Finite Volume Elements for Parabolic Optimal Control Problems Based on Variational Discretization

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***Abstract*—This paper presents a finite volume element method (FVEM) to solve the parabolic optimal control problems. Variational discretization is applied to discrete the optimality conditions and the derivation of the estimate becomes very simple. It reflects the power of variational discretizaton. Semi-discrete and fully discrete error estimates are presented separately by using both the transposition techniques [J. L. Lions et al. Non-homogeneous boundary value problems and applications, volume 1. Springer Berlin, 1972.] and the variational discretization. Numerical examples are given to confirm the theoretical results.**

***Index Terms*—PDE-constrained optimization, variational discretization, finite volume elements, parabolic optimal control.**

I. Introduction

The optimal control problems governed by parabolic partial differential equations (PDEs) have many applications in different areas, including physical, social, and economic processes. The numerical research on optimal control problems with the PDE constraint is becoming more important in science and engineering technology [11, 16]. Many works on the parabolic optimal control problems [1, 2, 10, 12, 13, 15] have been published to attract people's attention. However, there are only a few papers adopt the finite volume element method (FVEM). The FVEM [7] as an numerical tool to treat the numerical solutions of PDEs has been proposed. It's accuracy is higher than the finite difference method and almost equal to the finite element method (FEM), while its computational cost is less than that of the FEM. In this paper, the following linear-quadratic optimal control problem [9] for the control u and state y is considered:

$$\begin{aligned} \text{Min } J(y,u) =& \frac{1}{2}\int_0^T \int_\Omega (y-y_d)^2 dxdt \\ &+ \frac{\alpha}{2}\int_0^T \int_\Omega u^2 dxdt \end{aligned} \tag{1}$$

subject to

$$\tfrac{\partial y}{\partial t} - \mu\Delta y + \sigma y = f + u, \ (x,t) \in \Omega \times [0,T], \tag{2}$$

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{3}$$

$$y(x,0) = \phi(x), \quad x \in \Omega. \tag{4}$$

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Here u obeys the control constraints $u_a \le u \le u_b$.

In recent article[10], the authors give some estimates of the FVEM for parabolic optimal control problems. We differ from it on the deducing process of the error estimates. It seems to be more simple that we derive the error estimates using the transposition techniques [8] and the variational discretization [3] in optimal control with control constraints. Once the optimal adjoint state which had been treated numerically is known, variable discretization tailors the discretization of the control to the discrete treatment of the adjoint through the projection equation.

This paper aimes at proposing a FVEM for the parabolic PDE constrained optimization problem and presenting the error estimates of optimal order in the L^2 norm. While for elliptic PDE constrained optimization problems the error estimates of numerical methods are developed far, less have been done for numerical treatment of time-dependent optimization problems. Firstly, the Lagrange multiplier method [4, 5, 6] is applied to the deduction of the Karush-Kuhn-Tucker (KKT) conditions. The KKT-system contains the state equation, the adjoint equation and the variational inequality. Then, we apply a concept called variational discretization proposed by Hinze [3]to discritize the control space implicitly via the state equation. Due to the variational discretization, the error estimate becomes so simple and that reflects the power of variational discretizaton. At last, the L^2-error estimates of optimal order and the fully discrete error estimates are obtained. Some examples are provided to test and verify the error estimates.

In Section II, we give the KKT conditions for the problem considered. It is discretized by the variational discretization concept and the FVEM in Section III. The error estimates in the sense of L^2 norm are presented in Section IV. In Section V, some examples are given to test and verify the effectiveness of the above method and results of error estimates. In the last section, conclusions are made.

II. Model problem and optimality conditions

Consider the following parabolic PDE equations:

$$\tfrac{\partial y}{\partial t} - \mu\Delta y + \sigma y = f + u, \ (x,t) \in \Omega \times [0,T], \tag{5}$$

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{6}$$

$$y(x,0) = \phi(x), \quad x \in \Omega, \tag{7}$$

where $\Omega \subset R^2$ is a bound convex domain, $\partial\Omega$ is the boundary of Ω, μ, σ are constants, and f,ϕ are smooth functions.

For $y(t), v(t) \in H_0^1(\Omega)$, we may use Green's formulation and write (5) in an integral form:

$$(y_t, v) + a(y, v) = (f + u, v), \tag{8}$$

where $y(\cdot, t)$ is written as $y(t)$ for short, $(y, v) = \int_\Omega yvdx$, $y_t = \frac{\partial y}{\partial t}$ and $a(y, v) = \int_\Omega (\mu \nabla y \cdot \nabla v + \sigma yv)dx$.

With the transposition techniques ([8]), problem (5)-(7) has a unique solution $y \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$ in the sense that

$$\begin{aligned} &-\int_0^T \left(y, \frac{\partial v}{\partial t}\right) dt - \int_0^T \int_\Omega (\mu y \Delta v - \sigma yv) dxdt \\ &= \int_0^T (f + u, v)dt, \end{aligned} \tag{9}$$

for all $v \in V$, where

$$\begin{aligned} V := \{&v \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \\ &: \ v(T) = 0\}. \end{aligned} \tag{10}$$

Problem II.1. $(\mathbf{P})$ *Consider the optimization problem of minimizing*

$$J(y, u) = \frac{1}{2} \int_0^T \int_\Omega (y - y_d)^2 dxdt + \frac{\alpha}{2} \int_0^T \int_\Omega u^2 dxdt \tag{11}$$

over all $(y, u) \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$ *subject to the parabolic system (5)-(7) and the control constraints*

$$u_a \le u \le u_b. \tag{12}$$

Here α is a fixed positive number which called regularization parameter and $y_d \in L^2(0, T; L^2(\Omega))$ is a given function. The admissible controls set of $(\mathbf{P})$ is written by

$$\begin{aligned} U_{ad} = \{&u \in L^2(0, T; L^2(\Omega)) : u_a \le u \le u_b \\ &\text{almost everywhere in } (0, T]\}, \end{aligned} \tag{13}$$

here $u_a < u_b$ are given constants.

Due to the linear-quadratic property and convexity, the standard techniques in [4, 14] and be used and then the existence of solutions and KKT conditions can be proofed directly.

Theorem II.2. *The problem* $(\mathbf{P})$ *admittes a unique optimal control* u^**, an associated state* y^* *and an adjoint state* p^* *that satisfy the KKT conditions which contain the following state equation*

$$\left(\frac{\partial y^*}{\partial t}, v\right) + a(y^*, v) = (f + u^*, v), \ \forall v \in H_0^1(\Omega), \tag{14}$$

the adjoint equation

$$-\left(\frac{\partial p^*}{\partial t}, v\right) + a(p^*, v) = (y^* - y_d, v), \ \forall v \in H_0^1(\Omega), \tag{15}$$

and the variational inequality

$$\int_0^T (\alpha u^* + p^*, w - u^*)dt \ge 0 \quad \forall\, w \in U_{ad}. \tag{16}$$

Here, the variational inequality can be replaced by the projection equation

$$u^* = \mathcal{P}_{[u_a, u_b]}\left(-\frac{1}{\alpha} p^*\right), \tag{17}$$

where $\mathcal{P}_{[u_a, u_b]}(v)$ *means the projection of* $v \in \mathbb{R}$ *onto the interval* $[u_a, u_b]$.

The above adjoint equation (15) is the weak form of the following parabolic equations which run backwards in time:

$$-\tfrac{\partial p^*}{\partial t} - \mu \triangle p^* + \sigma p^* = y^* - y_d, \ \Omega \times [0, T], \tag{18}$$

$$p^*(x, t) = 0, \quad (x, t) \in \Gamma \times [0, T], \tag{19}$$

$$p^*(x, T) = 0, \quad x \in [0, L]. \tag{20}$$

III. Discretization

The optimization problem $(\mathbf{P})$ is the one in infinite dimensional space, and its solution is not easy to be computed on the computer. Applying the FVEM to solve the problem $(\mathbf{P})$, so we discretize it to get a finite dimensional one which can be computed numerically. Then the variational disretization concept [3] is used, and with it the state becomes a finite volume element approximation of the state equation (5)-(7).

We place a quasi-uniform triangulation T_h of Ω. We choose T_h^* to be the barycenter or circumcenter dual decomposition relative to T_h. We choose the linear element space to be the trial space U_h related to the triangulation T_h, and we take the piecewise constant function space as the test function space V_h which is related to the dual decomposition T_h^*. N_h denotes the set of the interior nodes of T_h. For $p_i \in N_h$, $T_i^* \in T_h^*$ denotes the dual element containing p_i. The operator $I_h : U_h \to V_h$ is defined as follows to connect the trial and test spaces

$$I_h v_h = \sum_{p_i \in N_h} v_h(p_i)\chi_i, \quad I_h v_h|_{T_i^*} = v_h(p_i) \ \ \forall p_i \in N_h,$$

where χ_i is the characteristic function of T_i^*.

Define the bilinear form, for all $u_h \in U_h, v_h \in U_h$,

$$\begin{aligned} &A_h(u_h, v_h) \doteq a_h(u_h, I_h v_h) \\ &= \sum_{p_i \in N_h} \left(v_h(p_i) \int_{\partial T_i^*} \mu \nabla u_h \cdot n ds + \int_{T_i^*} \sigma u_h v_h dx \right). \end{aligned} \tag{21}$$

The semi-discrete finite volume element scheme of (8) is: Find $y_h = y_h(\cdot, t) \in U_h$ $(0 \le t \le T)$, $\forall v_h \in U_h$ such that

$$(\frac{\partial y_h}{\partial t}, v_h) + A_h(y_h, v_h) = (f + u, v_h). \tag{22}$$

Using the variational discretization concept presented in [3], we define the discrete problem $(\mathbf{P_h})$.

Problem III.1. $(\mathbf{P_h})$ *Consider the problem*

$$\begin{aligned} min J_h(y_h, u) = &\frac{1}{2} \int_0^T \int_\Omega (y_h - y_d)^2 dxdt \\ &+ \frac{\alpha}{2} \int_0^T \int_\Omega u^2 dxdt \end{aligned} \tag{23}$$

subject to

$$(\frac{\partial y_h}{\partial t}, v_h) + A_h(y_h, v_h) = (f + u, v_h), \ \forall v_h \in U_h, \tag{24}$$

and the control constraints

$$u_a \le u \le u_b. \tag{25}$$

Lemma III.2. *For a constant coefficient μ, the bilinear form $A_h(\cdot,\cdot)$ is symmetric, that is*

$$A_h(u_h, v_h) = A_h(v_h, u_h) \quad \forall u_h \in U_h, \forall v_h \in U_h.$$

Proof: Please see [7, page 122]. ■

Theorem III.3. *The optimal control $u_h^* \in L^2(\Omega)$, associated state $y_h^* \in U_h$ and adjoint state $p_h^* \in U_h$ of $(\mathbf{P_h})$ satisfy the state equation*

$$(\frac{\partial y_h^*}{\partial t}, v_h) + A_h(y_h^*, v_h) = (f + u_h^*, v_h), \ \forall v_h \in U_h, \quad (26)$$

the adjoint equation

$$-(\frac{\partial p_h^*}{\partial t}, v_h) + A_h(p_h^*, v_h) = (y_h^* - y_d, v_h), \ \forall v_h \in U_h, \quad (27)$$

and the projection equation

$$u_h^* = \mathcal{P}_{[u_a,u_b]}\left(-\frac{1}{\alpha}p_h^*\right). \quad (28)$$

In addition, (28) is equivalent to

$$\int_0^T (\alpha u_h^* + p_h^*, w - u_h^*)dt \geq 0, \quad \forall w \in U_{ad}. \quad (29)$$

IV. Error analysis of optimal control problem

First, we introduce auxiliary functions $\widetilde{y}$ and $\widetilde{p}$ satisfying

$$\begin{aligned}&(\widetilde{y}_t, v) + a(\widetilde{y}, v) = (f + u_h^*, v), \ \forall v \in H^1(\Omega),\\ &-(\widetilde{p}_t, v) + a(\widetilde{p}, v) = (y_h^* - y_d, v), \ \forall v \in H^1(\Omega),\end{aligned} \quad (30)$$

where u_h^* and y_h^* are the solutions to $(\mathbf{P_h})$.

A. Error estimates for the semi-discrete schemes

The following lemma, we restate some results of the finite volume discretization for the state equation.

Lemma IV.1. *Let y and y_h be the solutions to the problem (8) and the semi-discrete generalized scheme (22) respectively. Then there holds the following error estimates*

$$\|y - y_h\|_0 \leq Ch^2. \quad (31)$$

Proof: Please see [7, Lem. 5.1.3]. ■

Now we will prove the error estimates for the proposed method in the following theorem.

Theorem IV.2. *Let (u^*, y^*, p^*) and (u_h^*, y_h^*, p_h^*) be the solutions of the problems (P) and (P_h) respectively. Then there exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned}&\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(\Omega))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(\Omega))}\\ &+ \|p^* - p_h^*\|_{L^2(0,T;L^2(\Omega))} \leq Ch^2.\end{aligned} \quad (32)$$

Proof: Testing (16) with u_h^* and (29) with u^*, we obtain

$$\int_0^T (\alpha(u^* - u_h^*) + (p^* - p_h^*), u_h^* - u^*)dt \geq 0. \quad (33)$$

Therefore,

$$\begin{aligned}&\alpha\|u^* - u_h^*\|^2_{L^2(0,T;L^2(\Omega))}\\ \leq&\int_0^T (p^* - p_h^*, u_h^* - u^*)dt\\ \leq&\int_0^T (p^* - \widetilde{p}, u_h^* - u^*)dt\\ &+ \int_0^T (\widetilde{p} - p_h^*, u_h^* - u^*)dt.\end{aligned} \quad (34)$$

Following (9), (14) and (30) we have

$$\begin{aligned}&-\int_0^T (y^* - \widetilde{y}, v_t)dt - \int_0^T \int_\Omega [\mu(y^* - \widetilde{y})\Delta v\\ &- \sigma(y^* - \widetilde{y})v]dxdt = \int_0^T (u^* - u_h^*, v)dt.\end{aligned} \quad (35)$$

Setting $v = \widetilde{p} - p^*$ in (35) yields

$$\begin{aligned}&\int_0^T (\widetilde{p} - p^*, u^* - u_h^*)dt\\ =&-\int_0^T (\widetilde{y} - y^*, \widetilde{p}_t - p_t^*)dt - \int_0^T \int_\Omega [\mu(y^* - \widetilde{y})\Delta(\widetilde{p} - p^*)\\ &- \sigma(y^* - \widetilde{y})(\widetilde{p} - p^*)]dxdt\\ =&-\int_0^T (\widetilde{p}_t, \widetilde{y} - y^*)dt + \int_0^T a(\widetilde{p}, \widetilde{y} - y^*)dt\\ &+ \int_0^T (p_t^*, \widetilde{y} - y^*)dt - \int_0^T a(p^*, \widetilde{y} - y^*)dt\\ =&\int_0^T (y_h^* - y_d, \widetilde{y} - y^*)dt - \int_0^T (y^* - y_d, \widetilde{y} - y^*)dt\\ =&\int_0^T (y_h^* - y^*, \widetilde{y} - y^*)dt\\ =&-\int_0^T \|y_h^* - y^*\|_0^2 dt + \int_0^T (y_h^* - y^*, y_h^* - \widetilde{y})dt\\ \leq&-\frac{1}{2}\|y_h^* - y^*\|^2_{L^2(0,T;L^2(\Omega))}\\ &+ \frac{1}{2}\|y_h^* - \widetilde{y}\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \quad (36)$$

With Young's inequality, we have

$$\begin{aligned}&\int_0^T (\widetilde{p} - p_h^*, u_h^* - u^*)dt\\ \leq&\frac{1}{2}\alpha\|u_h^* - u^*\|^2_{L^2(0,T;L^2(\Omega))}\\ &+ C(\alpha)\|\widetilde{p} - p_h^*\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \quad (37)$$

Therefore, we have

$$\begin{aligned}&\alpha\|u^* - u_h^*\|^2_{L^2(0,T;L^2(\Omega))} + \|y_h^* - y^*\|^2_{L^2(0,T;L^2(\Omega))}\\ \leq&C\left(\|y_h^* - \widetilde{y}\|^2_{L^2(0,T;L^2(\Omega))} + \|\widetilde{p} - p_h^*\|^2_{L^2(0,T;L^2(\Omega))}\right).\end{aligned} \quad (38)$$

From Lemma IV.1, there exists a constant $C > 0$, independent of h such that

$$\|\widetilde{y} - y_h^*\|_0 \leq Ch^2, \ \|\widetilde{p} - p_h^*\|_0 \leq Ch^2. \quad (39)$$

Combining (38) and (39) we get

$$\begin{aligned}&\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(\Omega))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(\Omega))}\\ \leq&Ch^2.\end{aligned} \quad (40)$$

Using (15) and (30) we obtain

$$-(p_t^* - \widetilde{p}_t, v) + a(p^* - \widetilde{p}, v) = (y^* - y_h^*, v). \quad (41)$$

This implies

$$\|p^* - \widetilde{p}\|_{L^2(0,T;L^2(\Omega))} \leq C\|y^* - y_h^*\|_{L^2(0,T;L^2(\Omega))}. \quad (42)$$

With (39), (40) and (42) we now conclude

$$\begin{aligned}&\|p^* - p_h^*\|_{L^2(0,T;L^2(\Omega))}\\ \leq&\|p^* - \widetilde{p}\|_{L^2(0,T;L^2(\Omega))} + \|\widetilde{p} - p_h^*\|_{L^2(0,T;L^2(\Omega))}\\ \leq&Ch^2.\end{aligned} \quad (43)$$

This, together with (40), completes the proof of the theorem. ■

B. Fully discrete schemes and error estimates

Problem IV.3. $(\mathbf{P_{hk}})$ *Consider the problem of minimizing*

$$J_h(y_{hk}, u) = \frac{1}{2}\int_0^T \int_\Omega (y_{hk} - y_d)^2 dxdt + \frac{\alpha}{2}\int_0^T \int_\Omega u^2 dxdt \tag{44}$$

subject to

$$\int_\Omega \frac{y_{hk}^n - y_{hk}^{n-1}}{\Delta t} v_h dx + A_h(y_{hk}^n, v_h) = \int_\Omega (f^n + u) v_h dx,\ \forall\ v_h \in U_h, \tag{45}$$

where $k = \Delta t$, $t_n = n\Delta t$, $y_{hk}^n = y_h(x, t_n) \in U_h$, *and* $f^n = f(x, t_n)$, *and the control constraints*

$$u_a \le u \le u_b. \tag{46}$$

Lemma IV.4. *Let* y *and* y_{hk} *be the solutions to and the fully discrete scheme (45) of the problem (8) respectively. Then there holds the following error estimates*

$$\|y - y_{hk}\|_0 \le C(k + h^2). \tag{47}$$

Proof: Please see [7, Chap. 5.2]. ■

Theorem IV.5. *Let* (u^*, y^*, p^*) *and* $(u_{hk}^*, y_{hk}^*, p_{hk}^*)$ *be the solutions of the problems* $(\mathbf{P})$ *and* $(\mathbf{P_{hk}})$ *respectively. Then there exists a constant* $C > 0$, *independent of* h *and* k, *such that*

$$\sqrt{\alpha}\|u^* - u_h^*\|_{L^2(0,T;L^2(\Omega))} + \|y^* - y_h^*\|_{L^2(0,T;L^2(\Omega))} \le C(k + h^2). \tag{48}$$

Proof: Define the auxiliary functions $\widehat{y}$ and $\widehat{p}$ satisfying

$$(\widehat{y}_t, v) + a(\widehat{y}, v) = (f + u_{hk}^*, v),\ \forall v \in H^1(\Omega), \tag{49}$$

$$-(\widehat{p}_t, v) + a(\widehat{p}, v) = (y_{hk}^* - y_d, v),\ \forall v \in H^1(\Omega). \tag{50}$$

Replacing u_h^* in the proof of Theorem IV.2 by u_{hk}^*, we have

$$\alpha\|u^* - u_{hk}^*\|^2_{L^2(0,T;L^2(\Omega))} + \|y_{hk}^* - y^*\|^2_{L^2(0,T;L^2(\Omega))} \le C(\|y_{hk}^* - \widehat{y}\|^2_{L^2(0,T;L^2(\Omega))} + \|\widehat{p} - p_{hk}^*\|^2_{L^2(0,T;L^2(\Omega))}). \tag{51}$$

By Lemma IV.4 and similar to the proof of Theorem IV.2, we obtain the desired results. ■

V. Numerical experiments

In this section, we provide some numerical examples to confirm the theoretical results. We set $\alpha = 1$ in all computations. The convergence order of the finite volume element schemes is evaluated as

$$\text{Order} = \log_2\left(\frac{\|y_{2h} - y_{exa}\|}{\|y_h - y_{exa}\|}\right),$$

where y_{exa} means the exact solution. The L^2 and L^∞ errors can be evaluated as

$$L^2 = \sqrt{\sum_{j=1}^N |(y_h)_j - (y_{exa})_j|^2 h}, \quad L^\infty = \max|y_h - y_{exa}|.$$

A. Example 1

Let

$$u^* = (T-t)^2 \sin(\pi x_1)\sin(\pi x_2),$$
$$p^* = -\alpha(T-t)^2 \sin(\pi x_1)\sin(\pi x_2),$$
$$y^* = e^t \sin(\pi x)\sin(\pi y).$$

Then the functions f, y_0 and y_d can be determined accordingly.

In Figure 1, the discrete solution u_h^* versus the exact solution u^* at $T = 1$ are plotted. From the numerical results reported in Table I, we can get that the method achieves $O(h^2)$ in the L^2 norm which are consisted with Theorem IV.2. Errors and convergence order for the state, control and adjoint in the L^∞-norm are reported in Table II.

B. Example 2

In this example, we choose $u_a = 0.2$, $u_b = 0.5$. Then we set the optimal control and the associated adjoint by

$$u^* = -\max\{u_a,\ \min\{u_b, (T-t)^2 \sin(\pi x_1)\sin(\pi x_2)\}\}, \tag{52}$$

$$p^* = -\alpha(T-t)^2 \sin(\pi x_1)\sin(\pi x_2). \tag{53}$$

We also take the optimal state as

$$y^* = e^t \sin(\pi x)\sin(\pi y).$$

In Figure 2, the exact solution u^* versus the discrete solution u_h^* obtained by the FVEM at $t = 1$ are plotted. Numerical results reported in Table III also show $O(h^2)$ convergence orders. In Tabel IV, we also present the errors and convergence order for the state, control and adjoint in the sense of L^∞-norm.

VI. Conclusion

In this paper, we proposed an FVEM for optimal control problems governed by parabolic equation. Error analysis shows that under reasonable assumptions the approximation solutions obtained using the finite volume element schemes have optimal error order in the L^2-norm. Numerical examples are provided to validate the theoretical results.

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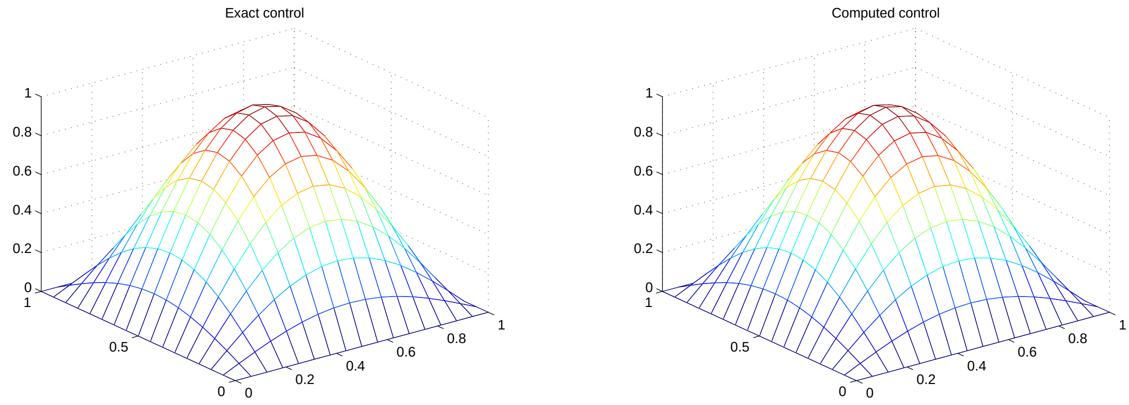

Fig. 1. The exact control u^* versus the computed control u_h^* obtained by the FVEM at $T = 1$ with $N = 16$ in Example 1.

TABLE I
L^2-ERROR OBTAINED BY THE FVEM AT $T = 1$ WITH $\Delta t = h^2$ IN EXAMPLE 1.

N	$\|u^* - u_h^*\|_{L^2}$	Order	$\|p^* - p_h^*\|_{L^2}$	Order	$\|y^* - y_h^*\|_{L^2}$	Order
8	2.7748e-003		2.7748e-003		1.6662e-003	
16	7.1268e-004	1.96	7.1268e-004	1.96	4.3167e-004	1.95
32	1.7937e-004	1.99	1.7937e-004	1.99	1.0880e-004	1.99
64	4.4917e-005	2.00	4.4917e-005	2.00	2.7252e-005	1.99

TABLE II
L^∞-ERROR OBTAINED BY THE FVEM AT $T = 1$ WITH $\Delta t = h^2$ IN EXAMPLE 1.

N	$\|u^* - u_h^*\|_{L^\infty}$	Order	$\|p^* - p_h^*\|_{L^\infty}$	Order	$\|y^* - y_h^*\|_{L^\infty}$	Order
8	5.5100e-003		5.5100e-003		3.4166e-003	
16	1.4164e-003	1.96	1.4164e-003	1.96	8.7952e-004	1.96
32	3.5656e-004	2.00	3.5656e-004	2.00	2.2609e-004	1.96
64	8.9294e-005	2.00	8.9294e-005	2.00	5.6623e-005	2.00

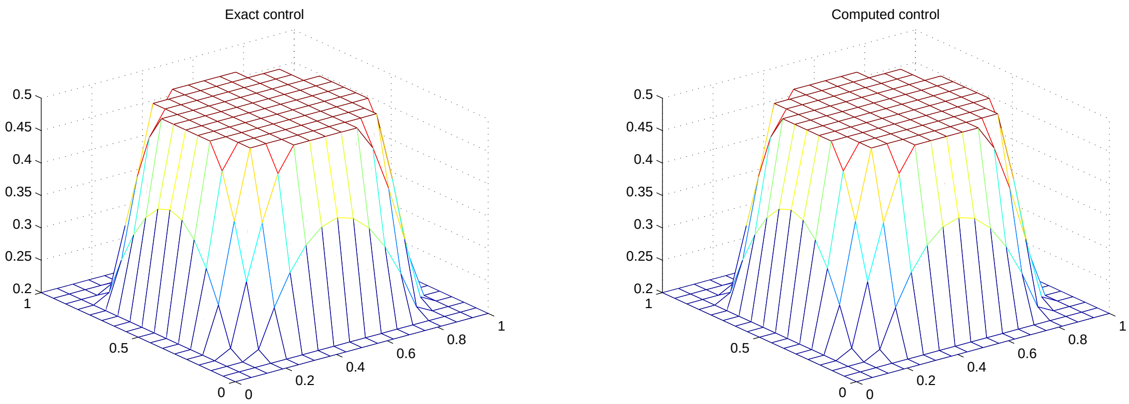

Fig. 2. The exact control u^* versus the computed control u_h^* obtained by the FVEM at $T = 1$ with $N = 16$ in Example 2.

TABLE III
L^2-ERROR OBTAINED BY THE FVEM AT $T = 1$ WITH $\Delta t = h^2$ IN EXAMPLE 2.

N	$\|u^* - u_h^*\|_{L^2}$	Order	$\|p^* - p_h^*\|_{L^2}$	Order	$\|y^* - y_h^*\|_{L^2}$	Order
8	1.0365e-003		2.7663e-002		1.7161e-003	
16	2.5273e-004	2.04	7.0988e-004	1.96	4.5320e-004	1.92
32	7.0633e-005	1.84	1.7871e-004	1.99	1.1350e-004	2.00
64	1.7653e-004	2.00	4.4761e-005	2.00	2.8515e-005	1.99

TABLE IV
L^∞-ERROR OBTAINED BY THE FVEM AT $T = 1$ WITH $\Delta t = h^2$ IN EXAMPLE 2.

N	$\|u^* - u_h^*\|_{L^\infty}$	Order	$\|p^* - p_h^*\|_{L^\infty}$	Order	$\|y^* - y_h^*\|_{L^\infty}$	Order
8	2.6741e-003		5.4923e-003		3.5300e-003	
16	7.7466e-004	1.55	1.4106e-003	1.96	9.0826e-004	1.96
32	2.0793e-004	1.90	3.5518e-004	1.99	2.3149e-004	1.97
64	5.2899e-005	1.99	8.8970e-005	2.00	5.7833e-005	2.00

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