for all } v, \vartheta \in \mathcal{H}$;

(3)

$$\begin{aligned} &\|\alpha v + (1-\alpha)\vartheta\|^2 \\ &= \alpha\|v\|^2 + (1-\alpha)\|\vartheta\|^2 - \alpha(1-\alpha)\|v - \vartheta\|^2, \end{aligned} \quad (21)$$

where $\alpha \in [0,1]$;

(4)

$$\begin{aligned} &\|\alpha v + \beta\vartheta + \gamma\eta\|^2 \\ &= \alpha\|v\|^2 + \beta\|\vartheta\|^2 + \gamma\|\eta\|^2 - \alpha\beta\|v - \vartheta\|^2 \\ &\quad - \alpha\gamma\|v - \eta\|^2 - \beta\gamma\|\vartheta - \eta\|^2, \end{aligned} \quad (22)$$

where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma II.2 ([20]). *Let $\mathcal{Q}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$ and $\mathcal{S}:\mathcal{Q}\to\mathcal{Q}$ be a nonexpansive mapping. Set $\nabla f^{\mathcal{S}}=\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}$, then*

$$\langle v-\vartheta,\nabla f^{\mathcal{S}}(v)-\nabla f^{\mathcal{S}}(\vartheta)\rangle \geq \frac{1}{2\|\mathcal{A}\|^2}\|\nabla f^{\mathcal{S}}(v)-\nabla f^{\mathcal{S}}(\vartheta)\|^2. \tag{23}$$

Lemma II.3 ([21]). *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{C}$ be a closed convex subset of $\mathcal{H}$. Let $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ be a continuous pseudo-contractive mapping. Then*

(i) *$\mathcal{F}(\mathcal{T})$ is a closed convex subset of $\mathcal{C}$;*
(ii) *$\mathcal{I}-\mathcal{T}$ is demiclosed at zero.*

III. Main results

We propose the generalized SP-type extragradient with regularization iterative method for pseudo-contractive mappings with Lipschitz assumption for solving the split feasibility and fixed point problems.

Theorem III.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $\mathcal{C}$ and $\mathcal{Q}$ be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{A}:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator with its adjoint $\mathcal{A}^*$. Let $\mathcal{S}:\mathcal{Q}\to\mathcal{Q}$ be a nonexpansive mapping and $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ be an $\mathcal{L}$-Lipschitzian pseudo-contractive mapping. For $v_0\in\mathcal{H}_1$ arbitrarily, let $\{v_n\}$ be a sequence defined by*

$$\begin{cases} q_n=\mathcal{P}_{\mathcal{C}}(v_n-\gamma_n(\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}+\rho_n\mathcal{I})v_n),\\ w_n=\mathcal{P}_{\mathcal{C}}(v_n-\gamma_n(\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}+\rho_n\mathcal{I})q_n),\\ s_n=(1-\delta_n)w_n+\delta_n\mathcal{T}w_n,\\ z_n=(1-\beta_n)s_n+\beta_n\mathcal{T}s_n,\\ v_{n+1}=(1-\alpha_n)z_n+\alpha_n\mathcal{T}z_n,\end{cases} \tag{24}$$

where $\{\gamma_n\}\subset[a,b]$ for some $a,b\in(0,\frac{1}{\rho_n+2\|\mathcal{A}\|^2})$, $\{\rho_n\}\subset(0,\infty)$, $\sum_{n=0}^{\infty}\rho_n<\infty$ and $0<a<\alpha_n<b<\beta_n<c<\delta_n<d<\frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{v_n\}$ generated by (24) *have a fixed point.*

Proof: Firstly, we show that the sequence $\{v_n\}$ is bounded. Let $z\in\Upsilon$. We obtain that $z\in\mathcal{F}(\mathcal{T})\cap\mathcal{C}$ and $\mathcal{A}z\in\mathcal{F}(\mathcal{S})\cap\mathcal{Q}$. Setting $\mu_n=\mathcal{P}_{\mathcal{Q}}\mathcal{A}v_n$, $t_n=v_n-\gamma_n(\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}+\rho_n\mathcal{I})v_n$, $\nabla f^{\mathcal{S}\rho_n}=\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}+\rho_n\mathcal{I}$ and $\nabla f^{S}=\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}$, for all $n\geq 0$. From the nonexpansive property of $\mathcal{P}_{\mathcal{C}}$, we have

$$\begin{aligned}\|q_n-z\|^2&=\|\mathcal{P}_{\mathcal{C}}t_n-z\|^2\leq\|t_n-z\|^2\\ &=\|v_n-\gamma_n(\mathcal{A}^*(\mathcal{I}-\mathcal{S}\mathcal{P}_{\mathcal{Q}})\mathcal{A}+\rho_n\mathcal{I})v_n-z\|^2\\ &=\|v_n-z\|^2+2\gamma_n\langle v_n-z,\mathcal{A}^*(\mathcal{S}\mathcal{P}_{\mathcal{Q}}-\mathcal{I})\mathcal{A}v_n\rangle\\ &\quad+\gamma_n^2\|\mathcal{A}^*(\mathcal{S}\mathcal{P}_{\mathcal{Q}}-\mathcal{I})\mathcal{A}v_n\|^2\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle.\end{aligned} \tag{25}$$

Since $\{\gamma_n\}\subset[a,b]$ for some $a,b\in(0,\frac{1}{\|\mathcal{A}\|^2})$, we obtain

$$\begin{aligned}\|q_n-z\|^2&=\|v_n-z\|^2+\gamma_n^2\|\mathcal{A}\|^2\|\mathcal{S}\mathcal{P}_{\mathcal{Q}}\mathcal{A}v_n-\mathcal{A}v_n\|^2\\ &\quad+2\gamma_n\langle v_n-z,\mathcal{A}^*(\mathcal{S}\mathcal{P}_{\mathcal{Q}}-\mathcal{I})\mathcal{A}v_n\rangle\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\\ &=\|v_n-z\|^2+\gamma_n^2\|\mathcal{A}\|^2\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad+2\gamma_n\langle v_n-z,\mathcal{A}^*(\mathcal{S}\mathcal{P}_{\mathcal{Q}}-\mathcal{I})\mathcal{A}v_n\rangle\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle.\end{aligned} \tag{26}$$

Since $\mathcal{A}$ is a linear operator with its adjoint $\mathcal{A}^*$, we get

$$\begin{aligned}&\langle v_n-z,\mathcal{A}^*(\mathcal{S}\mu_n-\mathcal{A}v_n)\rangle\\ &=\langle\mathcal{A}v_n-\mathcal{A}z,\mathcal{S}\mu_n-\mathcal{A}v_n\rangle\\ &=\langle\mathcal{A}v_n-\mathcal{A}z+\mathcal{S}\mu_n-\mathcal{A}v_n-\mathcal{S}\mu_n+\mathcal{A}v_n,\mathcal{S}\mu_n-\mathcal{A}v_n\rangle\\ &=\langle\mathcal{S}\mu_n-\mathcal{A}z,\mathcal{S}\mu_n-\mathcal{A}v_n\rangle-\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2.\end{aligned} \tag{27}$$

Using combination (21) in (27), we obtain

$$\begin{aligned}&\langle\mathcal{S}\mu_n-\mathcal{A}z,\mathcal{S}\mu_n-\mathcal{A}v_n\rangle\\ &=\frac{1}{2}(\|\mathcal{S}\mu_n-\mathcal{A}z\|^2+\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2-\|\mathcal{A}v_n-\mathcal{A}z\|^2).\end{aligned} \tag{28}$$

From $\mathcal{S}$ is a nonexpansive mapping and (20), we get

$$\begin{aligned}\|\mathcal{S}\mu_n-\mathcal{A}z\|^2&=\|\mathcal{S}\mathcal{P}_{\mathcal{Q}}\mathcal{A}v_n-\mathcal{S}\mathcal{P}_{\mathcal{Q}}\mathcal{A}z\|^2\\ &\leq\|\mathcal{P}_{\mathcal{Q}}\mathcal{A}v_n-\mathcal{P}_{\mathcal{Q}}\mathcal{A}z\|^2\\ &\leq\|\mathcal{A}v_n-\mathcal{A}z\|^2-\|\mu_n-\mathcal{A}v_n\|^2.\end{aligned} \tag{29}$$

From (27), (28) and (29), we have

$$\begin{aligned}&\langle\mathcal{S}\mu_n-\mathcal{A}z,\mathcal{S}\mu_n-\mathcal{A}v_n\rangle\\ &=\frac{1}{2}(\|\mathcal{S}\mu_n-\mathcal{A}z\|^2+\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2-\|\mathcal{A}v_n-\mathcal{A}z\|^2)\\ &\quad-\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\leq\frac{1}{2}(\|\mathcal{A}v_n-\mathcal{A}z\|^2-\|\mu_n-\mathcal{A}v_n\|^2+\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad-\|\mathcal{A}v_n-\mathcal{A}z\|^2)-\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &=-\frac{1}{2}\|\mu_n-\mathcal{A}v_n\|^2-\frac{1}{2}\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2.\end{aligned} \tag{30}$$

Substituting (30) into (27), we get

$$\begin{aligned}&\langle v_n-z,\mathcal{A}^*(\mathcal{S}\mu_n-\mathcal{A}v_n)\rangle\\ &\leq-\frac{1}{2}\|\mu_n-\mathcal{A}v_n\|^2-\frac{1}{2}\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2.\end{aligned} \tag{31}$$

Substituting (31) into (26), we get

$$\begin{aligned}\|q_n-z\|^2&\leq\|v_n-z\|^2+\gamma_n^2\|\mathcal{A}\|^2\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad+2\gamma_n\left(-\frac{1}{2}\|\mu_n-\mathcal{A}v_n\|^2-\frac{1}{2}\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\right)\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\\ &=\|v_n-z\|^2+\gamma_n^2\|\mathcal{A}\|^2\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad-\gamma_n\|\mu_n-\mathcal{A}v_n\|^2-\gamma_n\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\\ &=\|v_n-z\|^2-\gamma_n\|\mu_n-\mathcal{A}v_n\|^2\\ &\quad-\gamma_n(1-\gamma_n\|\mathcal{A}\|^2)\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2\\ &\quad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\\ &\leq\|v_n-z\|^2-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\end{aligned} \tag{32}$$

Next, we show that

$$\begin{aligned}&\langle\nabla f^{\mathcal{S}\rho_n}(v)-\nabla f^{\mathcal{S}\rho_n}(\vartheta),v-\vartheta\rangle\\ &\geq\frac{1}{1-\gamma_n\|\mathcal{A}\|^2}\|\nabla f^{\mathcal{S}\rho_n}(v)-\nabla f^{\mathcal{S}\rho_n}(\vartheta)\|^2.\end{aligned} \tag{33}$$

Using Lemma II.2, we have

$$\langle v-\vartheta,\nabla f^{\mathcal{S}}(v)-\nabla f^{\mathcal{S}}(\vartheta)\rangle\geq\frac{1}{2\|\mathcal{A}\|^2}\|\nabla f^{\mathcal{S}}(v)-\nabla f^{\mathcal{S}}(\vartheta)\|^2. \tag{34}$$

So,

$$
\begin{aligned}
&(\gamma_n + 2\|\mathcal{A}\|^2)\langle \nabla f^{\mathcal{S}\rho_n}(v) - \nabla f^{\mathcal{S}\rho_n}(\vartheta), v - \vartheta\rangle \\
&= (\gamma_n + 2\|\mathcal{A}\|^2)\langle \gamma_n \|v - \vartheta\|^2 + \langle \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta), v - \vartheta\rangle \\
&= \gamma_n^2 \|v - \vartheta\|^2 + \gamma_n \langle \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta), v - \vartheta\rangle \\
&\quad + 2\gamma_n \|\mathcal{A}\|^2 \|v - \vartheta\|^2 + 2\|\mathcal{A}\|^2 \langle \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta), v - \vartheta\rangle \\
&\geq \gamma_n^2 \|v - \vartheta\|^2 + \gamma_n \langle \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta), v - \vartheta\rangle \\
&\quad + 2\gamma_n \|\mathcal{A}\|^2 \|v - \vartheta\|^2 + \|\nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta)\|^2 \\
&\geq \gamma_n^2 \|v - \vartheta\|^2 + 2\gamma_n \langle \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta), v - \vartheta\rangle \\
&\quad + \|\nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta)\|^2 \\
&= \|\gamma_n (v - \vartheta) + \nabla f^{\mathcal{S}}(v) - \nabla f^{\mathcal{S}}(\vartheta)\|^2 \\
&= \|\nabla f^{\mathcal{S}\rho_n}(v) - \nabla f^{\mathcal{S}\rho_n}(\vartheta)\|^2.
\end{aligned} \tag{35}
$$

Using Proposition II.1(ii), we obtain

$$
\begin{aligned}
&\|w_n - z\|^2 \\
&\leq \|v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(q_n) - z\|^2 \\
&\quad - \|v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(q_n) - w_n\|^2 \\
&= \|v_n - z\|^2 - 2\gamma_n \langle v_n - z, \nabla f^{\mathcal{S}\rho_n}(q_n)\rangle \\
&\quad + \gamma_n^2 \|\nabla f^{\mathcal{S}\rho_n}(q_n)\|^2 - \|v_n - w_n\|^2 \\
&\quad + 2\gamma_n \langle v_n - w_n, \nabla f^{\mathcal{S}\rho_n}(q_n)\rangle - \gamma_n^2 \|\nabla f^{\mathcal{S}\rho_n}(q_n)\|^2 \\
&= \|v_n - z\|^2 - \|v_n - w_n\|^2 + 2\gamma_n \langle \nabla f^{\mathcal{S}\rho_n}(q_n), z - w_n\rangle \\
&= \|v_n - z\|^2 - \|v_n - w_n\|^2 \\
&\quad - 2\gamma_n \Big(\langle \nabla f^{\mathcal{S}\rho_n}(q_n) - \nabla f^{\mathcal{S}\rho_n}(z), q_n - z\rangle \\
&\quad + \langle \nabla f^{\mathcal{S}\rho_n}(z), z - q_n + \nabla f^{\mathcal{S}\rho_n}(q_n), q_n - w_n\rangle \Big) \\
&\leq \|v_n - z\|^2 - \|v_n - w_n\|^2 + 2\gamma_n \langle \nabla f^{\mathcal{S}\rho_n}(q_n), q_n - w_n\rangle \\
&= \|v_n - z\|^2 - \|v_n - q_n\|^2 - \|q_n - w_n\|^2 \\
&\quad + 2\|v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(q_n) - q_n, w_n - q_n\rangle.
\end{aligned} \tag{36}
$$

Using Proposition II.1(i) in (34), we get

$$
\begin{aligned}
&\langle v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(q_n) - q_n, w_n - q_n\rangle \\
&= \langle v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(v_n) - q_n, w_n - q_n\rangle \\
&\quad + \gamma_n \langle \nabla f^{\mathcal{S}\rho_n}(v_n) - \nabla f^{\mathcal{S}\rho_n}(q_n), w_n - q_n\rangle \\
&\leq \gamma_n \langle \nabla f^{\mathcal{S}\rho_n}(v_n) - \nabla f^{\mathcal{S}\rho_n}(q_n), w_n - yn\rangle \\
&\leq \gamma_n \|\nabla f^{\mathcal{S}\rho_n}(v_n) - \nabla f^{\mathcal{S}\rho_n}(q_n)\| \|w_n - q_n\| \\
&\leq \gamma_n (\rho_n + 2\|\mathcal{A}\|^2)\|v_n - q_n\| \|w_n - q_n\|.
\end{aligned} \tag{37}
$$

It follows from the hypothesis on $\{\gamma_n\}$ and (37), we obtain

$$
\begin{aligned}
\|w_n - z\|^2 &\leq \|v_n - z\|^2 - \|v_n - q_n\|^2 - \|q_n - w_n\|^2 \\
&\quad + 2\langle v_n - \gamma_n \nabla f^{\mathcal{S}\rho_n}(q_n) - q_n, w_n - q_n\rangle \\
&\leq \|v_n - z\|^2 - \|v_n - q_n\|^2 - \|q_n - w_n\|^2 \\
&\quad + 2\gamma_n (\rho_n + 2\|\mathcal{A}\|^2)\|v_n - q_n\| \|w_n - q_n\| \\
&\leq \|v_n - z\|^2 - \|v_n - q_n\|^2 - \|q_n - w_n\|^2 \\
&\quad + \|w_n - q_n\|^2 + \gamma_n^2 (\rho_n + 2\|\mathcal{A}\|^2)^2 \|v_n - q_n\|^2 \\
&= \|v_n - z\|^2 - (1 - \gamma_n^2 (\rho_n + 2\|\mathcal{A}\|^2)^2 \|v_n - q_n\|^2 \\
&\leq \|v_n - z\|^2.
\end{aligned} \tag{38}
$$

Since $\mathcal{T}$ is a pseudo-contractive mapping, we get

$$
\|\mathcal{T} w_n - z\|^2 \leq \|w_n - z\|^2 + \|w_n - \mathcal{T} w_n\|^2. \tag{39}
$$

Then,

$$
\begin{aligned}
&\|\mathcal{T} s_n - z\|^2 \\
&= \|\mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n) - z\|^2 \\
&\leq \|(1 - \delta_n)(w_n - z) + \delta_n (\mathcal{T} w_n - z)\|^2 \\
&\quad + \|(1 - \delta_n) w_n + \delta_n \mathcal{T} w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2
\end{aligned} \tag{40}
$$

Using (21) and $\mathcal{T}$ is an $\mathcal{L}$-Lipschitzian pseudo-contractive mapping, we get

$$
\begin{aligned}
&\|(1 - \delta_n) w_n + \delta_n \mathcal{T} w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&= \|(1 - \delta_n)(w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n) \\
&\quad + \delta_n (\mathcal{T} w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&= (1 - \delta_n)\|w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\quad + \delta_n \|\mathcal{T} w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\quad - \delta_n (1 - \delta_n)\|w_n - \mathcal{T} w_n\|^2 \\
&\leq (1 - \delta_n)\|w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\quad + \delta_n^3 \mathcal{L}^2 \|w_n - \mathcal{T} w_n\|^2 - \delta_n (1 - \delta_n)\|w_n - \mathcal{T} w_n\|^2 \\
&= (1 - \delta_n)\|w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\quad - \delta_n (1 - \delta_n - \delta_n^2 \mathcal{L}^2)\|w_n - \mathcal{T} w_n\|^2.
\end{aligned} \tag{41}
$$

Using (21), (39) and (38), we get

$$
\begin{aligned}
&\|(1 - \delta_n)(w_n - z) + \delta_n (\mathcal{T} w_n - z)\|^2 \\
&= (1 - \delta_n)\|w_n - z\|^2 + \delta_n \|\mathcal{T} w_n - z\|^2 \\
&\quad - \delta_n (1 - \delta_n)\|w_n - \mathcal{T} w_n\|^2 \\
&\leq (1 - \delta_n)\|w_n - z\|^2 + \delta_n \big(\|w_n - z\|^2 + \|w_n - \mathcal{T} w_n\|^2\big) \\
&\quad - \delta_n (1 - \delta_n)\|w_n - \mathcal{T} w_n\|^2 \\
&= \|w_n - z\|^2 + \delta_n^2 \|w_n - \mathcal{T} w_n\|^2 \\
&\leq \|v_n - z\|^2 + \delta_n^2 \|w_n - \mathcal{T} w_n\|^2.
\end{aligned} \tag{42}
$$

Using (41) and (42), we get

$$
\begin{aligned}
&\|\mathcal{T} s_n - z\|^2 \\
&= \|\mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n) - z\|^2 \\
&\leq \|(1 - \delta_n)(w_n - z) + \delta_n (\mathcal{T} w_n - z)\|^2 \\
&\quad + \|(1 - \delta_n) w_n + \delta_n \mathcal{T} w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\leq \|v_n - z\|^2 + (1 - \delta_n)\|w_n - \mathcal{T}((1 - \delta_n) w_n + \delta_n \mathcal{T} w_n)\|^2 \\
&\quad - \delta_n (1 - 2\delta_n - \delta_n^2 \mathcal{L}^2)\|w_n - \mathcal{T} w_n\|^2.
\end{aligned} \tag{43}
$$

Since $\mathcal{T}$ is a pseudo-contractive mapping, we obtain

$$
\|\mathcal{T} z_n - z\|^2 \leq \|z_n - z\|^2 + \|z_n - \mathcal{T} z_n\|^2. \tag{44}
$$

Then,

$$
\begin{aligned}
&\|\mathcal{T} z_n - z\|^2 \\
&= \|\mathcal{T}((1 - \beta_n) s_n + \beta_n \mathcal{T} s_n) - z\|^2 \\
&\leq \|(1 - \beta_n)(s_n - z) + \beta_n (\mathcal{T} s_n - z)\|^2 \\
&\quad + \|(1 - \beta_n) s_n + \beta_n \mathcal{T} s_n - \mathcal{T}((1 - \beta_n) s_n + \beta_n \mathcal{T} s_n)\|^2.
\end{aligned} \tag{45}
$$

Using (21) and $\mathcal{T}$ is an $\mathcal{L}$-Lipschitzian pseudo-contractive mapping, we get

$$\begin{aligned}
&\|(1-\beta_n)s_n+\beta_n\mathcal{T}s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&=\|(1-\beta_n)(s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\\
&\quad+\beta_n(\mathcal{T}s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n))\|^2\\
&=(1-\beta_n)\|s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&\quad+\beta_n\|\mathcal{T}s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&\quad-\beta_n(1-\beta_n)\|s_n-\mathcal{T}s_n\|^2\\
&\le(1-\beta_n)\|s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&\quad+\beta_n^3\mathcal{L}^2\|s_n-\mathcal{T}s_n\|^2-\beta_n(1-\beta_n)\|s_n-\mathcal{T}s_n\|^2\\
&=(1-\beta_n)\|s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&\quad-\beta_n(1-\beta_n-\beta_n^2\mathcal{L}^2)\|s_n-\mathcal{T}s_n\|^2.
\end{aligned}\tag{46}$$

Using (21) and (44), we get

$$\begin{aligned}
&\|(1-\beta_n)(s_n-z)+\beta_n(\mathcal{T}s_n-z)\|^2\\
&=(1-\beta_n)\|s_n-z\|^2+\beta_n\mathcal{T}s_n-z\|^2\\
&\quad-\beta_n(1-\beta_n)\|s_n-\mathcal{T}s_n\|^2\\
&\le(1-\beta_n)\|s_n-z\|^2+\beta_n\big(\|s_n-z\|^2+\|s_n-\mathcal{T}s_n\|^2\big)\\
&\quad-\beta_n(1-\beta_n)\|s_n-\mathcal{T}s_n\|^2\\
&=\|s_n-z\|^2+\beta_n^2\|s_n-\mathcal{T}s_n\|^2.
\end{aligned}\tag{47}$$

Using (46) and (47), we get

$$\begin{aligned}
&\|\mathcal{T}z_n-z\|^2\\
&=\|\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)-z\|^2\\
&\le\|(1-\beta_n)(s_n-z)+\beta_n(\mathcal{T}s_n-z)\|^2\\
&\quad+\|(1-\beta_n)s_n+\beta_n\mathcal{T}s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&=\|s_n-z\|^2+(1-\beta_n)\|s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|^2\\
&\quad-\beta_n(1-2\beta_n-\beta_n^2\mathcal{L}^2)\|s_n-\mathcal{T}s_n\|^2.
\end{aligned}\tag{48}$$

Consider,

$$\begin{aligned}
&\|s_n-z\|^2\\
&=\|(1-\delta_n)w_n+\delta_n\mathcal{T}w_n-z\|^2\\
&=(1-\delta_n)\|w_n-z\|^2+\delta_n\|\mathcal{T}w_n-z\|^2\\
&\quad-\delta_n(1-\delta_n)\|w_n-\mathcal{T}w_n\|^2\\
&\le(1-\delta_n)\|w_n-z\|^2+\delta_n(\|w_n-z\|^2+\|w_n-\mathcal{T}w_n\|^2)\\
&\quad-\delta_n(1-\delta_n)\|w_n-\mathcal{T}w_n\|^2\\
&=\|w_n-z\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\le\|v_n-z\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2.
\end{aligned}\tag{49}$$

Since $s_n=(1-\delta_n)w_n+\delta_n\mathcal{T}w_n$ and $\delta_n<d<\frac{1}{\sqrt{L^2+1}+1+L^2}$, we obtain

$$\begin{aligned}
&\|s_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n\|^2\\
&=\|(1-\delta_n)w_n+\delta_n\mathcal{T}w_n-\mathcal{T}((1-\beta_n)s_n+\beta_n\mathcal{T}s_n)\|\\
&=\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\langle w_n-\mathcal{T}s_n,w_n-\mathcal{T}w_n\rangle\\
&=\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\langle w_n-\mathcal{T}s_n+\mathcal{T}w_n-\mathcal{T}w_n,w_n-\mathcal{T}w_n\rangle\\
&=\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\langle\mathcal{T}w_n-\mathcal{T}s_n,w_n-\mathcal{T}w_n\rangle\\
&\le\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\|w_n-\mathcal{T}w_n\|^2\\
&\quad+2\beta_n\delta_n\|\mathcal{T}w_n-\mathcal{T}s_n\|\|w_n-\mathcal{T}w_n\|\\
&\le\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\|w_n-\mathcal{T}w_n\|^2\\
&\quad+2\beta_n\delta_n^2\mathcal{L}\|w_n-\mathcal{T}w_n\|\|w_n-\mathcal{T}w_n\|\\
&=\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n\|w_n-\mathcal{T}w_n\|^2\\
&\quad+2\beta_n\delta_n^2\mathcal{L}\|w_n-\mathcal{T}w_n\|^2\\
&=\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad-2\beta_n\delta_n(1-\delta_n\mathcal{L})\|w_n-\mathcal{T}w_n\|^2\\
&\le\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2
\end{aligned}\tag{50}$$

Combining (48) with (50), we get

$$\begin{aligned}
&\|\mathcal{T}z_n-z\|^2\\
&\le\|v_n-z\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\\
&\quad+(1-\beta_n)\big(\beta_n^2\|w_n-\mathcal{T}s_n\|^2+\delta_n^2\|w_n-\mathcal{T}w_n\|^2\big)\\
&\quad-\beta_n(1-2\beta_n-\beta_n^2\mathcal{L}^2)\|s_n-\mathcal{T}s_n\|^2\\
&\le\|v_n-z\|^2+(2-\beta_n)(\beta_n^2+\delta_n^2)\|w_n-\mathcal{T}w_n\|^2\\
&\quad-\beta_n(1-2\beta_n-\beta_n^2\mathcal{L}^2)\|s_n-\mathcal{T}s_n\|^2.
\end{aligned}\tag{51}$$

Since $\beta_n<c<\delta_n<d<\frac{1}{\sqrt{L^2+1}+1+L^2}$, we have

$$1-2\beta_n-\beta_n^2\mathcal{L}^2>0.$$

Hence,

$$\|\mathcal{T}z_n-z\|^2\le\|v_n-z\|^2+(2-\beta_n)(\beta_n^2+\delta_n^2)\|w_n-\mathcal{T}w_n\|^2.\tag{52}$$

Consider,

$$\begin{aligned}
&\|z_n-z\|^2\\
&=\|(1-\beta_n)w_n+\beta_n\mathcal{T}w_n\|^2\\
&=(1-\beta_n)\|w_n-z\|^2+\beta_n\|\mathcal{T}w_n-z\|^2\\
&\quad-\beta_n(1-\beta_n)\|w_n-\mathcal{T}w_n\|^2\\
&\le(1-\beta_n)\|w_n-z\|^2+\beta_n(\|w_n-z\|^2+\|w_n-\mathcal{T}w_n\|^2)\\
&\quad-\beta_n(1-\beta_n)\|w_n-\mathcal{T}w_n\|^2\\
&=\|w_n-z\|^2+\beta_n^2\|w_n-\mathcal{T}w_n\|^2
\end{aligned}\tag{53}$$

Using (21), (24), (52) and (53), we have

$$\begin{aligned}&\|v_{n+1}-z\|^2\\&=\|(1-\alpha_n)z_n+\alpha_n\mathcal{T}z_n-z\|^2\\&=(1-\alpha_n)\|z_n-z\|^2+\alpha_n\|\mathcal{T}z_n-z\|^2\\&\quad-\alpha_n(1-\alpha_n)\|z_n-\mathcal{T}z_n\|^2\\&\leq(1-\alpha_n)(\|w_n-z\|^2+\beta_n^2\|w_n-\mathcal{T}w_n\|^2)\\&\quad+\alpha_n(\|w_n-z\|^2+(2-\beta_n)(\beta_n^2+\delta_n^2)\|w_n-\mathcal{T}w_n\|^2)\\&\quad-\alpha_n(1-\alpha_n)\|z_n-\mathcal{T}z_n\|^2\\&=\|w_n-z\|^2-\alpha_n(\beta_n^2-(2-\beta_n)(\beta_n^2+\delta_n^2))\|w_n-\mathcal{T}w_n\|^2)\\&\quad-\alpha_n(1-\alpha_n)\|z_n-\mathcal{T}z_n\|^2\\&\leq\|w_n-z\|^2.\end{aligned}\tag{54}$$

Equation (54) together with (38)

$$\|v_{n+1}-z\|^2\leq\|v_n-z\|^2,\tag{55}$$

for every $z\in\Upsilon$ and for all $n\geq 0$. Hence, the sequence $\{v_n\}$ generated by algorithm (24) is Féjermonotone with respect to Υ. Therefore, we obtain $\lim_{n\to\infty}\|v_n-z\|^2$ exists immediately, it follows that $\{v_n\}$ is bounded and the sequence $\{\|v_n-z\|\}$ is monotonically decreasing. Consequently, from (32) and (38), $\{q_n\}$ and $\{w_n\}$ are also bounded.

Using (54) and (38), we obtain

$$\begin{aligned}&\|v_{n+1}-z\|^2\\&\leq\|w_n-z\|^2\\&\leq\|v_n-z\|^2-(1-\gamma_n^2(\rho_n+2\|\mathcal{A}\|^2)^2\|v_n-q_n\|^2.\end{aligned}\tag{56}$$

Then, we have

$$\begin{aligned}&(1-\gamma_n^2(\rho_n+2\|\mathcal{A}\|^2)^2\|v_n-q_n\|^2\\&\quad\leq\|v_n-z\|^2-\|v_{n+1}-z\|^2\end{aligned}\tag{57}$$

and

$$\lim_{n\to\infty}\|v_n-q_n\|^2=0.\tag{58}$$

Similarly, we have

$$\lim_{n\to\infty}\|w_n-q_n\|^2=0.\tag{59}$$

Using (58), (32) and $\lim_{n\to\infty}\rho_n=0$, we get

$$\begin{aligned}&\gamma_n(1-\gamma_n\|\mathcal{A}\|^2)\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2+\gamma_n\|\mu_n-\mathcal{A}v_n\|^2\\&\quad\leq\|v_n-z\|^2-\|q_n-z\|^2\\&\qquad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle\\&\quad\leq(\|v_n-z\|-\|q_n-z\|)\|v_n-q_n\|\\&\qquad-\gamma_n\rho_n\langle 2(t_n-z)+\gamma_n\rho_nv_n,v_n\rangle.\end{aligned}\tag{60}$$

This implies that

$$\lim_{n\to\infty}\|\mathcal{S}\mu_n-\mathcal{A}v_n\|^2=0\quad\text{and}\quad\lim_{n\to\infty}\|\mu_n-\mathcal{A}v_n\|^2=0.\tag{61}$$

Hence, $\lim_{n\to\infty}\|\mu_n-\mathcal{S}\mu_n\|^2=0$.

Using (54), we obtain

$$\begin{aligned}&\alpha_n(\beta_n^2-(2-\beta_n)(\beta_n^2+\delta_n^2))\|w_n-\mathcal{T}w_n\|^2)\\&\quad+\alpha_n(1-\alpha_n)\|z_n-\mathcal{T}z_n\|^2\\&\leq\|v_n-z\|^2-\|v_{n+1}-z\|^2.\end{aligned}\tag{62}$$

This implies that

$$\lim_{n\to\infty}\|w_n-\mathcal{T}w_n\|^2=0\quad\text{and}\quad\lim_{n\to\infty}\|z_n-\mathcal{T}z_n\|^2=0.\tag{63}$$

For all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\|s_n-\mathcal{T}s_n\|&\leq\|s_n-\mathcal{T}w_n\|+\|\mathcal{T}w_n-\mathcal{T}s_n\|\\&\leq\|s_n-\mathcal{T}w_n\|+\mathcal{L}\|w_n-s_n\|.\end{aligned}\tag{64}$$

Since $s_n=(1-\delta_n)w_n+\delta_n\mathcal{T}w_n$, we get

$$\begin{aligned}&\|s_n-\mathcal{T}s_n\|\\&\leq\|(1-\delta_n)w_n+\delta_n\mathcal{T}w_n-\mathcal{T}w_n\|\\&\quad+\mathcal{L}\|w_n-((1-\delta_n)w_n+\delta_n\mathcal{T}w_n)\|\\&=\|(1-\delta_n)(w_n-\mathcal{T}w_n)\|+\mathcal{L}\|\delta_n(\mathcal{T}w_n-w_n)\|\\&\leq(1-\delta_n)\|w_n-\mathcal{T}w_n\|+\delta_n\mathcal{L}\|\mathcal{T}w_n-w_n\|.\end{aligned}\tag{65}$$

This implies that

$$\lim_{n\to\infty}\|s_n-\mathcal{T}s_n\|=0.\tag{66}$$

Using the firmly nonexpansiveness of $\mathcal{P}_\mathcal{C}$, (20) and (32), we obtain

$$\begin{aligned}\|q_n-z\|^2=\|\mathcal{P}_\mathcal{C}t_n-z\|^2&\leq\|t_n-z\|^2-\|\mathcal{P}_\mathcal{C}t_n-t_n\|^2\\&\leq\|v_n-z\|^2-\|q_n-t_n\|^2.\end{aligned}\tag{67}$$

So,

$$\begin{aligned}\|q_n-t_n\|^2&\leq\|v_n-z\|^2-\|q_n-z\|^2\\&\leq(\|v_n-z\|+\|q_n-z)\|v_n-q_n\|.\end{aligned}\tag{68}$$

Using (58), we have

$$\lim_{n\to\infty}\|q_n-t_n\|=0.\tag{69}$$

Since the sequence $\{v_n\}$ is bounded, we can choose a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i}\rightharpoonup\widehat{z}$.

From the above conclusions, we can obtain that

$$\begin{cases}v_{n_i}\rightharpoonup\widehat{z},\\q_{n_i}\rightharpoonup\widehat{z},\\t_{n_i}\rightharpoonup\widehat{z},\end{cases}\quad\text{and}\quad\begin{cases}z_{n_i}\rightharpoonup\widehat{z},\\\mathcal{A}v_{n_i}\rightharpoonup\mathcal{A}\widehat{z},\\\mu_{n_i}\rightharpoonup\mathcal{A}\widehat{z}.\end{cases}\tag{70}$$

Using Lemma II.3,

$$\widehat{z}\in\mathcal{F}(\mathcal{T})\quad\text{and}\quad\mathcal{A}\widehat{z}\in\mathcal{F}(\mathcal{S})$$

From $q_{n_i}=\mathcal{P}_\mathcal{C}t_{n_i}\in\mathcal{C}$ and $\mu_{n_i}=\mathcal{P}_\mathcal{Q}\mathcal{A}v_{n_i}$ and using (70), we obtain

$$\widehat{z}\in\mathcal{C}\quad\text{and}\quad\mathcal{A}\widehat{z}\in\mathcal{Q}.$$

Hence,

$$\widehat{z}\in\mathcal{C}\cap\mathcal{F}(\mathcal{T})\quad\text{and}\quad\mathcal{A}\widehat{z}\in\mathcal{Q}\cap\mathcal{F}(\mathcal{S}).$$

This is $\widehat{z}\in\Upsilon$ and shows that $\omega_W(v_n)\subset\Upsilon$. Since the $\lim_{n\to\infty}\|v_n-z\|$ exists for every $z\in\Upsilon$ and every subsequence of$\{v_n\}$ converges weakly to $z\in\Upsilon$, it is immediate from Lemma 2 that $\{v_n\}$ converges weakly to $z\in\Upsilon$. The proof is completed. ■

Example III.2 ([28]). Let $\mathcal{H}$ be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $v=(a,b)\in\mathcal{H}$, define $v^\perp\in\mathcal{H}$ to be $(b,-a)$. Let $\mathcal{K}:=\{v\in\mathcal{H}:v\leq 1\}$. and setting

$$\mathcal{K}_1:=\{v\in\mathcal{H}:\|v\|\leq\frac{1}{2}\}\text{ and }K_2:=\{v\in\mathcal{H}:\frac{1}{2}\leq\|v\|\leq 1\}.$$

Define $\mathcal{T}:\mathcal{K}\to\mathcal{K}$ as follows:

$$\mathcal{T}v=\begin{cases}v+v^\perp, & \text{if } v\in\mathcal{K}_1,\\\frac{v}{\|v\|}-v+v^\perp, & \text{if } v\in\mathcal{K}_2,\end{cases}\tag{71}$$

Then $\mathcal{T}$ is an $\mathcal{L}$-Lipschitzian pseudo-contractive mapping with $\mathcal{L} = 5$ and $\mathcal{F}(\mathcal{T}) = \{0\}$.

Example III.3. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^2$ under the usual Euclidean inner product. Let $\mathcal{C} = \{v \in \mathcal{H} : v \leq 1\}$ and $\mathcal{T}$ as in Example III.2. Let $\mathcal{Q} = \mathbb{R}^2$ and $\mathcal{S}v = \frac{1}{5}v$ for all $v \in \mathbb{R}^2$. Setting $\mathcal{A}v = \frac{1}{4}v$ for all $v \in \mathbb{R}^2$. Let $\gamma_n = \frac{n+1}{n+7}$, $\rho_n = \frac{1}{(n+3)^2}$, $\delta_n = 0.05$, $\beta_n = 0.035$, $\alpha_n = 0.01$ for all $n \geq 1$. is easy to see that $\Upsilon = \{0\}$. Let $x_0 = (-0.5, 0.7)$, then the sequence $\{v_n\}$ generated iteratively by (24) converges to 0.

TABLE I: Results of Example III.3

Number of Iterations	(x_1, x_2)	$\|x_{n+1} - x_n\|$
5	(-0.121765, 0.569494)	0.072246
10	(0.139452, 0.465847)	0.050984
50	(-0.138005, -0.062421)	0.016084
100	(0.004327, -0.030433)	0.003282
150	(0.005884, -0.000843)	0.000636
200	(0.000465, 0.001024)	0.000120
250	(-0.000159, 0.000137)	0.000023
300	(-0.000033, -0.000021)	0.000004
343	(0.000008, -0.000005)	0.000001

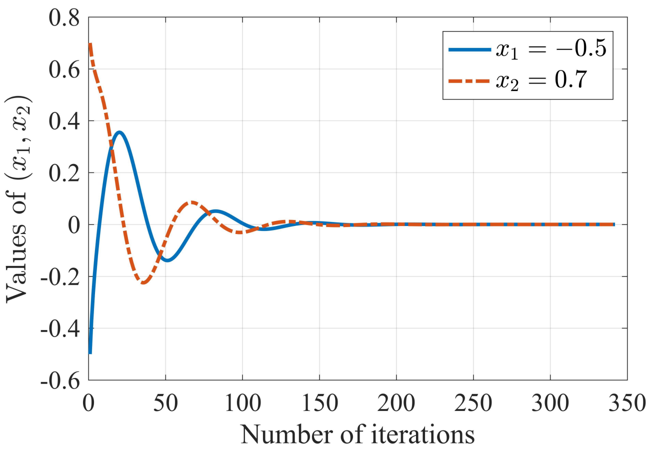

Fig. 1: Graph of Example III.3

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