Sum of Powers of Natural Numbers via Stirling Numbers

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Abstract—**The current paper is dedicated to proving the many properties of falling numbers and Stirling numbers. Then, these properties are implemented to construct the recurrence relations of the sums of powers $\sigma_m(n) = \sum_{k=1}^{n} k^m$.**

Index Terms—**Sum of powers, difference operators, Stirling numbers of the first kind, Stirling numbers of the second kind, diagonalizable matrices.**

I. Introduction

Assume that $\mathbf{N}$ is the set of natural numbers and let $m \in \mathbf{N}$, the sequence of the sums of the m-th powers of the first n positive integers denoted as $\{\sigma_m(n)\}_{m=0}^{\infty}$, i.e.:

$$\sigma_m(n) = \sum_{k=1}^{n} k^m.$$

Using mathematical induction, it is easy to prove that
$\sigma_1(n) = \frac{n(n+1)}{2}$,
$\sigma_2(n) = \frac{n(n+1)(2n+1)}{6}$,
$\sigma_3(n) = \left[\frac{n(n+1)}{2}\right]^2$,
$\sigma_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$,
$\sigma_5(n) = \frac{2n^6+6n^5+5n^4-n^2}{12}$,
$\sigma_6(n) = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$.
For further properties of $\sigma_m(n)$, see [10], [12], and [13]. The sum of the m-th power for the first n integers as initiated by Faulhaber's formula. This formula, named after Johann Faulhaber, expresses the sum of the m-th powers of the first n positive integers

$$1^m + 2^m + 3^m + \cdots + n^m$$

as a $(m+1)$ th-degree polynomial function of n.
The coefficients of these polynomials are related to Bernoulli numbers B_k by the following formula

$$\sigma_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} (-1)^k \binom{m+1}{k} B_k n^{m+1-k}.$$

More details about this formula and Bernoulli numbers can be found in [9]. In this paper, we will express the sum of powers as a polynomial function using Stirling numbers of the second kind. Stirling numbers of the first and second kinds play a crucial role in combinatorial problems, specifically depicting close correlations with Bernoulli numbers and other combinatorial numbers. The definitions and properties of the classical Stirling numbers are given in the handbook by Abramowitz and Stegun [1]. In this paper, we give definitions for the related sequences to $\sigma_m(n)$ and then use these properties to answer the following questions:

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Problem 1.1: Can we find $\sigma_{m+1}(n)$ using $\sigma_m(n)$? can we find $\sigma_{m+3}(n)$ using $\sigma_{m+1}(n)$ and $\sigma_{m+2}(n)$?

II. Preliminaries

In this section, we give definitions and properties of several sequences that will be used to construct recurrence relations for $\sigma_m(n)$.

A. The falling numbers

Let n be a non-negative integer. The falling numbers, commonly denoted as $(x)_n$, have been introduced by Leo August Pochhammer (see [11]) and can be given as:

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}, n \in \mathbf{N}_0, \quad (1)$$

where $\mathbf{N}_0$ be the set of nonnegative integers. Clearly, $(x)_0 = 1$. Furthermore, for $m, n \in \mathbf{N}_0$ and $m \geq n$, it is easy to prove that $(m)_n = \frac{m!}{(m-n)!}$.
Similarly, one can easily see the following result is true.

Proposition 2.1: For $m \in \mathbf{N}_0$, the falling numbers $(x)_n$ satisfies

$$(x+m)_m (x)_j = (x+m)_{m+j}.$$

In particular,

$$(x+1)(x)_j = (x+1)_{j+1}. \quad (2)$$

Proof:

$$\begin{aligned}(x+m)_m (x)_j &= \frac{\Gamma(x+m+1)}{\Gamma(x+1)} \frac{\Gamma(x+1)}{\Gamma(x+1-j)} \\ &= \frac{\Gamma(x+m+1)}{\Gamma(x+m+1-(j+m))} \\ &= (x+m)_{m+j}.\end{aligned}$$

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For additional properties of these numbers, see [11] and [17].

B. The forward difference operator

Let $S(\mathbf{N})$ be the set of complex valued sequences over $\mathbf{N}$. For $u \in S(\mathbf{N})$, the forward difference operator $\Delta : S(\mathbf{N}) \to S(\mathbf{N})$ can be defined as

$$(\Delta u)(k) = u(k+1) - u(k).$$

Therefore, it is easy to prove the following proposition accordingly

Proposition 2.2: If $u(k) = (k+s)_n$, then $(\Delta u)(k) = n(k+s)_{n-1}$.

Now, for any scalar c we have:

$$\Delta\left(c + \sum_{j=1}^{k-1} u(j)\right) = c + \sum_{j=1}^{k} u(j) - (c + \sum_{j=1}^{k-1} u(j)) = u(k).$$

This equation defines the inverse of the forward difference operator as:

$$(\Delta^{-1}u)(k) = c + \sum_{j=1}^{k-1} u(j). \tag{3}$$

For example, by using (2.2) and Proposition 2.2, one can obtain:

Proposition 2.3: If $u(k) = (k+s)_n$, then $(\Delta^{-1}u)(k) = \frac{(k+s)_{n+1}}{n+1} + c$.

Besides, it is easy to show that the forward difference operator satisfies the following proposition

$$\sum_{n=m}^{N-1} (\triangle f)(n) = f(N) - f(m). \tag{4}$$

For additional properties for difference operators and their applications, see [14], [4], [5], [6], [8] and [7].

C. Stirling numbers

Stirling numbers were conceptually established following James Stirling in the 18th century, which arises throughout various analytic and combinatorial problems. It can be differentiated into two different types, namely Stirling numbers of the first kind and Stirling numbers of the second kind (see [18]). The Stirling numbers of the first kind can be defined as follows:

Definition 2.1: For $n \in \mathbf{N}_0$, the Stirling numbers of the first kind as denoted by $\left[{n \atop k}\right]$ can be defined as

$$(x+n-1)_n = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k.$$

Using this definition, the Stirling numbers of the first kind form a transient matrix, which can be found between the two bases of P_n (the vector space of polynomials of degree n). For example, the equations

$$\begin{aligned} x^3 + 3x^2 + 2x &= (x+2)(x+1)x \\ &= (x+2)_3 \\ &= \left[{3 \atop 0}\right] x^0 + \left[{3 \atop 1}\right] x + \left[{3 \atop 2}\right] x^2 + \left[{3 \atop 3}\right] x^3 \end{aligned}$$

gives $\left[{3 \atop 0}\right] = 0, \left[{3 \atop 1}\right] = 2, \left[{3 \atop 2}\right] = 3$ and $\left[{3 \atop 3}\right] = 1$. Now , it is clear that for $n > 0$, $\left[{0 \atop 0}\right] = 1$ and $\left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0$. Using this fact and Definition 2.1, it is easy to get the following: For all $r \in \mathbf{N}$

$$(k+r)_r = \sum_{j=1}^{r+1} \left[{r+1 \atop j}\right] k^{j-1}. \tag{5}$$

Meanwhile, the definition for the Stirling numbers of the second kind is as follows:

Definition 2.2: For $n \in \mathbf{N}_0$, the sequence $\left\{{n \atop k}\right\}, k = 0, 1, 2 \cdots n$, which satisfies

$$x^n = \sum_{j=0}^{n} \left\{{n \atop j}\right\} (x)_j \tag{6}$$

is called the Stirling numbers of the second kind.

Using this definition, it is easy for one to conclude that:

$$\left\{{0 \atop 0}\right\} = 1 \text{ and } \left\{{n \atop 0}\right\} = 0 \text{ for } n > 0. \tag{7}$$

Besides, the definition will easily depict the manner in which the Stirling numbers of the second kind also forms another transient matrix between the two bases of P_n.

Example 2.1: Using

$$\begin{aligned} x^2 &= x + x(x-1) = (x)_1 + (x)_2 \\ &= \left\{{2 \atop 0}\right\} (x)_0 + \left\{{2 \atop 1}\right\} (x)_1 + \left\{{2 \atop 2}\right\} (x)_2, \end{aligned}$$

we get that $\left\{{2 \atop 0}\right\} = 0, \left\{{2 \atop 1}\right\} = 1$, and $\left\{{2 \atop 2}\right\} = 1$.

In [15], the following explicit formula is given to calculate the Stirling numbers of the second kind

$$\left\{{n \atop k}\right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (k-j)^n. \tag{8}$$

Using (7) and for $k, m \in \mathbf{N}$, the following statement is thus held true:

$$k^m = \sum_{j=1}^{m} \left\{{m \atop j}\right\} (k)_j. \tag{9}$$

Taking the sum from $k = 1$ to n and then using Proposition 2.3 and (4), we get:

$$\sigma_m(n) = \sum_{j=1}^{m} \left\{{m \atop j}\right\} \frac{(n+1)_{j+1}}{j+1}. \tag{10}$$

Example 2.2:

$$\begin{aligned} \sum_{j=1}^{n} k^2 = \sigma_2(n) &= \sum_{j=1}^{2} \left\{{2 \atop j}\right\} \frac{(n+1)_{j+1}}{j+1} \\ &= \left\{{2 \atop 1}\right\} \frac{(n+1)_2}{2} + \left\{{2 \atop 2}\right\} \frac{(n+1)_3}{3} \\ &= \frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{3} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Now, interchanging the sums gives

$$\begin{aligned} \sum_{n=1}^{N} \sigma_m(n) &= \sum_{n=1}^{N} \sum_{k=1}^{n} k^m = \sum_{k=1}^{N} \sum_{n=k}^{N} k^m \\ &= \sum_{k=1}^{N} (N-k+1) k^m \\ &= (N+1)\sigma_m(N) - \sigma_{m+1}(N). \end{aligned}$$

Therefore,

$$\sum_{n=1}^{N} \sigma_m(n) = (N+1)\sigma_m(N) - \sigma_{m+1}(N). \tag{11}$$

For the additional properties of the Stirling numbers, see [16] and [3].

III. Main Results

In this section, we construct recurrence relations for the sequence of the sums of powers $\sigma_m(n) = \sum_{k=1}^{n} k^m$. In particular, it is proven that using $\sigma_m(n)$, one can find $\sigma_{m+1}(n), \sigma_{m+2}(n)$, and $\sigma_{m+3}(n)$.

Let

$$a(m; r; k) = \sum_{j=1}^{m} \left\{{m \atop j}\right\} (k+r)_{j+r},$$

and

$$b(m;r;n) = \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{(n+r+1)_{j+r+1}}{j+r+1}.$$

It is easy to notice that $a(m;0;k) = k^m$ and $b(m;0;k) = \sigma_m(n)$. Now, using Proposition 2.3 and (4), we get that :

$$\begin{aligned} \sum_{k=1}^{n} a(m;r;k) &= \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \Big(\sum_{k=1}^{n} (k+r)_{j+r} \Big) \\ &= \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{(n+r+1)_{j+r+1}}{j+r+1} \\ &= b(m;r;n). \end{aligned}$$

Therefore, we have the following result

Proposition 3.1: For $r \in \mathbf{N}_0$ and $m, n \in \mathbf{N}$,

$$b(m;r;n) = \sum_{k=1}^{n} a(m;r;k).$$

Theorem 3.2: For $r, m, k \in \mathbf{N}$

$$a(m;r;k) = (k+r)_r k^m.$$

Proof: Using Proposition 2.1

$$\begin{aligned} (k+r)_r k^m &= \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (k+r)_r (k)_j \\ &= \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (k+r)_{r+j} \\ &= a(m;r;k). \end{aligned}$$

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Theorem 6.1 with $r = 1, 2, 3,$ and 4 implies

Corollary 3.3: For $m, k \in \mathbf{N}$

1) $a(m;1;k) = k^{m+1} + k^m$.
2) $a(m;2;k) = k^{m+2} + 3k^{m+1} + 2k^m$.
3) $a(m;3;k) = k^{m+3} + 6k^{m+2} + 11k^{m+1} + 6k^m$.
4) $a(m;4;k) = k^{m+4} + 10k^{m+3} + 35k^{m+2} + 50k^{m+1} + 24k^m$.

Meanwhile, the following theorem establishes recurrence relations for the sequence of the sums of powers $\sigma_m(n)$.

Theorem 3.4: For $r \in \mathbf{N}$

$$\sigma_{m+r}(n) = b(m;r;n) - \sum_{i=1}^{r} \begin{bmatrix} r+1 \\ i \end{bmatrix} \sigma_{m+i-1}(n).$$

Proof: Equation 5 implies

$$(k+r)_r k^m = k^{m+r} + \sum_{j=1}^{r} \begin{bmatrix} r+1 \\ j \end{bmatrix} k^{m+j-1}. \tag{12}$$

Using Theorem 6.1, (12), and (3.1) give:

$$k^{m+r} = a(m;r;k) - \sum_{j=1}^{r} \begin{bmatrix} r+1 \\ j \end{bmatrix} k^{m+j-1}. \tag{13}$$

Morover, by taking the sum from $k = 1$ to $k = n$ and using Proposition 2.3 and Equation 4, we will get the following results.

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Theorem 3.4 with $r = 1, 2, 3,$ and 4 implies

Corollary 3.5: For $m \in \mathbf{N}$,

1) $\sigma_{m+1}(n) = b(m;1;n) - \sigma_m(n)$.
2) $\sigma_{m+2}(n) = b(m;2;n) - 3\sigma_{m+1}(n) - 2\sigma_m(n)$
3) $\sigma_{m+3}(n) = b(m;3;n) - 6\sigma_{m+2}(n) - 11\sigma_{m+1}(n) - 6\sigma_m(n)$.
4) $\sigma_{m+4}(n) = b(m;4;n) - 10\sigma_{m+3}(n) - 35\sigma_{m+2}(n) - 50\sigma_{m+1}(n) - 24\sigma_m(n)$

Corollary 2.4 gives the answer to Problem 1.1. For example, one can find $\sigma_2(n)$ using $\sigma_1(n)$. Indeed, using part (1), we have

$$\begin{aligned} \sigma_2(n) &= b(1;1;n) - \sigma_1(n) \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Now, let $\mathbf{A}= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 \\ 24 & 50 & 35 & 10 & 1 \end{pmatrix}$, $\mathbf{X}= \begin{pmatrix} \sigma_m(n) \\ \sigma_{m+1}(n) \\ \sigma_{m+2}(n) \\ \sigma_{m+3}(n) \\ \sigma_{m+4}(n) \end{pmatrix}$,

and $\mathbf{B} = \begin{pmatrix} b(m;0;k) \\ b(m;1;k) \\ b(m;2;k) \\ b(m;3;k) \\ b(m;4;k) \end{pmatrix}$.

Here, Corollary 6.5 can be rearranged and written as $\mathbf{AX} = \mathbf{B}$, which will lead to $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. Therefore, $\sigma_{m+1}(n)$, $\sigma_{m+2}(n), \sigma_{m+3}(n)$, and $\sigma_{m+4}(n)$ are given explicitly as

Corollary 3.6: 1) $\sigma_{m+1}(n) = b(m;1;k) - b(m;0;k)$.
2) $\sigma_{m+2}(n) = b(m;2;k) - 3b(m;1;k) + b(m;0;k)$.
3) $\sigma_{m+3}(n) = b(m;3;k) - 6b(m;2;k) + 7b(m;1;k) - b(m;0;k)$.
4) $\sigma_{m+4}(n) = b(m;4;k) - 15b(m;3;k) + 25b(m;2;k) - 10b(m;1;k) + b(m;0;k)$.

Taking the sum from $n = 1$ to N and using (11) and (4), new recurrence relations are generated. For example, by applying these steps to part(3), it gives that:

$$\begin{aligned} &(N+1)\sigma_{m+3}(N) - \sigma_{m+4}(N) \\ &= \sum_{j=1}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \Big(\frac{(N+5)_{j+5}}{(j+4)(j+5)} - 6\frac{(N+4)_{j+4}}{(j+3)(j+4)} \\ &+ 7\frac{(N+3)_{j+3}}{(j+2)(j+3)} - \frac{(N+2)_{j+2}}{(j+1)(j+2)} \Big). \end{aligned} \tag{14}$$

IV. Matrix analysis for main results

Let

$$\mathbf{X}(m,n) = \begin{pmatrix} \sigma_m(n) \\ \sigma_{m+1}(n) \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ -2 & -3 \end{pmatrix}, \mathbf{C}(m,n) = \begin{pmatrix} b(m;1;k) \\ b(m;2;k) \end{pmatrix},$$

and

$$\begin{aligned} \mathbf{B}(m,n) &= \begin{pmatrix} b(m;2;k) \\ b(m;3;k) - 6b(m;2;k) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -6 \end{pmatrix} b(m;2;k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} b(m;3;k). \end{aligned} \tag{15}$$

Then, Corollary 6.5 may be denoted in matrix notation as:

Proposition 4.1: For $m \in \mathbf{N}$,

1) $\mathbf{X}(m+1,n) = \mathbf{DX}(m,n) + \mathbf{C}(m,n)$.

2) $\mathbf{X}(m+2,n) = \mathbf{A}\mathbf{X}(m,n) + \mathbf{B}\ (m,n)$.

The following results generate recurrence equations of Stirling numbers accordingly.

Theorem 4.2: For $s \in \mathbf{N}$,

1) $\mathbf{X}(m+s,n) = \mathbf{D}^s\mathbf{X}(m,n) + \sum_{j=0}^{s-1}\mathbf{D}^{s-1-j}\mathbf{C}\ (m+j,n)$.
2) $\mathbf{X}(m+2s,n) = \mathbf{A}^s\mathbf{X}(m,n) + \sum_{j=0}^{s-1}\mathbf{A}^{s-1-j}\mathbf{B}\ (m+2j,n)$.

Proof: The proof is done via induction. Proposition 4.1 implies that the statement is true $s = 1$. Assume the statement is true for s, we can prove that the statement is also true for $s+1$. Similarly, Proposition 4.1 implies

$$\mathbf{X}(m+2s+2,n) = \mathbf{A}\mathbf{X}(m+2s,n) + \mathbf{B}(m+2s,n).$$

Hence, by the s-statement

$$\begin{aligned}
&\mathbf{X}(m+2s+2,n)\\
&= \mathbf{A}\left(\mathbf{A}^s\mathbf{X}(m,n) + \sum_{j=0}^{s-1}\mathbf{A}^{s-1-j}\mathbf{B}(m+2j,n)\right)\\
&+ \mathbf{B}(m+2s,n)\\
&= \mathbf{A}^{s+1}\mathbf{X}(m,n) + \sum_{j=0}^{s}\mathbf{A}^{s-j}\mathbf{B}(m+2j,n).
\end{aligned}$$

The proof of part (1) can be achieved by similar steps. ■

V. Numerical results

In this section, we find $\sigma_5(n)$ and $\sigma_6(n)$ by using $\sigma_3(n)$ and $\sigma_2(n)$.

Since $\mathbf{X}(4,n) = \mathbf{A}\mathbf{X}(2,n) + \mathbf{B}(2,n)$ and $\mathbf{X}(2,n) = \begin{pmatrix} \sigma_2(n) \\ \sigma_3(n) \end{pmatrix} = \begin{pmatrix} \frac{n(n+1)(2n+1)}{6} \\ (\frac{n(n+1)}{2})^2 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{X}(4,n) &= \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} \frac{n(n+1)(2n+1)}{6} \\ (\frac{n(n+1)}{2})^2 \end{pmatrix}\\
&+ \begin{pmatrix} 1 \\ -6 \end{pmatrix} b(2;2;k)\\
&+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} b(2;3;k)\\
&= \begin{pmatrix} -2\frac{n(n+1)(2n+1)}{6} - 3(\frac{n(n+1)}{2})^2 \\ 6\frac{n(n+1)(2n+1)}{6} + 7(\frac{n(n+1)}{2})^2 \end{pmatrix}\\
&+ \begin{pmatrix} b(2;2;k) \\ b(2;3;k) - 6b(2;2;k) \end{pmatrix}\\
&= \begin{pmatrix} \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30} \\ \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \end{pmatrix}
\end{aligned}$$

Therefore, $\sigma_4(n) = \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30}$ and $\sigma_5(n) = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

Using this result, one can find $\sigma_6(n)$. Now, we start with $\mathbf{X}(5,n) = \mathbf{D}\mathbf{X}(4,n) + \mathbf{C}(4,n)$. Besides,

$$\begin{aligned}
\mathbf{X}(5,n) &= \begin{pmatrix} -1 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30} \\ \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \end{pmatrix}\\
&+ \begin{pmatrix} b(4;1;k) \\ b(4;2;k). \end{pmatrix}\\
&= \begin{pmatrix} \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \\ \frac{n(2n+1)(n+1)(3n^4+6n^3-3n+1)}{42}. \end{pmatrix}
\end{aligned}$$

Therefore, $\sigma_6(n) = \frac{n(2n+1)(n+1)(3n^4+6n^3-3n+1)}{42}$.

VI. An extension for the main results

For $l \in \mathbf{N}_0$, define the generalized sum of powers

$$\sigma_{l,f}(n) = \sum_{k=0}^{n} k^l f(k). \tag{16}$$

In this section, we construct recurrence relations for $\sigma_f(n)$. If $f(k) = k^m$, then $\sigma_{l,f}(n) = \sigma_{m+l}(n)$.

The falling numbers occur in a formula which represents polynomials using the forward difference operator Δ and which is formally similar to Taylor's theorem:

$$f(x) = \sum_{n=0}^{\infty}\left[\frac{(\Delta^n f)(0)}{n!}\right](x)_n. \tag{17}$$

If $k \in \mathbf{N}_0$ and $f : \mathbf{N}_0 \longrightarrow \mathbf{C}$, then

$$f(k) = \sum_{n=0}^{\infty}\left[\frac{(\Delta^n f)(0)}{n!}\right](k)_n = \sum_{n=0}^{k}\left[\frac{(\Delta^n f)(0)}{n!}\right](k)_n. \tag{18}$$

To construct recurrence relations for the sequence of the generalized sum of powers $\sigma_{m,f}(n)$, the following theorem is needed

Theorem 6.1: For $r, k \in \mathbf{N}$

$$(k+r)_r f(k) = \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r}.$$

Proof: Using Proposition 2.1

$$\begin{aligned}
(k+r)_r f(k) &= \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+r)_r(k)_j\\
&= \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r}.
\end{aligned}$$

■

Theorem 6.1 with $r = 1, 2, 3$, and 4 implies

Corollary 6.2: For $m, k \in \mathbf{N}$

1) $kf(k) + f(k) = \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+1)_{j+1}$.
2) $k^2 f(k) + 3kf(k) + 2f(k) = \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+2)_{j+2}$.
3) $k^3 f(k) + 6k^2 f(k) + 11kf(k) + 6f(k)$ $= \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+3)_{j+3}$.
4) $k^4 f(k) + 10k^3 f(k) + 35k^2 f(k) + 50kf(k) + 24f(k) = \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+4)_{j+4}$.

The following Theorem generalizes the conclusion of Corollary 6.2

Theorem 6.3: For $r \in \mathbf{N}_0$,

$$k^r f(k) = \sum_{j=0}^{k}\frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r} - \sum_{j=1}^{r}\begin{bmatrix} r+1 \\ j \end{bmatrix} k^{j-1} f(k).$$

Proof: Using (5),

$$\begin{aligned}
(k+r)_r f(k) &= \sum_{j=1}^{r+1}\begin{bmatrix} r+1 \\ j \end{bmatrix} k^{j-1} f(k)\\
&= k^r f(k) + \sum_{j=1}^{r}\begin{bmatrix} r+1 \\ j \end{bmatrix} k^{j-1} f(k).
\end{aligned}$$

Now, Theorem 6.1 gives

$$k^r f(k) = \sum_{j=0}^{k} \frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r} - \sum_{j=1}^{r} \begin{bmatrix} r+1 \\ j \end{bmatrix} k^{j-1} f(k).$$

■

Now,

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r} \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} \frac{(\Delta^j f)(0)}{j!}(k+r)_{j+r} \\
&= \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\Big(\frac{(n+r+1)_{j+r+1}}{j+r+1} - \frac{(j+r)_{j+r+1}}{j+r+1}\Big) \\
&= \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+r+1)_{j+r+1}}{j+r+1}. \qquad (19)
\end{aligned}$$

By taking the sum from $k = 0$ to $k = n$ for Theorem 6.3 and using Proposition 2.3, Equation 4, and interchanging the sums (19), we will get the following theorem:

Theorem 6.4: For $r \in \mathbf{N}_0$,

$$\sigma_{r,f}(n) = \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+r+1)_{j+r+1}}{j+r+1} - \sum_{j=1}^{r} \begin{bmatrix} r+1 \\ j \end{bmatrix} \sigma_{j-1,f}(n).$$

Corollary 6.5: The sequence $\sigma_{r,f}(n); r = 0, 1, 2, 3,$ and 4 satisfy

1) $\sigma_{1,f}(n) = \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+2)_{j+2}}{j+2} - \sigma_{0,f}(n)$.
2) $\sigma_{2,f}(n) = \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+3)_{j+3}}{j+3} - 3\sigma_{1,f}(n) - 2\sigma_{0,f}(n)$.
3) $\sigma_{3,f}(n) = \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+4)_{j+4}}{j+4} - 6\sigma_{2,f}(n) - 11\sigma_{1,f}(n) - 6\sigma_0(n)$.
4) $\sigma_{4,f}(n) = \sum_{j=0}^{n} \frac{(\Delta^j f)(0)}{j!}\frac{(n+5)_{j+5}}{j+5} - 10\sigma_{3,f}(n) - 35\sigma_{2,f}(n) - 50\sigma_{1,f}(n) - 24\sigma_{0,f}(n)$.

Example 6.1: For $r \in \mathbf{N}$, consider $f(k) = (k)_r$, then clearly $(\Delta^r f)(0) = r!$ and $(\Delta^j f)(0) = 0$ for $j \neq r$. Using this fact and Corollary 6.5, we conclude that

$$\sum_{k=1}^{n} k(k)_r = \sigma_{1,f}(n) = \frac{(n+2)_{r+2}}{r+2} - \sum_{k=1}^{n}(k)_r = \frac{(n+2)_{r+2}}{r+2} - \frac{(n+1)_{r+1}}{r+1}.$$

Therefore,

$$\sigma_{1,f}(n) = \frac{(n+2)_{r+2}}{r+2} - \frac{(n+1)_{r+1}}{r+1}. \qquad (20)$$

In particular,

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} k(k)_1 = \frac{(n+2)_3}{3} - \frac{(n+1)_2}{2} = \frac{1}{6}n(n+1)(2n+1).$$

Using (20),

$$\begin{aligned}
\sum_{k=1}^{n} k^2(k)_r &= \sigma_{2,f}(n) \\
&= \frac{(n+3)_{r+3}}{r+3} - 3\left(\frac{(n+2)_{r+2}}{r+2} - \frac{(n+1)_{r+1}}{r+1}\right) \\
&\quad - 2\frac{(n+1)_{r+1}}{r+1} \\
&= \frac{(n+3)_{r+3}}{r+3} - 3\frac{(n+2)_{r+2}}{r+2} + \frac{(n+1)_{r+1}}{r+1}.
\end{aligned}$$

In particular,

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k^2(k)_1 = \frac{(n+3)_4}{4} - (n+2)_3 + \frac{(n+1)_2}{2} = \left(\frac{n(n+1)}{2}\right)^2.$$

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