Explicit Error Estimates for the Raviart-Thomas Mixed Finite Element

Yongwei Yang*, and Qi Liu

***Abstract*—In this paper, a general form of the Poincaré inequality is established, and an explicit relationship between the interpolation error constants of the Raviart-Thomas element and the geometric characters of the triangulation is firstly obtained, which is consistent with the maximal angle condition. In addition, explicit representations of the constants in the final error estimates of the Raviart-Thomas mixed finite element method are studied in detail. This explicit prior error estimation can be effectively used as an error bounds calculation.**

***Index Terms*—Raviart-Thomas element, Explicit error estimate, Maximal angle condition, Poincaré inequality.**

I. Introduction

As a method of numerically solving elliptic boundary value problems, the finite element method (FEM) has been widely used in the numerical solution of various structural problems for the field of engineering mechanics. Moreover, it also establishes a solid theoretical foundation in terms of the priori and posterior error estimation ([1], [2], [3]). In the classical finite element prior error analysis, interpolation error based on two-points is the key to derive the final error estimation. One of the two points here is the transformation inequality of the function semi-norm between the general element and the reference element, and the other is the interpolation error on the reference element [4]. Various positive constants appear in this process, which we call interpolation error constants. The interpolation error constant is independent of the element size, but may be related to the sine value of the minimum angle of the triangulation for the 2D case [5]. In fact, if the minimum angle condition for the finite elements is relaxed, it can be obtained that the anisotropic elements which are found for a long time before in [6]. Since late 1980's, many different methods of dealing with the anisotropic interpolation error estimation have emerged, and some important results have been obtained ([7], [8], [9]).

It is well known that various constants are likely to appear in the process of deriving the final finite element error estimate. For the purpose of quantitative error bounds, it is good to evaluate or constrain these constants as well as interpolation error constants. Recently, some research work on the estimation of the error constants for the finite elements has appeared, for example, ([8], [10]) for linear triangular finite elements, ([11], [12], [13]) for bilinear quadrilateral finite elements and [14] for the lowest order Raviart-Thomas element and nonconforming Crouzeix-Raviart element. However, all of them are only concentrated on low-order interpolations, such as the constant of interpolation, linear and bilinear interpolation, and constant L^2-projection. As mentioned in [14], there is no explicit error bounds for general Raviart-Thomas mixed finite elements are given at present. Therefore, it is necessary to further give explicit bounds for discrete inf-sup constants.

The purpose of this paper is to derive an explicit error bound for the mixed finite element in the Poisson problem. We first give some results of the error constants for arbitrary order Raviart-Thomas interpolation, which play an important role in the prior error estimation of finite element methods. Our way is based on Durán's some anisotropic results about Rarviart-Thomas elements [15]. In this process, it is vital to introduce the Poincaré inequality [16] and sharp trace theorem [14]. However, we need to establish a generalized form of the Poincaré inequality that will be frequently used to get higher-order norms of the function and its some partial derivatives with vanishing averages. At the same time, we get an explicit expression of the discrete inf-sup constant. Based on the above results, a constructive error bound for the mixed finite element is derived. The explicit prior error estimation obtained in this paper not only provides a computable error boundary, but also can be used as a posterior error estimation of the finite element method. Moreover, our explicit interpolation error estimates for the Raviart-Thomas element are proved to be consistent with the maximal angle condition.

The rest of the paper is organized as follows. In Section II, we establish a general form of the Poincaré inequality and explicit error estimates of auxiliary interpolation operators. Then we present some primary results about the set-up and approximation of the model problem. Section III gives bounds of interpolation error constants for the Raviart-Thomas element. We study the explicit expression of the discrete inf-sup constant for the Raviart-Thomas element in Section IV. Then we present explicit error estimates for the Raviart-Thomas mixed finite element method in Section V. In order to test the error bounds for lowest order mixed finite element method, the numerical experiment given in [14] is introduce in Section VI. Finally, some comments and extensions of the results are described in Section VII.

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II. Discretization of the Model Problem

In this paper, let $\Omega \subseteq R^2$ be a bounded convex polygonal domain and $P_k(\Omega)$ the space of all polynomials of degree $\leq k$ on Ω. Define the semi-norm in Sobolev space $H^m(\Omega)$ as follows

$$|v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} \|D^\alpha v\|_{0,\Omega}^2 \Big)^{\frac{1}{2}}, \quad \forall v \in H^m(\Omega), \quad \text{(II.1)}$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1!\alpha_2!$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$.

We observe that the coefficients in (II.1) are helpful to simplify the final forms of our results.

The following classic Poincaré inequality which correctly proved by Bebendorf in [17], was firstly presented in [16].

Lemma 2.1: Let $v \in H^1(\Omega)$ be a function with vanishing average. Then

$$\|v\|_{0,\Omega} \le \frac{d_\Omega}{\pi}|v|_{1,\Omega}, \quad \text{(II.2)}$$

where d_Ω is the diameter of Ω.

As mentioned in Introduction, we can establish a generalized form of the Poincaré inequality (II.2) as follows.

Lemma 2.2: Let $v \in H^{k+1}(\Omega)$ be a function satisfying

$$\int_\Omega D^\alpha v \mathrm{d}x = 0, \quad |\alpha| \le k. \quad \text{(II.3)}$$

Then there holds

$$|v|_{m,\Omega} \le (\frac{d_\Omega}{\pi})^{k+1-m}|v|_{k+1,\Omega}, \quad \text{(II.4)}$$

where $0 \le m \le k+1$.

Proof: (II.4) is obvious for $m = k+1$. Consider for $0 \le m < k+1$. According to (II.3) and Lemma 2.1, we have (II.5), that is, $|v|_{m,\Omega} \le \frac{d_\Omega}{\pi}|v|_{m+1,\Omega}$. Consequently, we can get (II.4) for all $0 \le m \le k+1$ by analogy. ■

Let us denote by $\hat{T}$ the reference unit triangle in the $(\hat{x}_1, \hat{x}_2)$ space with vertices $\hat{a}_0 = (0,0), \hat{a}_1 = (1,0)$, and $\hat{a}_2 = (0,1)$. Let $\hat{\gamma}_i = \hat{a}_0\hat{a}_i,\ i = 1,2$. For each edge $\hat{\gamma}_i$, we define the auxiliary interpolation operator $\hat{I}_k^{(i)} : H^1(\hat{T}) \to P_k(\hat{T})$ satisfying

$$\begin{cases} \int_{\hat{T}}(v - \hat{I}_k^{(i)}v)p\mathrm{d}\hat{x}, & \forall p \in P_{k-1}(\hat{T}), \\ \int_{\hat{\gamma}_i}(v - \hat{I}_k^{(i)}v)q\mathrm{d}\hat{s} = 0, & \forall q \in P_k(\hat{\gamma}_i), \end{cases} \quad \text{(II.6)}$$

where $v \in H^1(\hat{T})$, $i = 1,2$ and k is a non-negative integer. The interpolation problem (II.6) is well posed, ref. [15]. In order to explicitly estimate the interpolation error $\|v - \hat{I}_k^{(i)}v\|_{0,\hat{T}}$, we introduce a sharp trace theorem on $\hat{T}$ given in [14].

Lemma 2.3: $\forall v \in H^1(\hat{T})$, $\forall \varepsilon > 0$, then there holds

$$\|v\|^2_{0,\hat{\gamma}_i} \le (2 + \frac{2}{\varepsilon^2})\|v\|^2_{0,\hat{T}} + \varepsilon^2|v|^2_{1,\hat{T}}, \quad i = 1,2. \quad \text{(II.7)}$$

For convenience, we define two index sets as follows:

$$A = \{1,2,\ldots,\frac{1}{2}(k+1)k\}, B = \{\frac{1}{2}(k+1)k+1, \frac{1}{2}(k+1)k+2,\cdots,\frac{1}{2}(k+2)(k+1)\}. \quad \text{(II.8)}$$

Let $\{p_j;\ j \in A\}$ be a basis of $P_{k-1}(\hat{T})$ and $\{p_j;\ j \in B\}$ a basis of $P_k(\hat{\gamma}_i)$ such that $\|p_j\|_{0,\hat{T}} = 1$ for $j \in A$ and $\|p_j\|_{0,\hat{\gamma}_i} = 1$ for $j \in B$. Then for each interpolation operator $\hat{I}_k^{(i)}$, there exists a corresponding basis $\{\hat{\phi}_j,\ j \in A \cup B\}$ in $P_k(\hat{T})$ such that

$$\hat{I}_k^{(i)}v = \sum_{j\in A}(\int_{\hat{T}} vp_j\mathrm{d}\hat{x})\hat{\phi}_j + \sum_{j\in B}(\int_{\hat{\gamma}_i} vp_j\mathrm{d}\hat{s})\hat{\phi}_j. \quad \text{(II.9)}$$

Combining Lemma 2.3 and the Hölder inequality, we get

$$\begin{aligned} &\|v - \hat{I}_k^{(i)}v\|^2_{0,\hat{T}} \\ &= \int_{\hat{T}}(v - \hat{I}_k^{(i)}v)^2\mathrm{d}\hat{x} \\ &\le [1 + \frac{1}{2}(k+2)(k+1)](\|v\|^2_{0,\hat{T}} \\ &\qquad + \sum_{j\in A}\|\hat{\phi}_j\|^2_{0,\hat{T}}\|v\|^2_{0,\hat{T}} + \sum_{j\in B}\|\hat{\phi}_j\|^2_{0,\hat{T}}\|v\|^2_{0,\hat{\gamma}_i}) \\ &\le N[(M_1 + 2M_2(1 + \frac{1}{\varepsilon^2}))\|v\|^2_{0,\hat{T}} + M_2\varepsilon_2|v|^2_{1,\hat{T}}], \end{aligned} \quad \text{(II.10)}$$

where $M_1 = 1 + \sum_{j\in A}\|\hat{\phi}_j\|^2_{0,\hat{T}}$, $M_2 = \sum_{j\in B}\|\hat{\phi}_j\|^2_{0,\hat{T}}$, $N = 1 + \frac{1}{2}(k+2)(k+1)$.

Let $p_v \in P_k(\hat{T})$ such that $\int_{\hat{T}} D^\alpha(v+p_v)\mathrm{d}\hat{x} = 0,\ \ |\alpha| \le k$. According to Lemma 2.2, it implies that

$$\begin{aligned} &\|v - \hat{I}_k^{(i)}v\|^2_{0,\hat{T}} \\ &= \|v + p_v - \hat{I}_k^{(i)}(v+p_v)\|^2_{0,\hat{T}} \\ &\le N[(M_1 + 2M_2(1 + \frac{1}{\varepsilon^2}))\|v \\ &\qquad + p_v\|^2_{0,\hat{T}} + M_2\varepsilon^2|v+p_v|^2_{1,\hat{T}}] \\ &\le N[\frac{2}{\pi^2}(M_1 + 2M_2(1 + \frac{1}{\varepsilon^2}))|v+p_v|^2_{1,\hat{T}} \\ &\qquad + M_2\varepsilon^2|v+p_v|^2_{1,\hat{T}}] \\ &\le N[\frac{2}{\pi^2}(M_1 + 2M_2) + \frac{4M_2}{\pi^2\varepsilon^2} + M_2\varepsilon^2]|v+p_v|^2_{1,\hat{T}}. \end{aligned} \quad \text{(II.11)}$$

Let $\varepsilon^2 = \frac{2}{\pi}$ in the above inequality, that implies

$$\begin{aligned} &\|v - \hat{I}_k^{(i)}v\|^2_{0,\hat{T}} \\ &\le \frac{2N}{\pi^2}[M_1 + 2(1+\pi)M_2]|v+p_v|^2_{1,\hat{T}} \\ &\le \frac{2^{k+1}N}{\pi^{2(k+1)}}[M_1 + 2(1+\pi)M_2]|v|^2_{k+1,\hat{T}}. \end{aligned} \quad \text{(II.12)}$$

Therefore, we establish the following lemma.

Lemma 2.4: $\forall v \in H^{k+1}(\hat{T})$, there holds

$$\|v - I_k^{(i)}v\|_{0,\hat{T}} \le d_k|v|_{k+1,\hat{T}},\ i = 1,2, \quad \text{(II.13)}$$

where

$$d_k = \frac{1}{\pi^{k+1}}\sqrt{2^{k+1}N[M_1 + 2(1+\pi)M_2]}, \quad \text{(II.14)}$$

and N, M_1, M_2 are given in (II.12).

Next, we consider the following Poisson problem: Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad \text{(II.15)}$$

It is known that (II.15) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ for the convex domain with $\partial\Omega \in C^2$. Moreover, the Miranda-Talente estimate is valid, ref. ([18], [19]),

$$|u|_{2,\Omega} \le \|f\|_{0,\Omega}, \quad \text{(II.16)}$$

which is extended to a general convex polygonal domain in [20].

In order to get a mixed variational formulation of the Poisson problem (II.15), we introduce the Hilbert space $H(\mathrm{div}) \triangleq \{\underline{\varphi};\ \underline{\varphi} = (\varphi_1, \varphi_2) \in L^2(\Omega)^2,\ \mathrm{div}\underline{\varphi} \in L^2(\Omega)\}$ and the norm in $H(\mathrm{div})$ which is defined by

$$\|\underline{\varphi}\|_{H(\mathrm{div})} \triangleq (\|\underline{\varphi}\|^2_{0,\Omega} + \|\mathrm{div}\underline{\varphi}\|^2_{0,\Omega})^{\frac{1}{2}}, \quad \text{(II.17)}$$

$$
\begin{aligned}
|v|_{m,\Omega}^2 &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}\|D^\alpha v\|_{0,\Omega}^2 \\
&\le (\frac{d_\Omega}{\pi})^2 \sum_{|\alpha|=m} |D^\alpha v|_{1,\Omega}^2 \\
&= (\frac{d_\Omega}{\pi})^2 \sum_{|\alpha|=m} \frac{m!}{\alpha!}[\|D^{\alpha+(1,0)}v\|_{0,\Omega}^2 + \|D^{\alpha+(0,1)}v\|_{0,\Omega}^2 + \sum_{|\alpha|=m} \frac{m!}{\alpha!}\|D^{\alpha+(0,1)}v\|_{0,\Omega}^2] \\
&= (\frac{d_\Omega}{\pi})^2 [\sum_{\substack{|\alpha|=m+1\\ \alpha_1>0}} \frac{m!}{(\alpha_1-1)!\alpha_2!}\|D^\alpha v\|_{0,\Omega}^2 + \sum_{\substack{|\alpha|=m+1\\ \alpha_2>0}} \frac{m!}{\alpha_1!(\alpha_2-1)!}\|D^\alpha v\|_{0,\Omega}^2] \\
&= (\frac{d_\Omega}{\pi})^2 [\sum_{\substack{|\alpha|=m+1\\ \alpha_1>0,\alpha_2>0}} (\frac{m!}{(\alpha_1-1)!\alpha_2!} + \frac{m!}{\alpha_1!(\alpha_2-1)!})\|D^\alpha v\|_{0,\Omega}^2 + \|D^{(m+1,0)}v\|_{0,\Omega}^2 + \|D^{(0,m+1)}v\|_{0,\Omega}^2] \\
&= (\frac{d_\Omega}{\pi})^2 [\sum_{\substack{|\alpha|=m+1\\ \alpha_1>0,\alpha_2>0}} \frac{m!(\alpha_1+\alpha_2)}{\alpha!}\|D^\alpha v\|_{0,\Omega}^2 + \|D^{(m+1,0)}v\|_{0,\Omega}^2 + \|D^{(0,m+1)}v\|_{0,\Omega}^2] \\
&= (\frac{d_\Omega}{\pi})^2 |v|_{m+1,\Omega}^2,
\end{aligned}
\tag{II.5}
$$

where $\mathrm{div}\underline{\varphi} = \frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2}$.

Let $\underline{\psi} = -\nabla u$. Then the mixed variational formulation of (II.15) is to find $(\underline{\psi}, u) \in H(\mathrm{div}) \times L^2(\Omega)$ such that

$$
\begin{cases} a(\underline{\psi}, \underline{\varphi}) + b(\underline{\varphi}, u) = 0, & \forall \underline{\varphi} \in H(\mathrm{div}), \\ b(\underline{\psi}, v) = F(v), & \forall v \in L^2(\Omega), \end{cases} \tag{II.18}
$$

where $a(\underline{\psi}, \underline{\varphi}) = \int_\Omega \underline{\psi} \cdot \underline{\varphi} \mathrm{d}x,\ b(\underline{\varphi}, v) = -\int_\Omega v \mathrm{div}\underline{\varphi} \mathrm{d}x$ and $F(v) = -\int_\Omega f v \mathrm{d}x$.

The well-posedness of the problem (II.18) has been well established in [21]. Generally speaking, the final error constants for the mixed finite element method will be dependent on the constant in the inf-sup condition. The following lemma give an explicit lower bound of the inf-sup constant, which can be found in [14].

Lemma 2.5: Let the domain Ω be star-shaped with respect to a point which we just choose to be the origin for simplicity, and the boundary of Ω be represented in the plane polar coordinates by $r = \rho(\theta)$. Then we get the following inf-sup condition

$$
\inf_{v \in L^2(\Omega)} \sup_{\underline{\varphi} \in H(\mathrm{div})} \frac{b(\underline{\varphi}, v)}{\|v\|_{0,\Omega}\|\underline{\varphi}\|_{H(\mathrm{div})}} \ge \beta, \tag{II.19}
$$

where

$$
\beta = \frac{1}{\sqrt{1 + \max \rho(\theta)}}. \tag{II.20}
$$

III. Bounds of interpolation error constants for the Raviart-Thomas element

We firstly give an introduction of the Raviart-Thomas mixed finite element space, ref. [15]. To this end, let $\mathcal{T}_h$ be a finite element triangulation of Ω. For a general element $T \in \mathcal{T}_h$ with vertices a_0, a_1 and a_2, we denote by h_T and ρ_T the diameter of T and the supremum of the diameters of T, respectively. And we can get $h = \max_{T \in \mathcal{T}_h} h_T$. Without loss of generality, assume that the maximal angle of T is $\theta_T = \angle a_1 a_0 a_2$, $\gamma_i = a_0 a_i$, and n_i, τ_i, l_i are respectively the unit exterior normal, direction and length of the edge γ_i, where $i = 1, 2$.

For any $T \in \mathcal{T}_h$, the affine transformation $F_T : \hat{T} \to T$ is defined by

$$
x = F_T(\hat{x}) = B\hat{x} + a_0, \tag{III.1}
$$

where $B = (l_1\tau_1, l_2\tau_2)$ and $\hat{T}$ is the reference element given in Section II, seeing Fig. 1 as for an illustration.

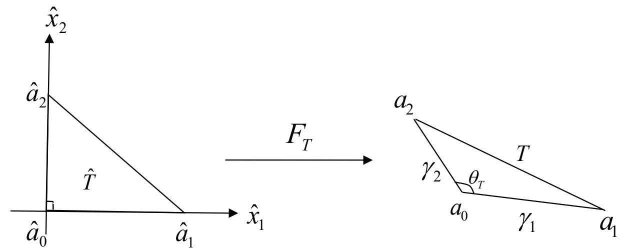

Fig. 1. An illustration of the affine transformation

The local Raviart-Thomas element on T of the order $k \ge 0$ is defined by

$$
H_T = P_k(T)^2 \oplus x\bar{P}_k(T), \tag{III.2}
$$

where $\bar{P}_k(T) = \mathrm{span}\{x_1^i x_2^j;\ i + j = k,\ i, j \ge 0\}$.

The corresponding local interpolation operator $RT_T^k : H^1(T) \to H_T$ is defined by

$$
\begin{cases} \int_\gamma q RT_T^k \underline{\varphi} \cdot n \mathrm{d}s = \int_\gamma q \underline{\varphi} \cdot n \mathrm{d}s, & \forall q \in P_k(\gamma),\ \gamma \subset \partial T, \\ \int_T RT_T^k \underline{\varphi} \cdot \underline{p} \mathrm{d}x = \int_T \underline{\varphi} \cdot \underline{p} \mathrm{d}x, & \forall \underline{p} \in P_{k-1}(T)^2, \end{cases} \tag{III.3}
$$

where n is the unit exterior normal vector of the edge γ.

Let $H_h = \{\underline{\varphi}_h \in H(\mathrm{div});\ \underline{\varphi}_h|_T \in H_T,\ \forall T \in \mathcal{T}_h\}$. Then the global interpolation operator is defined by

$$
RT_h^k|_T = RT_T^k, \quad \forall T \in \mathcal{T}_h. \tag{III.4}
$$

Furthermore, let M_h be the finite element space which is the discrete approximation of $L^2(\Omega)$. The operator $P_h^k : L^2(\Omega) \to M_h$ is defined by

$$
P_h^k|_T = P_T^k, \quad \forall T \in \mathcal{T}_h, \tag{III.5}
$$

where P_T^k is the k order L^2-projection operator on T.

It has been proved that the operator RT_h^k satisfies

$$\mathrm{div} RT_h^k \underline{\varphi} = P_h^k \mathrm{div}\underline{\varphi},\ \forall \underline{\varphi} \in H(\mathrm{div}). \tag{III.6}$$

For any $\underline{\varphi} \in H_T$, we introduce the relations between derivatives of $\underline{\varphi}$ and derivatives of its divergence, which are given in [15].

Lemma 3.1: $\forall T \in \mathcal{T}_h,\ \forall \underline{\varphi} \in H_T$, there holds

$$\begin{cases} \frac{\partial^{k+1}\underline{\varphi}}{\partial x_1^{k+1}} = (\frac{k+1}{k+2}\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_1^k}, 0), \\ \frac{\partial^{k+1}\underline{\varphi}}{\partial x_2^{k+1}} = (0, \frac{k+1}{k+2}\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_2^k}), \\ \frac{\partial^{k+1}\underline{\varphi}}{\partial x_1^i \partial x_2^j} = (\frac{i}{k+2}\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_1^{i-1}\partial x_2^j}, \frac{j}{k+2}\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_1^i \partial x_2^{j-1}}), \\ i+j = k+1, i > 0, j > 0. \end{cases} \tag{III.7}$$

For the projection operator $P_{\hat{T}}^k$, we have the following result that is also found in [15].

Lemma 3.2: $\forall v \in H^k(\hat{T}),\ i, j \geq 0,\ i+j = k$, there holds

$$\|\frac{\partial^k P_{\hat{T}}^k v}{\partial \hat{x}_1^i \partial \hat{x}_2^j}\|_{0,\hat{T}} \leq \hat{c}_{ij}\|\frac{\partial^k v}{\partial \hat{x}_1^i \partial \hat{x}_2^j}\|_{0,\hat{T}}, \tag{III.8}$$

where $\hat{c}_{ij} = \frac{1}{\|q_{ij}\|_{0,\hat{T}}^2}\|\frac{\partial^k q_{ij}}{\partial \hat{x}_1^i \partial \hat{x}_2^j}\|_{0,\hat{T}}\|\hat{x}_1^i \hat{x}_2^j (1-\hat{x}_1-\hat{x}_2)^k\|_{0,\hat{T}}$ and $q_{ij}(\hat{x}) = \frac{\partial^k}{\partial \hat{x}_1^i \partial \hat{x}_2^j}(\hat{x}_1^i \hat{x}_2^j (1-\hat{x}_1-\hat{x}_2)^k)$.

Remark 3.3: The expression of $\hat{c}_{ij}$ in Lemma 3.2 is not explicitly given in [15]. However, we can obtain it from the proof of Lemma 3.2 that is Lemma 2.3 in [15].

With the above preparations, we start to explicitly estimate error constants for the interpolation operator RT_T^k.

Theorem 3.4: $\forall T \in \mathcal{T}_h,\ \forall \underline{\varphi} \in H^{k+1}(T)^2$, there holds

$$\begin{aligned} &\|\underline{\varphi} - RT_T^k \underline{\varphi}\|_{0,T} \\ &\leq \frac{4d_k}{\sin\theta_T}[\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial_{k+1}\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2 \\ &+ \frac{3+2\sqrt{2}}{4}\frac{k+1}{k+2}b_k^2 h_T^2 \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2]^{\frac{1}{2}}, \end{aligned} \tag{III.9}$$

where $b_k = \max_{|\alpha|=k}\{\hat{c}_\alpha\}$, $l^{2\alpha} = l_1^{2\alpha_1} l_2^{2\alpha_2}$, $\hat{c}_\alpha$ is given in Lemma 3.2 and d_k is given in (II.14).

Proof: Let $N_0 = (n_1, n_2)$, then $|\mathrm{det}N_0| = \sin\theta_T$. Since $\underline{\varphi} - RT_T^k\underline{\varphi} = N_0^{-1}((\underline{\varphi} - RT_T^k\underline{\varphi})\cdot n_1, (\underline{\varphi} - RT_T^k\underline{\varphi})\cdot n_2)^T$, then it implies $\|\underline{\varphi} - RT_T^k\underline{\varphi}\|_{0,T}^2 \leq \frac{2}{\sin^2\theta_T}\sum_{i=1}^2 \|(\underline{\varphi} - RT_T^k\underline{\varphi})\cdot n_i\|_{0,T}^2$. Let $w_i = (\underline{\varphi} - RT_T^k\underline{\varphi})\cdot n_i$ and $\hat{w}_i = w_i \circ F_T$. Since (III.3) implies $I_k^{(i)}\hat{w}_i = 0$, according to Lemma 2.4 we have (III.10).

We need to estimate the second term in the above inequality. For a given $\underline{\varphi} \in H^{k+1}(T)^2$, we define the corresponding function $\hat{\underline{\varphi}} \in H^{k+1}(\hat{T})$ via the Piola transformation such that

$$\begin{aligned} &\underline{\varphi} = \frac{B}{|\mathrm{det}B|}\hat{\underline{\varphi}}, \hat{\nabla} = B^T\nabla, \\ &\hat{\nabla} = \Lambda\nabla_\tau, \nabla_\tau = B_0^T\nabla, \end{aligned} \tag{III.11}$$

where $\hat{\nabla} = (\frac{\partial}{\partial\hat{x}_1}, \frac{\partial}{\partial\hat{x}_2})^T$, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$, $\nabla_\tau = (\frac{\partial}{\partial\tau_1}, \frac{\partial}{\partial\tau_2})^T$, $B_0 = (\tau_1, \tau_2)$ and $\Lambda = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}$.

(III.11) implies

$$\mathrm{div}\underline{\varphi} = \nabla\cdot\underline{\varphi} = \frac{1}{|\mathrm{det}B|}\widehat{\mathrm{div}}\hat{\underline{\varphi}}. \tag{III.12}$$

Since $\|B\|_2 \leq \frac{h_T}{\hat{\rho}}$ and $\hat{\rho} = \rho_{\hat{T}}$, then we have

$$\begin{aligned} &\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\tau_2^{\alpha_2}}\|_{0,T}^2 \\ &= \frac{|\mathrm{det}B|}{l^{2\alpha}}\|\frac{B}{|\mathrm{det}B|}\frac{\partial^{k+1}RT_{\hat{T}}^k\hat{\underline{\varphi}}}{\partial\hat{x}_1^{\alpha_1}\partial\hat{x}_2^{\alpha_2}}\|_{0,\hat{T}}^2 \\ &\leq \frac{h_T^2}{\hat{\rho}^2 l^{2\alpha}|\mathrm{det}B|}\|\frac{\partial^{k+1}RT_{\hat{T}}^k\hat{\underline{\varphi}}}{\partial\hat{x}_1^{\alpha_1}\partial\hat{x}_2^{\alpha_2}}\|_{0,\hat{T}}^2. \end{aligned} \tag{III.13}$$

Combining Lemmas 3.1–3.2 and (III.6), for $\alpha = (k+1, 0)$ we get

$$\begin{aligned} &\|\frac{\partial^{k+1}RT_{\hat{T}}^k\hat{\underline{\varphi}}}{\partial\hat{x}_1^{\alpha_1}\partial\hat{x}_2^{\alpha_2}}\|_{0,\hat{T}}^2 \\ &= (\frac{k+1}{k+2})^2\|\frac{\partial^k(\widehat{\mathrm{div}}RT_{\hat{T}}^k\hat{\underline{\varphi}})}{\partial\hat{x}_1^k}\|_{0,\hat{T}}^2 \\ &= (\frac{k+1}{k+2})^2\|\frac{\partial^k(P_{\hat{T}}^k\widehat{\mathrm{div}}\hat{\underline{\varphi}})}{\partial\hat{x}_1^k}\|_{0,\hat{T}}^2 \\ &= (\frac{k+1}{k+2})^2\hat{c}_{k0}^2 l_1^{2k}|\mathrm{det}B|\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^k}\|_{0,T}^2. \end{aligned} \tag{III.14}$$

For $\alpha = (0, k+1)$, similarly there exists

$$\begin{aligned} &\|\frac{\partial^{k+1}RT_{\hat{T}}^k\hat{\underline{\varphi}}}{\partial\hat{x}_1^{\alpha_1}\partial\hat{x}_2^{\alpha_2}}\|_{0,\hat{T}}^2 \\ &\leq (\frac{k+1}{k+2})^2\hat{c}_{0k}^2 l_2^{2k}|\mathrm{det}B|\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_2^k}\|_{0,T}^2. \end{aligned} \tag{III.15}$$

For $\alpha > 0$ ($\alpha_1 > 0$ and $\alpha_2 > 0$), we get III.16

According to (III.13–III.16), we obtain III.17.

Let $b_k = \max_{|\alpha|=k}\{\hat{c}_\alpha\}$. Then there holds

$$\begin{aligned} &\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2 \\ &\leq \frac{k+1}{k+2}(\frac{b_k h_T}{\hat{\rho}})^2\sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2. \end{aligned} \tag{III.18}$$

Combining (III.10) and (III.18), and observing $\hat{\rho} = 2-\sqrt{2}$, we obtain (III.9). ■

Remark 3.5: If $\theta_T = \frac{\pi}{2}$, we can get better results. Without loss of generality, let the right triangle T has two edges respectively parallel to the coordinate axes. Then (III.10) can be replaced by

$$\begin{aligned} \|\underline{\varphi} - RT_T^k\underline{\varphi}\|_{0,T}^2 \leq 4d_k^2 \sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}(&\|\frac{\partial^{k+1}\underline{\varphi}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|_{0,T}^2 \\ &+ \|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|_{0,T}^2). \end{aligned}$$

Corollary 3.6: $\forall T \in \mathcal{T}_h,\ \forall \underline{\varphi} \in H^{k+1}(T)^2$, suppose $\theta_T = \frac{\pi}{2}$, then there holds

$$\begin{aligned} &\|\underline{\varphi} - RT_T^k\underline{\varphi}\|_{0,T} \\ &\leq 2d_k[\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial_{k+1}\underline{\varphi}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|_{0,T}^2 \\ &\quad + \frac{(3+2\sqrt{2})(k+1)}{2(k+2)}b_k^2 h_T^2 \\ &\quad \cdot \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|_{0,T}^2]^{\frac{1}{2}}, \end{aligned} \tag{III.19}$$

$$
\begin{aligned}
\|\underline{\varphi}-RT_T^k\underline{\varphi}\|_{0,T}^2 &\le \frac{2d_k^2}{\sin^2\theta_T}\sum_{i=1}^{2}\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}(\underline{\varphi}-RT_T^k\underline{\varphi})\cdot n_i}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2\\
&\le \frac{4d_k^2}{\sin^2\theta_T}\sum_{i=1}^{2}\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}(\|\frac{\partial^{k+1}(\underline{\varphi}\cdot n_i)}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2+\|\frac{\partial^{k+1}(RT_T^k\underline{\varphi}\cdot n_i)}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2)\\
&\le \frac{4d_k^2}{\sin^2\theta_T}\sum_{i=1}^{2}\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}(2\|\frac{\partial^{k+1}\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2+\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2)\\
&= \frac{16d_k^2}{\sin^2\theta_T}\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2+\frac{8d_k^2}{\sin^2\theta_T}\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2.
\end{aligned}
\tag{III.10}
$$

$$
\begin{aligned}
&\|\frac{\partial^{k+1}RT_{\hat T}^k\hat{\underline{\varphi}}}{\partial\hat x_1^{\alpha_1}\partial\hat x_2^{\alpha_2}}\|_{0,\hat T}^2\\
&=(\frac{\alpha_1}{k+2})^2\|\frac{\partial^k(\widehat{\mathrm{div}}RT_{\hat T}^k\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1-1}\partial\hat x_2^{\alpha_2}}\|_{0,\hat T}^2+(\frac{\alpha_2}{k+2})^2\|\frac{\partial^k(\widehat{\mathrm{div}}RT_{\hat T}^k\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1}\partial\hat x_2^{\alpha_2-1}}\|_{0,\hat T}^2\\
&=(\frac{\alpha_1}{k+2})^2\|\frac{\partial^k P_{\hat T}^k(\widehat{\mathrm{div}}\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1-1}\partial\hat x_2^{\alpha_2}}\|_{0,\hat T}^2+(\frac{\alpha_2}{k+2})^2\|\frac{\partial^k P_{\hat T}^k(\widehat{\mathrm{div}}\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1}\partial\hat x_2^{\alpha_2-1}}\|_{0,\hat T}^2\\
&\le(\frac{\alpha_1}{k+2}\hat c_{\alpha_1-1,\alpha_2})^2\|\frac{\partial^k(\widehat{\mathrm{div}}\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1-1}\partial\hat x_2^{\alpha_2}}\|_{0,\hat T}^2+(\frac{\alpha_2}{k+2}\hat c_{\alpha_1,\alpha_2-1})^2\|\frac{\partial^k(\widehat{\mathrm{div}}\hat{\underline{\varphi}})}{\partial\hat x_1^{\alpha_1}\partial\hat x_2^{\alpha_2-1}}\|_{0,\hat T}^2\\
&=(\frac{\alpha_1}{k+2}\hat c_{\alpha_1-1,\alpha_2})^2 l_1^{2(\alpha_1-1)}l_2^{2\alpha_2}|\mathrm{det}B|\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1-1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2+(\frac{\alpha_2}{k+2}\hat c_{\alpha_1,\alpha_2-1})^2 l_1^{2\alpha_1}l_2^{2(\alpha_2-1)}|\mathrm{det}B|\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2-1}}\|_{0,T}^2.
\end{aligned}
\tag{III.16}
$$

$$
\begin{aligned}
&\sum_{|\alpha|=k+1}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2\\
&=l_1^{2(k+1)}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{k+1}}\|_{0,T}^2+l_2^{2(k+1)}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_2^{k+1}}\|_{0,T}^2+\sum_{|\alpha|=k+1,\alpha>0}\frac{|\alpha|!}{\alpha!}l^{2\alpha}\|\frac{\partial^{k+1}RT_T^k\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2\\
&\le(\frac{h_Tl_1^k}{\hat\rho}\frac{k+1}{k+2}\hat c_{k0})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^k}\|_{0,T}^2+(\frac{h_Tl_2^k}{\hat\rho}\frac{k+1}{k+2}\hat c_{0k})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_2^k}\|_{0,T}^2\\
&\quad+\sum_{|\alpha|=k+1,\alpha>0}\frac{|\alpha|!}{\alpha!}(\frac{h_T}{\hat\rho})^2[(\frac{\alpha_1}{k+2}\hat c_{\alpha_1-1,\alpha_2}l_1^{\alpha_1-1}l_2^{\alpha_2})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau^{\alpha_1-1}\partial\tau^{\alpha_2}}\|_{0,T}^2\\
&\quad+(\frac{\alpha_2}{k+2}\hat c_{\alpha_1,\alpha_2-1}l_1^{\alpha_1}l_2^{\alpha_2-1})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau^{\alpha_1}\partial\tau^{\alpha_2-1}}\|_{0,T}^2]\\
&=(\frac{h_Tl_1^k}{\hat\rho}\frac{k+1}{k+2}\hat c_{k0})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^k}\|_{0,T}^2+(\frac{h_Tl_2^k}{\hat\rho}\frac{k+1}{k+2}\hat c_{0k})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_2^k}\|_{0,T}^2\\
&\quad+\sum_{|\alpha|=k,\alpha_1\ge0,\alpha_2>0}\frac{(k+1)!}{(\alpha_1+1)!\alpha_2!}(\frac{h_Tl^\alpha}{\hat\rho}\frac{\alpha_1+1}{k+2}\hat c_\alpha)^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau^{\alpha_1}\partial\tau^{\alpha_2}}\|_{0,T}^2\\
&\quad+\sum_{|\alpha|=k,\alpha_1>0,\alpha_2\ge0}\frac{(k+1)!}{\alpha_1!(\alpha_2+1)!}(\frac{h_Tl^\alpha}{\hat\rho}\frac{\alpha_2+1}{k+2}\hat c_\alpha)^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau^{\alpha_1}\partial\tau^{\alpha_2}}\|_{0,T}^2\\
&\le\frac{k+1}{k+2}(\frac{\hat c_{k0}h_Tl_1^k}{\hat\rho})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^k}\|_{0,T}^2+\frac{k+1}{k+2}(\frac{\hat c_{0k}h_Tl_2^k}{\hat\rho})^2\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_2^k}\|_{0,T}^2\\
&\quad+\sum_{|\alpha|=k,\alpha>0}\frac{|\alpha|!}{\alpha!}\frac{k+1}{(k+2)^2}(\frac{\hat c_\alpha h_Tl^\alpha}{\hat\rho})^2(\alpha_1+1+\alpha_2+1)\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2\\
&=\frac{k+1}{k+2}(\frac{h_T}{\hat\rho})^2\sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}\hat c_\alpha^2 l^{2\alpha}\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|_{0,T}^2.
\end{aligned}
\tag{III.17}
$$

---

where b_k is given in Theorem 3.4 and d_k is given in (II.14).

Due to Theorem 3.4, Corollary 3.6 and the specific triangulation, we can explicitly get corresponding estimates of the global interpolation error $\|\underline{\varphi}-RT_h^k\underline{\varphi}\|_{0,\Omega}$. However, we give general results for the global interpolation error.

Theorem 3.7: Let $\theta=\max\limits_{T\in\mathcal{T}_h}\theta_T$ and $\underline{\varphi}\in H^{k+1}(\Omega)$. Then we have

$$\begin{aligned}\|\underline{\varphi}-RT_h^k\underline{\varphi}\|_{0,\Omega}\le&\frac{4\cdot 2^{k/2}d_k}{\sin\theta}h^{k+1}(2|\underline{\varphi}|^2_{k+1,\Omega}\\&+\frac{3+2\sqrt{2}}{4}\frac{k+1}{k+2}b_k^2|\mathrm{div}\underline{\varphi}|^2_{k,\Omega})^{\frac12},\end{aligned}\tag{III.20}$$

where b_k is given in Theorem 3.4 and d_k given in (II.14).

Proof: Without loss of generality, for any fixed $T\in\mathcal{T}_h$, we suppose $\tau_1=(1,0)^T$ and $\tau_2=(\cos\theta_T,\sin\theta_T)^T$. Then (III.11) implies $\|B_0^T\|_2\le\sqrt{2}$ and

$$\begin{aligned}&\|\frac{\partial_{k+1}\underline{\varphi}}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|^2_{0,T}\le 2^{k+1}\|\frac{\partial_{k+1}\underline{\varphi}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|^2_{0,T},\\&\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial\tau_1^{\alpha_1}\partial\tau_2^{\alpha_2}}\|^2_{0,T}\le 2^k\|\frac{\partial^k(\mathrm{div}\underline{\varphi})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\|^2_{0,T}.\end{aligned}\tag{III.21}$$

Combining (III.9) and (III.21), we get (III.20) by simple calculations. ■

IV. The discrete inf-sup constants for the Raviart-Thomas element

In this section, we present the lower bounds of the inf-sup constants for a general Raviart-Thomas element. According to the Fortin's principle given by Fortin in [22], we need to estimate the constant c in the following inequality,

$$\|RT_h^k\underline{\varphi}\|_{H(\mathrm{div})}\le c\|\underline{\varphi}\|_{H(\mathrm{div})}.\tag{IV.1}$$

Let $\hat{\gamma}_3=\hat{a}_1\hat{a}_2$ and $\hat{n}_3,\hat{l}_3$ be the unit exterior normal vector and length of the edge $\hat{\gamma}_3$ on $\hat{T}$, respectively. Let A be the index set given in (II.8), $\{\underline{p}_j;\ j\in A\}$ a basis of $P_{k-1}(\hat{T})^2$ and $\{q_{ij};\ j=1,2,\ldots,k+1\}$ a basis of $P_k(\hat{\gamma}_i)$ for $i=1,2,3$. Then there exists a basis $\{\underline{\varphi}_{ij};\ i=1,2,3,\ j=1,2,\ldots,k+1\}\cup\{\underline{\psi}_j;\ j\in A\}$ in $H_{\hat{T}}$ such that

$$\begin{aligned}RT_{\hat{T}}^k\underline{\hat{\varphi}}=&\sum_{i=1}^3\sum_{j=1}^{k+1}(\int_{\hat{\gamma}_i}q_{ij}\underline{\hat{\varphi}}\cdot\hat{n}_i\mathrm{d}\hat{s})\underline{\varphi}_{ij}\\&+\sum_{j\in A}(\int_{\hat{T}}\underline{\hat{\varphi}}\cdot\underline{p}_j\mathrm{d}\hat{x})\underline{\psi}_j,\quad\forall\underline{\hat{\varphi}}\in H(\widehat{\mathrm{div}}),\end{aligned}\tag{IV.2}$$

where $H(\widehat{\mathrm{div}})=\{\underline{\hat{\varphi}};\ \underline{\hat{\varphi}}\in L^2(\hat{T})^2,\widehat{\mathrm{div}}\underline{\hat{\varphi}}\in L^2(\hat{T})\}$. Then

$$\begin{aligned}&\|RT_{\hat{T}}^k\underline{\hat{\varphi}}\|^2_{0,\hat{T}}\\&\le(k+1)(k+3)[\sum_{i=1}^3\sum_{j=1}^{k+1}(\int_{\hat{\gamma}_i}q_{ij}\underline{\hat{\varphi}}\cdot\hat{n}_i\mathrm{d}\hat{s})^2\|\underline{\varphi}_{ij}\|^2_{0,\hat{T}}\\&\quad+\sum_{j\in A}(\int_{\hat{T}}\underline{\hat{\varphi}}\cdot\underline{p}_j\mathrm{d}\hat{x})^2\|\underline{\psi}_j\|^2_{0,\hat{T}}]\\&\le(k+1)(k+3)[\sum_{i=1}^3\sum_{j=1}^{k+1}\|\underline{\hat{\varphi}}\cdot\hat{n}_i\|^2_{-\frac12,\partial\hat{T}}\|q_{ij}\|^2_{\frac12,\partial\hat{T}}\|\underline{\varphi}_{ij}\|^2_{0,\hat{T}}\\&\quad+\sum_{j\in A}\|\underline{\hat{\varphi}}\|^2_{0,\hat{T}}\|\underline{p}_j\|^2_{0,\hat{T}}\|\underline{\psi}_j\|^2_{0,\hat{T}}]\\&\le(k+1)(k+3)[\|\underline{\hat{\varphi}}\|^2_{H(\mathrm{div})}\sum_{i=1}^3\sum_{j=1}^{k+1}\|q_{ij}\|^2_{1,\hat{T}}\|\underline{\varphi}_{ij}\|^2_{0,\hat{T}}\\&\quad+\|\underline{\hat{\varphi}}\|^2_{0,\hat{T}}\sum_{j\in A}\|\underline{p}_j\|^2_{0,\hat{T}}\|\underline{\psi}_j\|^2_{0,\hat{T}}].\end{aligned}$$

In the last inequalities above, we use the following inequality given in [23],

$$\|\underline{\hat{\varphi}}\cdot\hat{n}\|_{-\frac12,\partial\hat{T}}\le\|\underline{\hat{\varphi}}\|_{H(\mathrm{div})}.\tag{IV.3}$$

Let

$$\begin{aligned}M_1^*=&\sum_{i=1}^3\sum_{j=1}^{k+1}\|q_{ij}\|^2_{1,\hat{T}}\|\underline{\varphi}_{ij}\|^2_{0,\hat{T}},\\M_2^*=&\sum_{j\in A}\|\underline{p}_j\|^2_{0,\hat{T}}\|\underline{\psi}_j\|^2_{0,\hat{T}}.\end{aligned}\tag{IV.4}$$

Then we have

$$\begin{aligned}&\|RT_{\hat{T}}^k\underline{\hat{\varphi}}\|^2_{0,\hat{T}}\\&\le(k+1)(k+3)\cdot(M_1^*\|\underline{\hat{\varphi}}\|^2_{H(\mathrm{div})}+M_2^*\|\underline{\hat{\varphi}}\|^2_{0,\hat{T}}).\end{aligned}\tag{IV.5}$$

Next we estimate $\|RT_T^k\underline{\varphi}\|^2_{0,T}$. Without loss of generality, we suppose $\tau_1=(1,0)^T$, $\tau_2=(\cos\theta_T,\sin\theta_T)^T$ and $a_T=\min\{l_1,l_2\}$. Then $B=(l_1\tau_1,l_2\tau_2)$ implies that

$$\|B\|_2\le\sqrt{2}h_T,\quad\|B^{-1}\|_2\le\frac{\sqrt{2}}{a_T}.\tag{IV.6}$$

According to (III.11–III.12), we have IV.7. On the other hand,

$$\|\mathrm{div}(RT_T^k\underline{\varphi})\|^2_{0,T}=\|P_T^k(\mathrm{div}\underline{\varphi})\|^2_{0,T}\le\|\mathrm{div}\underline{\varphi}\|^2_{0,T}.\tag{IV.8}$$

Since $\|RT_h^k\underline{\varphi}\|^2_{H(\mathrm{div})}=\sum_{T\in\mathcal{T}_h}(\|RT_T^k\underline{\varphi}\|^2_{0,T}+\|\mathrm{div}(RT_T^k\underline{\varphi})\|^2_{0,T})$, then we establish the following lemma.

Lemma 4.1: Suppose that there is a constant $\bar{c}>0$ such that $\frac{h_T}{a_T}\le\bar{c}$ for any $T\in\mathcal{T}_h$, then for any $\underline{\varphi}\in H(\mathrm{div})$, we have

$$\|RT_h^k\underline{\varphi}\|_{H(\mathrm{div})}\le d_k^*\|\underline{\varphi}\|_{H(\mathrm{div})},\tag{IV.9}$$

where

$$\begin{aligned}d_k^*=\max\{&2\bar{c}\sqrt{(k+1)(k+3)(M_1^*+M_2^*)},\\&\sqrt{1+2(k+1)(k+3)M_1^*h^2}\},\end{aligned}\tag{IV.10}$$

and M_1^*,M_2^* are given in (IV.4).

Remark 4.2: Similar to d_k in Lemma 2.4, for a given k, the value of d_k^* is depended on the choice of the basis functions.

$$
\begin{aligned}
\|RT_T^k\underline{\varphi}\|_{0,T}^2 &\le \frac{\|B\|^2}{|\mathrm{det}B|}\|RT_{\hat{T}}^k\hat{\underline{\varphi}}\|_{0,\hat{T}}^2\\
&\le (k+1)(k+3)\frac{\|B\|^2}{|\mathrm{det}B|}(M_1^*\|\hat{\underline{\varphi}}\|_{H(\widehat{\mathrm{div}})}^2+M_2^*\|\hat{\underline{\varphi}}\|_{0,\hat{T}}^2)\\
&= (k+1)(k+3)\frac{\|B\|^2}{|\mathrm{det}B|}[(M_1^*+M_2^*)\|\hat{\underline{\varphi}}\|_{0,\hat{T}}^2+M_1^*\|\widehat{\mathrm{div}}\hat{\underline{\varphi}}\|_{0,\hat{T}}^2]\\
&\le (k+1)(k+3)\frac{\|B\|^2}{|\mathrm{det}B|}[(M_1^*+M_2^*)\|B^{-1}\|^2|\mathrm{det}B|\|\underline{\varphi}\|_{0,T}^2+M_1^*|\mathrm{det}B|\|\mathrm{div}\underline{\varphi}\|_{0,T}^2]\\
&= (k+1)(k+3)[(M_1^*+M_2^*)\|B\|^2\|B^{-1}\|^2\|\underline{\varphi}\|_{0,T}^2+M_1^*\|B\|^2\|\mathrm{div}\underline{\varphi}\|_{0,T}^2]\\
&\le (k+1)(k+3)[4(M_1^*+M_2^*)\frac{h_T^2}{a_T^2}\|\underline{\varphi}\|_{0,T}^2+2M_1^*h_T^2\|\mathrm{div}\underline{\varphi}\|_{0,T}^2]
\end{aligned} \tag{IV.7}
$$

We now consider the mixed finite element formulation (II.18), and try to seek $(\underline{\psi}_h, u_h)\in H_h\times M_h$ such that

$$
\begin{cases} a(\underline{\psi}_h,\underline{\varphi}_h)+b(\underline{\varphi}_h,u_h)=0, & \forall\underline{\varphi}_h\in H_h,\\ b(\underline{\psi}_h,v_h)=F(v_h), & \forall v_h\in M_h.\end{cases} \tag{IV.11}
$$

In order to test the existence and uniqueness of the discrete problem (IV.11), with the help of Fortin' principle, we display a characterization of the discrete inf-sup constant.

Lemma 4.3: For the Raviart-Thomas finite element space, we have the following discrete inf-sup condition

$$
\inf_{v_h\in M_h}\sup_{\underline{\varphi}_h\in H_h}\frac{b(\underline{\varphi}_h,v_h)}{\|v_h\|_{0,\Omega}\|\underline{\varphi}_h\|_{H(\mathrm{div})}}\ge\frac{\beta}{d_k^*}, \tag{IV.12}
$$

where β is given in (II.20) and d_k^* given in (IV.10).

V. Explicit error estimates for the Raviart-Thomas Element approximation

In this section, we estimate the approximation errors $\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}$, $\|\mathrm{div}\underline{\psi}-\mathrm{div}\underline{\psi}_h\|_{0,\Omega}$ and $\|u-u_h\|_{0,\Omega}$. As a preparation, we need to explicitly estimate the interpolation error for the L^2-projection operator P_T^k on T.

Lemma 5.1: $\forall T\in\mathcal{T}_h$, then

$$
\|v-P_T^kv\|_{0,T}\le(\frac{h}{\pi})^{k+1}|v|_{k+1,T},\quad\forall v\in H^{k+1}(T). \tag{V.1}
$$

Proof: Let $p_v\in P_k(T)$ such that $\int_{\hat{T}}D^\alpha(v+p_v)\mathrm{d}x=0$, $|\alpha|\le k$. According to Lemma 2.2, we have $\|v-P_T^kv\|_{0,T}=\|(v+p_v)-P_T^k(v+p_v)\|_{0,T}\le\|v+p_v\|_{0,T}\le(\frac{h}{\pi})^{k+1}|v|_{k+1,T}$. ■

Theorem 5.2: Let (ψ,u) and $(\underline{\psi}_h,u_h)$ be the solutions of (II.18) and (IV.11), respectively. Then we have

$$
\begin{aligned}
&\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}\le c_kh^{k+1}(|\underline{\psi}|_{k+1,\Omega}+|\mathrm{div}\underline{\psi}|_{k,\Omega}),\\
&\|u-u_h\|_{0,\Omega}\le C_kh^{k+1}(|\underline{\psi}|_{k+1,\Omega}+|\mathrm{div}\underline{\psi}|_{k,\Omega}+|u|_{k+1,\Omega}).
\end{aligned} \tag{V.2}
$$

In addition, if $u\in H^{k+3}(\Omega)$, then

$$
\|\mathrm{div}\underline{\psi}-\mathrm{div}\underline{\psi}_h\|_{0,\Omega}\le(\frac{h}{\pi})^{k+1}|\mathrm{div}\underline{\psi}|_{k+1,\Omega}, \tag{V.3}
$$

where $c_k=\frac{4\cdot2^{k/2}d_k}{\sin\theta}\max\{\sqrt{2},\ \frac{\sqrt{2}+1}{2}b_k\sqrt{\frac{k+1}{k+2}}\}, C_k=\max\{\frac{c_kd_k^*}{\beta},\frac{1}{\pi^{k+1}}\}$ and d_k,β,d_k^* are given in (II.4), (II.20) and (IV.10), respectively.

Proof: Taking $\underline{\varphi}=\underline{\varphi}_h$ in (II.18), and subtracting (IV.11) from (II.18), we get

$$
\begin{cases} a(\underline{\psi}-\underline{\psi}_h,\underline{\varphi}_h)+b(\underline{\varphi}_h,u-u_h)=0, & \forall\underline{\varphi}_h\in H_h,\\ b(\underline{\psi}-\underline{\psi}_h,v_h)=0, & \forall v_h\in M_h.\end{cases} \tag{V.4}
$$

The second equation of (V.4) implies that

$$
\mathrm{div}\underline{\psi}_h=P_h^k(\mathrm{div}\underline{\psi})=\mathrm{div}(RT_h^k\underline{\psi}), \tag{V.5}
$$

which together with Lemma 5.1, gives

$$
\begin{aligned}
\|\mathrm{div}\underline{\psi}-\mathrm{div}\underline{\psi}_h\|_{0,\Omega}&=\|\mathrm{div}\underline{\psi}-P_h^k\mathrm{div}\underline{\psi}\|_{0,\Omega}\\
&\le(\sum_{T\in\mathcal{T}_h}\|\mathrm{div}\underline{\psi}-P_T^k\mathrm{div}\underline{\psi}\|_{0,T}^2)^{\frac12}\\
&\le(\frac{h}{\pi})^{k+1}|\mathrm{div}\underline{\psi}|_{k+1,\Omega}.
\end{aligned} \tag{V.6}
$$

Combining (V.5) and the first equation of (V.4), we have

$$
\begin{aligned}
&\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}^2\\
&=a(\underline{\psi}-\underline{\psi}_h,\underline{\psi}-RT_h^k\underline{\psi})+a(\underline{\psi}-\underline{\psi}_h,RT_h^k\underline{\psi}-\underline{\psi}_h)\\
&=a(\underline{\psi}-\underline{\psi}_h,\underline{\psi}-RT_h^k\underline{\psi})+(\mathrm{div}(RT_h^k\underline{\psi}-\underline{\psi}_h),u-u_h)\\
&=a(\underline{\psi}-\underline{\psi}_h,\underline{\psi}-RT_h^k\underline{\psi}).
\end{aligned}
$$

According to Theorem 3.7, there holds

$$
\begin{aligned}
\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}&\le\|\underline{\psi}-RT_h^k\underline{\psi}\|_{0,\Omega}\\
&\le\frac{4\cdot2^{k/2}d_k}{\sin\theta}h^{k+1}(2|\underline{\psi}|_{k+1,\Omega}^2\\
&\quad+\frac{3+2\sqrt{2}}{4}\frac{k+1}{k+2}b_k^2|\mathrm{div}\underline{\psi}|_{k,\Omega}^2)^{\frac12},
\end{aligned} \tag{V.7}
$$

which implies (V.2) with $c_k=\frac{4\cdot2^{k/2}d_k}{\sin\theta}\max\{\sqrt{2},\frac{\sqrt{2}+1}{2}b_k\sqrt{\frac{k+1}{k+2}}\}$.

Combining Lemma 4.3 and the first equation of (V.4), it implies

$$
\begin{aligned}
\|P_h^ku-u_h\|_{0,\Omega}&\le\frac{d_k^*}{\beta}\sup_{\underline{\varphi}_h\in H_h}\frac{b(\underline{\varphi}_h,P_h^ku-u_h)}{\|\underline{\varphi}_h\|_{H(\mathrm{div})}}\\
&=\frac{d_k^*}{\beta}\sup_{\underline{\varphi}_h\in H_h}\frac{b(\underline{\varphi}_h,u-u_h)}{\|\underline{\varphi}_h\|_{H(\mathrm{div})}}\\
&=\frac{d_k^*}{\beta}\sup_{\underline{\varphi}_h\in H_h}\frac{a(\underline{\psi}-\underline{\psi}_h,\underline{\varphi}_h)}{\|\underline{\varphi}_h\|_{H(\mathrm{div})}}\\
&\le\frac{d_k^*}{\beta}\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}.
\end{aligned} \tag{V.8}
$$

TABLE I
NUMERICAL RESULTS FOR THE LOWEST ORDER MIXED FINITE ELEMENT METHOD

n^2	8×8	16×16	32×32
$\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}$	0.5507207331	0.2763366501	0.1382911861
C	0.2784718819	0.2794591972	0.2797076960
$\|f\|_{0,\Omega}$	5.5936471902	5.5936471902	5.5936471902
h	0.3535533906	0.1767766953	0.0883883476

Similar to (V.6), Lemma 5.1 implies that

$$\|u-P_h^k u\|_{0,\Omega} \le (\frac{h}{\pi})^{k+1}|u|_{k+1,\Omega}. \tag{V.9}$$

According to (V.7–V.9), there holds

$$\begin{aligned}\|u-u_h\|_{0,\Omega} &\le \|u-P_h^k u\|_{0,\Omega}+\|P_h^k u-u_h\|_{0,\Omega}\\ &\le C_k h^{k+1}(|\underline{\psi}|_{k+1,\Omega}+|\mathrm{div}\underline{\psi}|_{k,\Omega}+|u|_{k+1,\Omega}),\end{aligned}$$

where $C_k=\max\{\frac{c_k d_k^*}{\beta},\frac{1}{\pi^{k+1}}\}$. ■

VI. NUMERICAL EXPERIMENT

In the section, the error bounds for the lowest order mixed finite element method by numerical computation will be tested. Consider the following Poisson problem

$$\begin{cases} -\Delta u=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where $\Omega=[-1,1]\times[-1,1]$ and $f(x,y)=4-2x^2-2y^2$. The exact solution of this problem is $u(x,y)=(1-x^2)(1-y^2)$.

As is done in [14], we divide Ω into n^2 equal squares, which are further divided into triangles by the diagonals parallel to $x+y=1$, except in the top right and bottom left squares which are divided by the diagonals parallel to $x-y=0$. Numerical calculation is carried out by employing the lowest order mixed finite element method in [14] and $(\underline{\psi}_h,u_h)$ is the numerical solution. According to Theorem 5.2 and (II.16), by simple calculations we have, for $k=0$,

$$\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega} \quad\le\quad \frac{16\sqrt{2\pi+4}}{\pi}h\|f\|_{0,\Omega}.$$

Numerical results from [14] are listed in Table 1. Herein, the error constant C is defined by

$$C=\frac{\|\underline{\psi}-\underline{\psi}_h\|_{0,\Omega}}{h\|f\|_{0,\Omega}}.$$

It is easy to see from the numerical results that the experimentally determined constant is lower than the theoretical estimate.

VII. CONCLUSIONS

Base on a careful exploration, explicit error estimates for the Raviart-Thomas mixed finite element are developed. We obtain the explicit constants and the discrete inf-sup constants. The explicit estimates are very helpful to provide a computable error boundary and a posterior error estimation. Another feature of our error estimates for Raviart-Thomas interpolation is that we do not need to assume any mesh condition on the triangulation.

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