Affine Factorable Surfaces in the Three-Dimensional Simply Isotropic Space

Brahim Medjahdi and Hanifi Zoubir

Abstract—**In this paper, we classify affine factorable surfaces in the three dimensional simply isotropic space, which satisfy some algebraic equations in terms of the coordinate functions and the Laplacian operator with respect to the first and second fundamental forms of the surface. We also give explicit forms of these surfaces.**

Index Terms—**Isotropic space, Affine factorable surface, Laplace operator.**

I. Introduction

A Euclidean submanifold is said to be of finite Chen-type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian [11].
Chen posed the problem of classifying the finite type surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$. Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space.
A well-known result due to Takahashi [20] states that an n-dimensional submanifold of $\mathbb{E}^n$ is of 1-type if and only if it is either a minimal submanifold of $\mathbb{E}^n$, or a minimal submanifold of some hypersphere. In other words, the solutions of the equation

$$\Delta r = \lambda r, \lambda \in \mathbb{R}, \tag{1}$$

where Δ is the Laplace operator associated with the induced metric, and r is a position vector field of M in $\mathbb{E}^n$.
Bekkar and Senoussi [6] studied the factorable surfaces in the 3-dimensional Minkowski space under the condition

$$\Delta r_i = \lambda_i r_i, \tag{2}$$

where $\lambda_i \in \mathbb{R}$ and r_i are the coordinate functions of the surface.
The authors in [22], [16], [2], classified factorable surfaces in the 3-dimensional Minkowski, Euclidean and pseudo-Galilean spaces.
Lopez and Moruz [15] studied translation and homothetical surfaces with constant Gaussian curvature in $\mathbb{E}^3$. Zong, Xiao and Liu [23] defined affine factorable surfaces in $\mathbb{R}^3$, as the graphs

$$z = (f_1(x) f_2(y + ax)), a \in \mathbb{R}, a \neq 0. \tag{3}$$

Aydin, Erdur and Ergut [3] studied the problem of finding the affine factorable surfaces in a 3-dimensional isotropic space with a prescribed Gaussian curvature (K) and mean curvature (H).

Recently, in [4] Azzi, Zoubir and Bekkar studied surfaces as graphs of functions in $\widetilde{SL}(2,\mathbb{R})$ which has a finite type immersion.

In this paper, we classify affine factorable surfaces of Type 1 in the three dimensional simply Isotropic space under the condition

$$\mathbf{\Delta}^j \mathbf{r}_i = \lambda_\mathbf{i} \mathbf{r}_i,$$

where $\lambda_i \in \mathbb{R}$ and $\mathbf{\Delta}^j$ denotes the Laplace operator with respect to the fundamental forms I and II.

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II. Preliminaries

Differential geometry of isotopic spaces has been introduced by K. Strubacker [19], H. Sachs [17] and many others.

The three-dimensional isotropic space $\mathbb{I}^3$ is a Cayley-Klein space obtained from the three-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure which is an ordered (p, l_1, l_2) where p is a plane in $P(\mathbb{R}^3)$ and l_1, l_2 are two complex-conjugate straight lines in p. The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the absolute plane p is given by $x_0 = 0$ and the absolute lines l_1, l_2 by $x_0 = x_1 \pm i x_2 = 0$. The intersection point $P(0, 0, 0, 1)$ of these two lines is called the absolute point.

The group of motions of $\mathbb{I}^3$ is a six-parameter group given in affine coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ by

$$(x, y, z) \to (\overline{x}, \overline{y}, \overline{z}) = \begin{cases} \overline{x} = a + x\cos\varphi - y\sin\varphi, \\ \overline{y} = b + x\sin\varphi + y\cos\varphi, \\ \overline{z} = c_1 + c_2 x + c_3 y + z, \end{cases}$$

where $a, b, c_1, c_2, c_3, \varphi \in \mathbb{R}$.

Such affine transformations are called isotropic congruence transformations or merely i-motions.

On the orther hand, the isotropic distance, called i-distance of two points $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ is defined by

$$\|A - B\|_i = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

The i-motions is degenerate along the lines in z-direction. These lines are called isotropic lines.

Let $v_1 = (x_1, x_2, x_3)$ and $v_2 = (y_1, y_2, y_3)$ be vectors in $\mathbb{I}^3$. The isotropic inner product of v_1 and v_2 is defined by

$$\langle v_1, v_2 \rangle_i = \begin{cases} x_3 y_3 & \text{if} \quad x_i = y_i = 0 \\ x_1 y_1 + x_2 y_2 & \qquad\qquad \text{otherwise} \end{cases}$$

We call a vector of the form $v = (0, 0, x)$ in $\mathbb{I}^3$ an isotropic vector, and a non-isotropic vector otherwise.

A surface M^2 immersed in $\mathbb{I}^3$ is called admissible if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces.

Let M^2 be regular admissible graph surfaces in $\mathbb{I}^3$ locally parametrized by

$$r(u,v) = (u, v, z(u,v)).$$

The components E, F, G of the first fundamental form I of M^2 can be calculated via the metric induced from $\mathbb{I}^3$. We have

$$E = \langle r_u, r_u \rangle_i, F = \langle r_u, r_v \rangle_i, G = \langle r_v, r_v \rangle_i. \quad (4)$$

The unit normal vector of M^2 is completely isotropic. Morever, the components of the second fundamental form II are

$$\begin{cases} L = \frac{\det(r_{uu}, r_u, r_v)}{\sqrt{EG-F^2}}, \\ M = \frac{\det(r_{uv}, r_u, r_v)}{\sqrt{EG-F^2}}, \\ N = \frac{\det(r_{vv}, r_u, r_v)}{\sqrt{EG-F^2}}. \end{cases}$$

The isotropic Gaussian curvature K and the isotropic mean curvature H are respectively defined by

$$K = \frac{LN - M^2}{EG - F^2}, \quad 2H = \frac{EN - 2FM + GL}{EG - F^2}. \quad (5)$$

It is well known in terms of local coordinates $\{u, v\}$ of M^2 the Laplacian operators Δ of the first and the second fundamental form on M^2 are respectively defined by

$$\Delta^I r = -\frac{1}{W}\left[\frac{\partial}{\partial u}\left(\frac{Gr_u - Fr_v}{W}\right) - \frac{\partial}{\partial v}\left(\frac{Fr_u - Er_v}{W}\right)\right], \quad (6)$$

and

$$\Delta^{II} r = -\frac{1}{e}\left[\frac{\partial}{\partial u}\left(\frac{Nr_u - Mr_v}{e}\right) - \frac{\partial}{\partial v}\left(\frac{Mr_u - Lr_v}{e}\right)\right], \quad (7)$$

where $W = \sqrt{|EG - F^2|}$ and $e = \sqrt{|LN - M^2|}$.

A surface M^2 in the isotropic space $\mathbb{I}^3$ is said to be harmonic or isotropic minimal (resp. II-harmonic) if it satisfies the condition $\Delta^I r = 0$ (resp. $\Delta^{II} r = 0$).

III. Affine factorable surfaces in isotropic spaces

Let M^2 be an affine factorable surface of type 1 in $\mathbb{I}^3$ which is defined as a parameter surface in $\mathbb{I}^3$ which can be written as

$$r(u,v) = (u, v, f(u + av)g(v)), \quad (8)$$

for some non-zero constant a and smooth functions f and g. The coefficients of the first and the second fundamental forms are

$$E = 1, \ F = 0, \ G = 1, \quad (9)$$

and

$$L = f_{uu}g, \ M = af_{uv}g + f_u g_v, \ N = a^2 f_{vv} g + 2af_v g_v + fg_{vv}. \quad (10)$$

The mean curvature H and the Gaussian curvature K of M^2 are given by

$$2H = f_{uu}g + a^2 f_{vv} g + 2af_v g_v + fg_{vv}, \quad (11)$$

and

$$K = f_{uu}g(a^2 f_{vv} g + 2af_v g_v + fg_{vv}) - (af_{uv}g + f_u g_v)^2. \quad (12)$$

By the transformation

$$\begin{cases} x = u + av \\ y = v, \end{cases}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. Then (8) can be written as

$$r(x,y) = (x - ay, y, f(x)g(y)), \quad (13)$$

therefore,

$$E = 1, \quad F = -a, \ G = a^2 + 1, \quad (14)$$

and

$$L = f''(x)g(y), \ M = f'(x)g'(y), \ N = f(x)g''(y). \quad (15)$$

Therefore, (11) and (12) become

$$2H = \left(a^2 + 1\right) f''(x)g(y) + 2af'(x)g'(y) + f(x)g''(y),$$

and

$$K = f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2,$$

respectively.

IV. Affine factorable surfaces satisfying $\Delta^I r_i = \lambda_i r_i$

In this section, we classify affine factorable surface given by (13) in $\mathbb{I}^3$ satisfying the equation

$$\Delta^I r_i = \lambda_i r_i, \quad (16)$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^I r = \left(\Delta^I r_1, \Delta^I r_2, \Delta^I r_3\right), \quad (17)$$

where

$$r_1 = x - ay, \ r_2 = y, \ r_3 = f(x)g(y). \quad (18)$$

The determinant of the first fundamental form is given by

$$W = \sqrt{|EG - F^2|} = 1. \quad (19)$$

Suppose that the surface has non-zero Gaussian curvature, so

$$f(x)f''(x)g(y)g''(y) - (f'(x)g'(y))^2 \neq 0, \forall x, y \in \mathbb{R}.$$

By a straightforward computation, the Laplacian operator on M^2 with the help of (6) and (14) turns out to be

$$\Delta^I r = (0, 0, -[(a^2+1)f''g + 2af'g' + fg'']). \quad (20)$$

Next, suppose that M^2 satisfies (16). Then from (20), we have

$$\begin{cases} \lambda_1 (x - ay) = 0, \\ \lambda_2 y = 0, \\ \lambda_3 fg = -[(a^2+1)f''g + 2af'g' + fg''], \end{cases} \quad (21)$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. This means that M^2 is at most of $1-type$. We discuss two cases according to the values of λ_3.

Case 1: Let $\lambda_3 = 0$, the last equation in (21) becomes

$$(a^2+1)f''g + 2af'g' + fg'' = 0, \quad (22)$$

we have some cases to solves (22).

Case 1.1. f or g constant, both situations give rise to a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Case 1.2. $f' = c_0,\ c_0 \in \mathbb{R} - \{0\}$. Then (22) can be rewritten as

$$\frac{1}{f} = -\frac{g''}{2ac_0 g'}. \tag{23}$$

Since f is a non-constant function, the right side of (23) is either a constant or a function of y. Both cases are not possible and we deduce $f' \neq 0$. In a similar way $g'' \neq 0$ can be shown.

Case 1.3. $f''g'' \neq 0$. By dividing (22) by the product $f''g''$, one can write

$$(a^2+1)\frac{g}{g''} + 2a\frac{f'}{f''}\frac{g'}{g''} + \frac{f}{f''} = 0. \tag{24}$$

Taking partial derivatives of (24) with respect to x and y, we get

$$2a\left(\frac{f'}{f''}\right)'\left(\frac{g'}{g''}\right)' = 0,$$

and so we again have two situations.

Case 1.3.1. $f'' = c_1 f', c_1 \in \mathbb{R} - \{0\}$. We obtain $f(x) = \frac{c_2}{c_1}e^{c_1 x}, c_2 \neq 0$. By substituting it into (22), we obtain

$$g'' + 2ac_1 g' + (a^2+1)c_1^2 g = 0, \tag{25}$$

This differential equation admits the solutions

$$g(y) = e^{-ac_1 y}\left[c_3\cos(c_1 y) + c_4\sin(c_1 y)\right]$$

where $c_3, c_4 \in \mathbb{R}$.

Case 1.3.2. $g'' = c_5 g', c_5 \in \mathbb{R} - \{0\}$. In this case, we obtain $g(y) = \frac{c_6}{c_5}e^{c_5 y}, c_6 \neq 0$.By considering (22), we get

$$(a^2+1)f'' + 2ac_5 f' + c_5^2 f = 0, \tag{26}$$

After solving (26), we obtain

$$f(x) = e^{-aAx}\left(c_7\cos(Ax) + c_8\sin(Ax)\right),$$

where $A = \frac{c_5}{1+a^2}$ and $c_7, c_8 \in \mathbb{R}$.

Case 2: Let $\lambda_3 \neq 0$. By dividing the last equation in (21) by the product fg we get

$$\left(a^2+1\right)\frac{f''}{f} + 2a\frac{f'}{f}\frac{g'}{g} + \frac{g''}{g} = -\lambda_3. \tag{27}$$

Taking partial derivatives of (27) with respect to x and y, we find

$$2a\left(\frac{f'}{f}\right)'\left(\frac{g'}{g}\right)' = 0.$$

We have two cases:

Case 2.1. $f' = b_1 f,\ b_1 \in \mathbb{R} - \{0\}$. We get $f(x) = b_2 e^{b_1 x}, b_2 \in \mathbb{R} - \{0\}$. Considering Equation (27), we obtain

$$g'' + 2ab_1 g' + \left(\left(a^2+1\right)b_1^2 + \lambda_3\right)g = 0. \tag{28}$$

In order to solve (28), we have to consider three situations this time

Case 2.1.1. $\left(b_1^2 + \lambda_3\right) < 0$, the general solution of (28) is given by

$$g(y) = e^{-ab_1 y}\left(b_3 e^{\sqrt{-B}y} + b_4 e^{-\sqrt{-B}y}\right),$$

where $B = b_1^2 + \lambda_3$ and $b_3, b_4 \in \mathbb{R}$.

Case 2.1.2 $\left(b_1^2 + \lambda_3\right) > 0$, the general solution of (28) is given by

$$g(y) = e^{-ab_1 y}\left[b_5\cos(By) + b_6\sin(By)\right],$$

where $B = \sqrt{b_1^2 + \lambda_3}$ and $b_5, b_6 \in \mathbb{R}$.

Case 2.1.3. $b_1^2 = -\lambda_3$, the general solution of (28) is given by

$$g(y) = e^{-ab_1 y}\left(b_7 y + b_8\right),\ b_7, b_8 \in \mathbb{R}.$$

Case 2.2. $g' = d_1 g,\ d_1 \in \mathbb{R} - \{0\}$. Then we get $g(y) = d_2 e^{d_1 y}, d_2 \in \mathbb{R} - \{0\}$. Substituting this in Equation (27) we obtain

$$\left(a^2+1\right)f'' + 2ad_1 f' + \left(d_1^2 + \lambda_3\right)f = 0. \tag{29}$$

To solve this equation, we have the following situations to be discussed:

Case 2.2.1. $\left(d_1^2 + \left(a^2+1\right)\lambda_3\right) < 0$. The general solution of

(28) is given by

$$f(x) = e^{\frac{-ad_1}{a^2+1}x}\left(d_3 e^{\frac{\sqrt{-C}}{a^2+1}x} + d_4 e^{-\frac{\sqrt{-C}}{a^2+1}x}\right),$$

where $C = d_1^2 + \left(a^2+1\right)\lambda_3$, and $d_3, d_4 \in \mathbb{R}$.

Case 2.2.2. $d_1^2 + \left(a^2+1\right)\lambda_3 > 0$. The general solution of (28) is given by

$$f(x) = e^{-\frac{ad_1}{a^2+1}x}\left[d_5\cos\left(\frac{C}{a^2+1}x\right) + d_6\sin\left(\frac{C}{a^2+1}x\right)\right],$$

where $C = d_1^2 + \left(a^2+1\right)\lambda_3$, and $d_5, d_6 \in \mathbb{R}$.

Case 2.2.3. $d_1^2 = -\left(a^2+1\right)\lambda_3$. The general solution of (28) is given by

$$f(x) = e^{-ad_1 x}\left(d_7 x + d_8\right), d_7, d_8 \in \mathbb{R},$$

where $d_i \in \mathbb{R}$, $i \in \{3,4,5,6,7,8\}$.

Therefore, we have proven the following statements:

Theorem IV.1. *Let M^2 be a affine factorable surface given by (8) in $\mathbb{I}^3$. If M^2 is harmonic or isotropic minimal, then it is congruent to an open part of the surfaces*

- $z(u,v) = \frac{c_2}{c_1}e^{c_1 u + av + ac_1 v}\left[c_3\cos(c_1 v) + c_4\sin(c_1 v)\right],$
- $z(u,v) = \frac{c_6}{c_5}e^{\frac{c_5 v - ac_5 u}{1+a^2}}$ $\left[c_6\cos\left(A\left(u+av\right)\right) + c_7\sin\left(A\left(u+av\right)\right)\right],$

  where $A = \frac{c_5}{1+a^2}$, $c_1, ..., c_7 \in \mathbb{R}$ and $c_1, c_5 \neq 0$.

Theorem IV.2. *(Classification). Let M^2 be a non harmonic affine factorable surface given by (8) in the three dimensional isotropic space $\mathbb{I}^3$. The surfaces M^2 satisfy the condition $\mathbf{\Delta}^I\mathbf{r_i} = \lambda_\mathbf{i}\mathbf{r_i}$ where $\lambda_\mathbf{i} \in \mathbb{R}$, then it is congruent to an open part of the surfaces*

- $z(u,v) = b_2 e^{b_1 u}\left(b_3 e^{\sqrt{-B}v} + b_4 e^{-\sqrt{-B}v}\right),$

where $B = b_1^2 + \lambda_3 < 0$

- $z(u,v)=b_2e^{b_1u}\left[b_5\cos\left(\sqrt{B}v\right)+b_6\sin\left(\sqrt{B}v\right)\right]$,
where $B=b_1^2+\lambda_3>0$,
- $z(u,v)=b_2e^{b_1u}\,(b_7v+b_8)$,
where $b_1^2=-\lambda_3$,
- $z(u,v)=d_2e^{\frac{d_1v-ad_1u}{a^2+1}}\left[\begin{array}{c} d_3e^{\frac{\sqrt{-C}}{a^2+1}(u+av)} \\ +d_4e^{-\frac{\sqrt{-C}}{a^2+1}(u+av)}\end{array}\right]$,
where $C=d_1^2+\left(a^2+1\right)\lambda_3<0$,
- $z(u,v)=d_2e^{\frac{d_1v-ad_1u}{a^2+1}}\left[\begin{array}{c} d_5\cos\left(\alpha\left(u+av\right)\right) \\ +d_6\sin\left(\alpha\left(u+av\right)\right)\end{array}\right]$,
where $\alpha=\frac{\sqrt{C}}{a^2+1}$, and $C=d_1^2+\left(a^2+1\right)\lambda_3>0$,
- $z(u,v)=d_2e^{d_1v-ad_1(u+av)}\left(d_7\left(u+av\right)+d_8\right)$,
where $d_1^2=-\left(a^2+1\right)\lambda_3$, $b_i,d_i\in\mathbb{R}$, $i\in\{1,...,8\}$ and $b_1,b_2,d_1,d_2\neq 0$.

V. Affine factorable surfaces satisfying $\Delta^{II}r_i=\lambda_i r_i$

In this section, we classify affine factorable surfaces with non-degenerate second fundamental form in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta^{II}r_i=\lambda_i r_i, \tag{30}$$

where $\lambda_i\in\mathbb{R}$, $i=1,2,3$, and

$$\Delta^{II}r=\left(\Delta^{II}r_1,\Delta^{II}r_2,\Delta^{II}r_3\right), \tag{31}$$

and

$$r_1=x-ay,\ r_2=y,\ r_3=f(x)g(y). \tag{32}$$

Using (7) and (13), a straightforward computation, proves that the Laplacian operator on M^2 is given by

$$\Delta^{II}r=\left(\Delta^{II}\left(x-ay\right),\Delta^{II}y,\Delta^{II}\left(f(x)g(y)\right)\right), \tag{33}$$

where

$$\Delta^{II}(x-ay)=\frac{1}{2e^4}\left(\Omega_1\left(x,y\right)+a\Omega_2\left(x,y\right)\right), \tag{34}$$

$$\Delta^{II}y=\frac{1}{2e^4}\Omega_2\left(x,y\right), \tag{35}$$

$$\Delta^{II}fg=\frac{1}{2e^4}\left[\begin{array}{l} f'g\left(\Omega_1\left(x,y\right)+a\Omega_2\left(x,y\right)\right) \\ +\left(af'g+fg'\right)\Omega_2\left(x,y\right)\end{array}\right]-2, \tag{36}$$

where
$\Omega_1(x,y)=ff'f''gg''^2+f^2f'''gg''^2-3ff'f''g'^2g''-ff'f''gg'g'''+2f'^3g'^2g''$,
$\Omega_2(x,y)=-3f'^2f''gg'g''-ff'f'''gg'g''+2f'^2f''g'^3+ff''^2gg'g''+ff'^2g^2g'''$ and $e=\sqrt{|ff''gg''-f'^2g'^2|}$. The equation (30) by means of (33) gives rise to the following system of ordinary differential equations

$$\frac{1}{2e^4}\left(\Omega_1\left(x,y\right)+a\Omega_2\left(x,y\right)\right)=\lambda_1(x-ay), \tag{37}$$

$$\frac{1}{2e^4}\Omega_2\left(x,y\right)=\lambda_2 y, \tag{38}$$

$$\frac{1}{2e^4}\left[\begin{array}{l} f'g\left(\Omega_1\left(x,y\right)+a\Omega_2\left(x,y\right)\right) \\ +\left(af'g+fg'\right)\Omega_2\left(x,y\right)\end{array}\right]-2=\lambda_3 fg, \tag{39}$$

where $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$. This means that M^2 is most of $3-$*type.*
Combining Equations (37), (38) and (39), we have

$$\left(\lambda_1x+\left(\lambda_2-\lambda_1\right)ay\right)f'g+\lambda_2yfg'-2=\lambda_3fg. \tag{40}$$

According of the choices of constants λ_1, λ_2 and λ_3, we discuss all possible cases of λ_i, $i\in\{1,2,3\}$.

Case 1: Let $\lambda_1=\lambda_2$ and $\lambda_3\in\mathbb{R}$. From (40), we have

$$\lambda_1xf'g+\lambda_1yfg'-2=\lambda_3fg. \tag{41}$$

By dividing (41) by the produt fg, then taking partial derivatives of (27) with respect to x and y, we find

$$\left(\frac{1}{f}\right)'\left(\frac{1}{g}\right)'=0$$

which implies that f or g is constant. Both situations contradict our assumption stating that the solution must be non-degenerate second fundamental form. Therefore, in this case, there are no affine factorable surfaces satisfying (30).

It should be noted that, there exists no $II-$harmonic affine factorable surface in $\mathbb{I}_3^1$.

Case 2: Let $\lambda_1=\lambda_3=0$ and $\lambda_2\neq 0$. (40) can be rewritten as

$$\lambda_2ayf'g+\lambda_2yfg'-2=0. \tag{42}$$

By dividing (42) by the produt fg, then taking partial derivative with respect to y gives

$$\frac{\left(\frac{g'}{g}\right)'}{\left(\frac{2}{\lambda_2yg}\right)'}=\frac{1}{f}. \tag{43}$$

Both sides of (43) are equal to some nonzero constant, namely

$$\left(\frac{g'}{g}\right)'=c_1\left(\frac{2}{\lambda_2yg}\right)'\text{ and }\frac{1}{f}=c_1. \tag{44}$$

Therefore, the function f must be constant. This solution gives rise to a similar type of contradition as in case 1. Hence, there are no affine factorable surfaces satisfying (30) in this Case.

Case 3: Let $\lambda_2=\lambda_3=0$ and $\lambda_1\neq 0$. Hence we get from (40)

$$\left(x-ay\right)g=\frac{2}{\lambda_1f'}. \tag{45}$$

The partial derivative of (45) with respect to x leads to

$$g=\frac{2}{\lambda_1}\left(\frac{1}{f'}\right)' \tag{46}$$

Therefore the function g is constant. This solution gives rise to a similar type of contradition as in Case 1.

Case 4: Let $\lambda_1=0$, $\lambda_2\neq 0$ and $\lambda_3\neq 0$. By dividing (41) by the produt fg, we have

$$\lambda_2y\left(a\frac{f'}{f}+\frac{g'}{g}\right)-\frac{2}{fg}=\lambda_3. \tag{47}$$

Taking the partial derivative of (47) with respect to x gives

$$\left(\frac{f'}{f}\right)'-\frac{2}{a\lambda_2yg}\left(\frac{1}{f}\right)'=0, \tag{48}$$

which means that $g=\frac{c_2}{y}, c_2\in\mathbb{R}-\{0\}$. Considering it into (47) gives the following polynomial equation in y

$$\left(a\lambda_2f'-2c_2\right)y-\left(\lambda_2+\lambda_3\right)f=0, \tag{49}$$

where the coefficients must vanish, i.e.

$$(a\lambda_2 f' - 2c_2) = 0 \text{ and } (\lambda_2 + \lambda_3) f = 0. \tag{50}$$

Integrating the first equality leads to $f(x) = \frac{2c_2}{a\lambda_2}x + c_3,\ c_3 \in \mathbb{R}$. The second equality gives $\lambda_2 = -\lambda_3$.

Case 5: Let $\lambda_2 = 0$, $\lambda_1 \neq 0$ and $\lambda_3 \neq 0$. From (40), we obtain

$$\lambda_1 (x - ay) f' g - 2 = \lambda_3 f g. \tag{51}$$

By dividing (51) by the g and taking the partial derivative with respect to y gives

$$-\frac{\lambda_1 a}{2} f' = \left(\frac{1}{g}\right)'. \tag{52}$$

Since the left side of (52) is a function of x and the right side is a function of y, both sides have to be equal to a nonzero constant, i.e.

$$-\frac{\lambda_1 a}{2} f' = c_4 = \left(\frac{1}{g}\right)'. \tag{53}$$

From the right side of (53), we deduce $g(y) = \frac{2}{c_4 y + c_5}$. Substituting it into (51) leads to the polynomial equation in y

$$\lambda_1 x f' - \lambda_3 f - 2c_5 - (2c_4 + \lambda_1 a f') y = 0, \tag{54}$$

where the coefficients must vanish, i.e.

$$2c_4 + \lambda_1 a f' = 0 \tag{55}$$

$$\lambda_1 x f' - \lambda_3 f - 2c_5 = 0 \tag{56}$$

Integrating of (55) yields $f(x) = \frac{-c_4}{a\lambda_1}x + c_6, c_6 \in \mathbb{R}$. Substituting it into (56) gives $\lambda_1 = \lambda_3$ and $c_6 = \frac{-2c_5}{\lambda_3}$. Then

$$f(x) = \frac{-2c_4}{a\lambda_1}x - \frac{2c_5}{\lambda_1} = \frac{-2}{\lambda_1}\left(\frac{c_4}{a}x + c_5\right). \tag{57}$$

Case 6: Let $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$ and $\lambda_1 \neq \lambda_2$. From (40), we obtain

$$(\lambda_1 x + (\lambda_2 - \lambda_1) ay) f' g + \lambda_2 y f g' - 2 = 0. \tag{58}$$

By dividing (58) by the f', we get

$$[\lambda_1 x + (\lambda_2 - \lambda_1) ay] g + \lambda_2 y g' \frac{f}{f'} = \frac{2}{f'}. \tag{59}$$

Taking partial derivative of (59) with respect to x gives

$$\lambda_1 g + \lambda_2 y g' \left(\frac{f}{f'}\right)' = 2\left(\frac{1}{f'}\right)'. \tag{60}$$

Now, taking partial derivative of (60) with respect to y, we find

$$\lambda_1 g' + \lambda_2 (y g')' \left(\frac{f}{f'}\right)' = 0, \tag{61}$$

and so we have to consider two cases.

Case 6.1. $f' = c_7 f$, $c_7 \in \mathbb{R} - \{0\}$. Then (60) reduces to

$$\lambda_1 g = 2\left(\frac{1}{f'}\right)'. \tag{62}$$

The left side in (62) is a function of y while other sideis a constant or function of x. That is not possible.

Case 6.2. $f' \neq c_7 f$, $c_7 \in \mathbb{R} - \{0\}$. Then (61) can be rewritten as

$$\left(\frac{f}{f'}\right)' = -\frac{\lambda_1 g'}{\lambda_2 (y g')'}. \tag{63}$$

As in each side of this equation we have a function depending on x and the other depending on y, there exists $c_8 \in \mathbb{R} - \{0\}$, such that

$$\left(\frac{f}{f'}\right)' = c_8 \text{ and } \lambda_1 g' + c_8 \lambda_2 (y g')' = 0. \tag{64}$$

An integration of the first equation in (64) gives

$$f(x) = (c_8 x + c_9)^{\frac{1}{c_8}}, c_9 \in \mathbb{R}. \tag{65}$$

Substituting (65) in (60) leads to

$$\lambda_1 g + \lambda_2 c_8 y g' = 2(c_8 - 1)(c_8 x + c_9)^{-\frac{1}{c_8}}. \tag{66}$$

Thus, we have two situations:

Case 6.2.1. $c_8 \neq 1$. The left-hand side of (66) is a function of y or a constant whereas the right-hande side is a function of x. This is a contradiction.

Case 6.2.2. $c_8 = 1$. Then by (65) and (66) we get

$$f(x) = x + c_9 \text{ and } \lambda_1 g + \lambda_2 y g' = 0. \tag{67}$$

By considering (67) into (59) we get

$$(\lambda_2 - \lambda_1) a y g + \lambda_2 y g' c_9 = 2. \tag{68}$$

An integration of the second equation in (67) leads to $g(y) = y^{-\frac{\lambda_1}{\lambda_2}}$. Substituting it into (68) gives a contradiction.

Case 7: Let $\lambda_1 \lambda_2 \lambda_3 \neq 0$ and $\lambda = \lambda_1 - \lambda_2 \neq 0$. By dividing the equation (40) by fg we get

$$[\lambda_1 x + \lambda a y] \frac{f'}{f} + \lambda_2 y \frac{g'}{g} - \frac{2}{fg} = \lambda_3. \tag{69}$$

Taking partial derivatives of (69) with respect to x and y, leads to

$$\lambda a \left(\frac{f'}{f}\right)' = 2\left(\frac{1}{f}\right)'\left(\frac{1}{g}\right)', \tag{70}$$

we have to consider two cases.

Case 7.1. $f' = d_1 f, d_1 \in \mathbb{R} - \{0\}$. It follows from (70) that

$$\left(\frac{1}{f}\right)'\left(\frac{1}{g}\right)' = 0. \tag{71}$$

This implies that f or g is constant, which again leads to a contradiction.

Case 7.2. $f' \neq d_1 f, d_1 \in \mathbb{R} - \{0\}$. Therefore, (70) can be rewritten as

$$\frac{\left(\frac{f'}{f}\right)'}{\left(\frac{1}{f}\right)'} = \frac{2\left(\frac{1}{g}\right)'}{\lambda a}. \tag{72}$$

Both sides of (72) are equal to some non-zero constant, namely

$$\frac{\left(\frac{f'}{f}\right)'}{\left(\frac{1}{f}\right)'} = d_2 = \frac{2\left(\frac{1}{g}\right)'}{\lambda a}. \tag{73}$$

Form the left side of (73), we deduce $f(x) = (d_2 x + d_3)$, $d_3 \in \mathbb{R}$. We may assume $d_3 = 0$. Then we obtain $f(x) =$

d_2x. Considering it into (69) gives the following polynomial equotion in x

$$\left(\lambda_2 y\frac{g'}{g}-\lambda_3+\lambda_1\right)x+\lambda ay-\frac{2}{d_2 g}=0, \tag{74}$$

where the coefficients must vanish, i.e

$$\lambda_2 y\frac{g'}{g}-\lambda_3+\lambda_1=0, \tag{75}$$

and

$$\lambda ay-\frac{2}{d_2 g}=0. \tag{76}$$

From (76), we deduce $g(y)=\frac{2}{\lambda a d_2 y}$. Substituting it into (75) gives $\lambda_3=\lambda_1-\lambda_2$.

Therefore, we have shown the following theorems:

Theorem V.1. *There is no II-harmonic affine factorable surface given by* (8) *in three dimensional isotropic space* $\mathbb{I}^3$.

Theorem V.2. *(Classification) Let M^2 be a non$-$harmonic affine factorable surface with non-degenerate three fundamental form given by* (8) *in the three dimensional isotropic space $\mathbb{I}^3$. Assume that the surface M^2 satisfies the condition $\boldsymbol{\Delta}^{II}\mathbf{r_i}=\lambda_\mathbf{i}\mathbf{r_i}$ where $\lambda_\mathbf{i}\in\mathbb{R}$, then it is congruent to an open part of the surfaces,*

- $r(u,v)=\left(u,v,\frac{c_2}{v}\left(\frac{2c_2}{a\lambda_2}(u+av)+c_3\right)\right)$,
- $r(u,v)=\left(u,v,\frac{2}{c_4v+c_5}\left(\frac{-c_4}{a\lambda_1}(u+av)+c_6\right)\right)$, where $c_6=\frac{-2c_5}{\lambda_1}$,
- $r(u,v)=\left(u,v,\frac{2(u+av)}{a\lambda_3 v}\right)$.

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