Mechanism of Integration of Informatization and Industrialization Based on a Fuzzy Stochastic Model

Chaodong Yan and Jing Ma

Abstract—**This letter proposes and investigates a fuzzy stochastic model which depicts the mechanism of integration of informatization and industrialization (MIII for short). Sharp sufficient criteria for stagnation and prosperity of industrialization and informatization are provided. Some critical functions of imprecise parameters and stochastic perturbations on MIII are uncovered and numerically illustrated.**

Index Terms—**Industrialization, informatization, stochastic perturbations, imprecise parameter, stagnation, prosperity.**

I. Introduction

With the rapid development of information and communication technology, the integration of informatization and industrialization has been becoming more and more manifest ([1], [3], [4], [6]). Understanding the mechanism of integration of informatization and industrialization (MIII) is of great significance for making policy ([7], [9], [10], [14], [16]). As a result, in recent years, the researches on MIII have received much attention ([21]–[23], [25], [27]–[29], [31], [32]). Particularly, Wang and Du [22] used the following mathematical model to portray MIII:

$$\begin{cases} \dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} = N_1(t)\Big[\eta_1 - N_1(t) + \rho_1 N_2(t)\Big], \\ \dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} = N_2(t)\Big[\eta_2 - N_2(t) + \rho_2 N_1(t)\Big], \end{cases} \tag{1}$$

where $N_1(t)$ and $N_2(t)$ represent the diffusion rates of information technologies and industrial technologies, respectively. η_i is the growth rates of $N_i(t)$, ρ_i represents the influence rate of N_j to N_i, $i,j=1,2,\ i\neq j$. Due to the fact that the self-influence rates of N_i are larger than the influence rates between N_i and N_j, hence it is supposed that $0\le\rho_1,\rho_2\le 1$, $i,j=1,2,\ i\neq j$. The authors [22] analyzed the stability of model (1).

On the other hand, during the development of information and communication technology, the evolutions of information technologies and industrial technologies are inevitably influenced by environmental perturbations, consequently, Yan et al. [30] added white noise into model (1), and tested the following stochastic model:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\Big[\eta_1 - N_1(t) + \rho_1 N_2(t))\Big]\mathrm{d}t \\ \qquad + \psi_1 N_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\Big[\eta_2 - N_2(t) + \rho_2 N_1(t))\Big]\mathrm{d}t \\ \qquad + \psi_2 N_2(t)\mathrm{d}W_2(t). \end{cases} \tag{2}$$

where ψ_i^2 stands for the intensity of the environmental perturbations, $\{W_1(t)\}_{t\ge0}$ and $\{W_2(t)\}_{t\ge0}$ are two standard Brownian motions. For model (2), the authors [30] explored the existence of a unique stationary distribution.

In model (2), the authors hypothesized that all the parameters are precisely known. However, as a matter of fact, in the real world all parameter values could not be precisely known owing to the deficiency of real data and errors in the measurement process ([17], [19]). Several scholars ([17], [19]) pointed out that fuzzy models could fit reality better. Thus it is useful to test model (2) with imprecise parameters and to examine the influences of imprecise parameters on the properties of the model. Nevertheless, as far as we are concerned, few researches of this aspect have been carried out.

Motivated by these, in this paper, we introduce imprecise parameters into model (2) and pay attention to the following fuzzy stochastic model

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\Big[\hat\eta_1 - N_1(t) + \hat\rho_1 N_2(t))\Big]\mathrm{d}t \\ \qquad + \hat\psi_1 N_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\Big[\hat\eta_2 - N_2(t) + \hat\rho_2 N_1(t))\Big]\mathrm{d}t \\ \qquad + \hat\psi_2 N_2(t)\mathrm{d}W_2(t), \end{cases} \tag{3}$$

where $\hat g$ represents the interval counterpart of g, namely,

$$\hat g = [g_l, g_u] = \{y\in\mathbb{R} | g_l \le y \le g_u\}.$$

For arbitrary $y\in[g_l,g_u]$, there exists a $q\in[0,1]$ such that $y = g_l^{1-q} g_u^{q}$. As a result, in this letter, we shall test the

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following model:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\Big[\eta_{1l}^{1-q}\eta_{1u}^{q} - N_1(t) + \rho_{1l}^{1-q}\rho_{1u}^{q}N_2(t))\Big]\mathrm{d}t \\ \qquad + \psi_{1l}^{1-q}\psi_{1u}^{q}N_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\Big[\eta_{2l}^{1-q}\eta_{2u}^{q} - N_2(t) + \rho_{2l}^{1-q}\rho_{2u}^{q}N_1(t))\Big]\mathrm{d}t \\ \qquad + \psi_{2l}^{1-q}\psi_{2u}^{q}N_2(t)\mathrm{d}W_2(t). \end{cases} \tag{4}$$

II. The existence and uniqueness of the solution

Define

$$\mathbb{R}_+^2 = \{a = (a_1, a_2) \in \mathbb{R}^2 |\ a_i > 0,\ i = 1, 2\},$$
$$\alpha_i(q) = \psi_{il}^{2(1-q)}\psi_{iu}^{2q}/2,\ i = 1, 2;$$
$$\beta_i(q) = \eta_{il}^{1-q}\eta_{iu}^{q} - \psi_{il}^{2(1-q)}\psi_{iu}^{2q}/2,$$
$$\gamma(q) = 1 - \rho_{1l}^{1-q}\rho_{1u}^{q}\rho_{2l}^{1-q}\rho_{2u}^{q},$$
$$\gamma_1(q) = \beta_1(q) + \beta_2(q)\rho_{1l}^{1-q}\rho_{1u}^{q},$$
$$\gamma_2(q) = \beta_2(q) + \beta_1(q)\rho_{2l}^{1-q}\rho_{2u}^{q},$$
$$\Psi(q) = \alpha_1\eta_{2l}^{1-q}\eta_{2u}^{q} - \alpha_2\eta_{1l}^{1-q}\eta_{1u}^{q},$$
$$\theta_1(q) = \eta_{1l}^{1-q}\eta_{1u}^{q} + \rho_{1l}^{1-q}\rho_{1u}^{q}\eta_{2l}^{1-q}\eta_{2u}^{q},$$
$$\theta_2(q) = \eta_{2l}^{1-q}\eta_{2u}^{q} + \rho_{2l}^{1-q}\rho_{2u}^{q}\eta_{1l}^{1-q}\eta_{1u}^{q};$$
$$\sigma_1(q) = \alpha_1(q) + \rho_{1l}^{1-q}\rho_{1u}^{q}\alpha_2(q),$$
$$\sigma_2(q) = \alpha_2(q) + \rho_{2l}^{1-q}\rho_{2u}^{q}\alpha_1(q),$$
$$\pi_1(q) = \eta_{1l}^{1-q}\eta_{1u}^{q}/\alpha_1(q),\ \pi_2(q) = \theta_2(q)/\sigma_2(q);$$
$$\bar{h}(t) = t^{-1}\int_0^t h(s)ds,$$
$$h^* = \limsup_{t\to+\infty} h(t),\ h_* = \liminf_{t\to+\infty} h(t).$$

One can see that

$$\beta_i(q) = \eta_{il}^{1-q}\eta_{iu}^{q} - \alpha_i(q), \gamma_i(q) = \theta_i(q) - \sigma_i(q), i = 1, 2.$$

In this letter, we always hypothesize that

$$\eta_{1l}^{1-q}\eta_{1u}^{q}/\alpha_1(q) > \eta_{2l}^{1-q}\eta_{2u}^{q}/\alpha_2(q).$$

Consequently, $\pi_1(q) \geq \pi_2(q)$.

Since both $N_1(t)$ and $N_2(t)$ represent the growth rates, hence we should give some conditions under which $N_1(t) > 0$ and $N_2(t) > 0$ to be realistic.

Lemma 1. *For any $N(0) \in \mathbb{R}_+^2$, model (4) possesses a unique global positive solution $N(t) = (N_1(t), N_2(t))^{\mathrm{T}}$ almost surely (a.s.). Additional, for every $p > 1$, there exists a positive constant $K = K(p)$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}(N_i^p(t)) \leq K. \tag{5}$$

Proof. We can deduce from

$$\gamma(q) = 1 - \rho_{1l}^{1-q}\rho_{1u}^{q}\rho_{2l}^{1-q}\rho_{2u}^{q} > 0$$

that there exist two positive constants $c_1(q)$ and $c_2(q)$ such that

$$-2a(q) := \lambda_{max}(C(q)\rho(q) + \rho^T(q)C(q)) < 0, \tag{6}$$

where

$$\rho(q) = \begin{pmatrix} -1 & \rho_{1l}^{1-q}\rho_{1u}^{q} \\ \rho_{2l}^{1-q}\rho_{2u}^{q} & -1 \end{pmatrix},$$
$$C = \begin{pmatrix} c_1(q) & 0 \\ 0 & c_2(q) \end{pmatrix},$$

and $\lambda_{max}(C(q)\rho(q) + \rho^T(q)C(q))$ stands for the largest eigenvalue of $C(q)\rho(q)+\rho^T(q)C(q)$. Actually, one can find out two positive constants $c_1(q)$ and $c_2(q)$ such that

$$\rho_{1l}^{1-q}\rho_{1u}^{q}c_1(q) = \rho_{2l}^{1-q}\rho_{2u}^{q}c_2(q).$$

As a result,

$$|C(q)\rho(q) + \rho^T(q)C(q)| = 4c_1(q)c_2(q)\gamma(q) > 0.$$

It follows that

$$\lambda_{max}(C(q)\rho(q) + \rho^T(q)C(q)) < 0.$$

Define

$$U(N) = c_1(q)N_1 + c_2(q)N_2,\ N \in \mathbb{R}_+^2.$$

Then Itô's formula ([15]) means that

$$\begin{aligned} &\mathrm{d}U(N) \\ &= c_1(q)N_1\Big[\eta_{1l}^{1-q}\eta_{1u}^{q} - N_1 - \rho_{1l}^{1-q}\rho_{1u}^{q}N_2\Big]\mathrm{d}t \\ &\quad + c_2(q)N_2\Big[\eta_{2l}^{1-q}\eta_{2u}^{q} - N_2 - \rho_{2l}^{1-q}\rho_{2u}^{q}N_1\Big]\mathrm{d}t \\ &\quad + c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1\mathrm{d}W_1(t) \\ &\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2\mathrm{d}W_2(t) \\ &= \Big[c_1(q)\eta_{1l}^{1-q}\eta_{1u}^{q}N_1\eta_{1l}^{1-q}\eta_{1u}^{q} \\ &\quad + c_2(q)\eta_{2l}^{1-q}\eta_{2u}^{q}N_2 + N^{\mathrm{T}}C\rho(q)N\Big]\mathrm{d}t \\ &\quad + c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1\mathrm{d}W_1(t) \\ &\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2\mathrm{d}W_2(t) \\ &= \Big[c_1(q)\eta_{1l}^{1-q}\eta_{1u}^{q}N_1\eta_{1l}^{1-q}\eta_{1u}^{q} + c_2(q)\eta_{2l}^{1-q}\eta_{2u}^{q}N_2 \\ &\quad + \frac{1}{2}N^T(C(q)\rho(q) + \rho^{\mathrm{T}}(q)C(q))N\Big]\mathrm{d}t \\ &\quad + c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1\mathrm{d}W_1(t) \\ &\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2\mathrm{d}W_2(t) \\ &\leq \Big[c_1(q)\eta_{1l}^{1-q}\eta_{1u}^{q}N_1\eta_{1l}^{1-q}\eta_{1u}^{q} + c_2(q)\eta_{2l}^{1-q}\eta_{2u}^{q}N_2 \\ &\quad - a(q)(N_1^2 + N_2^2)\Big]\mathrm{d}t \\ &\quad + c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1\mathrm{d}W_1(t) \\ &\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2\mathrm{d}W_2(t) \\ &\leq k_1 dt + c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1\mathrm{d}W_1(t) \\ &\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2\mathrm{d}W_2(t) \end{aligned}$$

The following proof is standard and hence is left out ([12]). □

III. Stagnation and prosperity

Lemma 2. *The solution of model (1) obeys*

$$\limsup_{t\to+\infty} \frac{\ln N_i(t)}{\ln t} \leq 1, \quad a.s.,\ i = 1, 2. \tag{7}$$

Proof. We can deduce from Itô's formula that

$$\begin{aligned}
&\mathrm{d}[e^tV(N(t))] = e^tV(N(t))\mathrm{d}t + e^t\mathrm{d}U(N(t))\\
&\leq e^t\Big[c_1(q)N_1(t) + c_2(q)N_2(t) + \eta_{1l}^{1-q}\eta_{1u}^{q}c_1N_1(t)\\
&\quad + \eta_{2l}^{1-q}\eta_{2u}^{q}c_2N_2(t) - a(q)(N_1^2(t) + N_2^2(t))\Big]\mathrm{d}t\\
&\quad + e^t\Big[c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1(t)\mathrm{d}W_1(t)\\
&\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2(t)\mathrm{d}W_2(t)\Big]\\
&\leq k_1e^t\mathrm{d}t + e^t\Big[c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}N_1(t)\mathrm{d}W_1(t)\\
&\quad + c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}N_2(t)\mathrm{d}W_2(t)\Big].
\end{aligned}$$

Consequently

$$\limsup_{t\to+\infty}\mathbb{E}U(N(t)) \leq k_1. \tag{8}$$

It then follows from

$$|N| \leq N_1 + N_2 \leq U(N)/\min\{c_1(q), c_2(q)\}$$

that

$$\limsup_{t\to+\infty}\mathbb{E}|N(t)| \leq k_1/\min\{c_1(q), c_2(q)\} =: k_2. \tag{9}$$

An application of Itô's formula again,

$$\begin{aligned}
&\mathbb{E}U(N(t+1)) \leq \mathbb{E}U(N(t))\\
&+\mathbb{E}\int_t^{t+1}\Big[\eta_{1l}^{1-q}\eta_{1u}^{q}c_1(q)N_1(s)\\
&+\eta_{2l}^{1-q}\eta_{2u}^{q}c_2(q)N_2(s) - a(q)(N_1^2(s) + N_2^2(s))\Big]\mathrm{d}s\\
&\leq \mathbb{E}U(N(t)) + \varpi\mathbb{E}\int_t^{t+1}|N(s)|\mathrm{d}s\\
&\quad - a(q)\mathbb{E}\int_t^{t+1}|N(s)|^2\mathrm{d}s,
\end{aligned}$$

where

$$\varpi = \sqrt{2}\max\{\eta_{1l}^{1-q}\eta_{1u}^{q}c_1(q), \eta_{2l}^{1-q}\eta_{2u}^{q}c_2(q)\}.$$

Notice that

$$\mathbb{E}U(N(t+1)) \geq 0,$$

then

$$\limsup_{t\to+\infty}\mathbb{E}\int_t^{t+1}|N(s)|^2\mathrm{d}s \leq (k_1 + \varpi k_2)/a(q) =: k_3. \tag{10}$$

We then deduce from Itô's formula that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\leq u\leq t+1}U(N(u))\Big)\\
&\leq \mathbb{E}U(N(t)) + \varpi\mathbb{E}\int_t^{t+1}|N(s)|\mathrm{d}s\\
&+c_1(q)\psi_{1l}^{1-q}\psi_{1u}^{q}\mathbb{E}\Big(\sup_{t\leq u\leq t+1}\Big|\int_t^u f_1(s)\mathrm{d}W_1(s)\Big|\Big)\\
&+c_2(q)\psi_{2l}^{1-q}\psi_{2u}^{q}\mathbb{E}\Big(\sup_{t\leq u\leq t+1}\Big|\int_t^u f_2(s)\mathrm{d}W_2(s)\Big|\Big).
\end{aligned} \tag{11}$$

Define

$$\Gamma_1(t) = \int_t^u f_1(s)\mathrm{d}W_1(s),$$

$$\Gamma_2(t) = \int_t^u f_2(s)\mathrm{d}W_2(s).$$

By Burkholder-Davis-Gundy's inequality and the Hölder's inequality, we derive

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\leq u\leq t+1}|\Gamma_1(u)|\Big) \leq k_4\mathbb{E}\Big(\int_t^{t+1}N_1^2(s)\mathrm{d}s\Big)^{0.5}\\
&\leq k_4\Big(\mathbb{E}\int_t^{t+1}N_1^2(s)\mathrm{d}s\Big)^{0.5}\\
&\leq k_4\Big(\mathbb{E}\int_t^{t+1}|N(s)|^2\mathrm{d}s\Big)^{0.5}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\Big(\sup\nolimits_{t\leq u\leq t+1}|\Gamma_2(u)|\Big) \leq k_4\Big(\mathbb{E}\textstyle\int_t^{t+1}N_2^2(s)\mathrm{d}s\Big)^{0.5}\\
&\leq k_4\Big(\mathbb{E}\int_t^{t+1}|N(s)|^2\mathrm{d}s\Big)^{0.5},
\end{aligned}$$

where $k_4 > 0$ is a constant. Substituting the above two inequalities into (11), and then taking advantage of (8), (9) and (10), one gets

$$\begin{aligned}
&\limsup_{t\to+\infty}\mathbb{E}\Big(\sup_{t\leq u\leq t+1}U(N(u))\Big) \leq k_1\\
&+\varpi k_2 + [c_1\psi_{1l}^{1-q}\psi_{1u}^{q}k_4 + c_2\psi_{2l}^{1-q}\psi_{2u}^{q}k_4]k_3^{0.5}.
\end{aligned}$$

It follows that

$$\mathbb{E}\Big(\sup_{n\leq u\leq n+1}|N(u)|\Big) \leq k_5, \quad n = 1, 2, ...,$$

where $k_5 > 0$ is a constant. For any $\varepsilon > 0$, Chebyshev's inequality implies that

$$P\Big\{\sup_{n\leq t\leq n+1}|N(t)| > k^{1+\varepsilon}\Big\} \leq \frac{k_5}{k^{1+\varepsilon}},\ n = 1, 2, ...$$

Then Borel-Cantelli's lemma means that there is a n_0 such that for almost all $\omega \in \Omega$, if $n \geq n_0$ and $n \leq t \leq n+1$,

$$\sup_{n\leq t\leq n+1}|N(t)| \leq n^{1+\varepsilon}.$$

That is to say,

$$\frac{\ln|N(t)|}{\ln t} \leq \frac{(1+\varepsilon)\ln n}{\ln n} = 1 + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired assertion. □

Lemma 3. *([13]) Let $\Phi(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$.*
(I) If there are two positive constants T and ζ_0 such that

$$\ln\Phi(t) \leq \zeta t - \zeta_0\int_0^t\Phi(s)\mathrm{d}s + \sum_{i=1}^{2}\tau_iW_i(t), \quad a.s. \tag{12}$$

for all $t \geq T$, where τ_i, ζ and ζ_0 are constants, then

$$\begin{cases}
\limsup\limits_{t\to+\infty}t^{-1}\int_0^t\Phi(s)\mathrm{d}s \leq \zeta/\zeta_0 \quad a.s., \quad \text{if} \quad \zeta \geq 0;\\
\lim\limits_{t\to+\infty}\Phi(t) = 0 \quad a.s., \quad \text{if} \quad \zeta < 0,
\end{cases}$$

(II) If there are three positive constants T, ζ and ζ_0 such that

$$\ln\Phi(t) \geq \zeta t - \zeta_0\int_0^t\Phi(s)\mathrm{d}s + \sum_{i=1}^{2}\tau_iW_i(t), \quad a.s.$$

for all $t \geq T$, then

$$\liminf_{t\to+\infty}t^{-1}\int_0^t\Phi(s)\mathrm{d}s \geq \zeta/\zeta_0, \quad a.s..$$

Theorem 1. *For model (4),*

(i) *If* $\pi_2(q) > 1$, *then*
$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s = \frac{\gamma_1(q)}{\gamma(q)},$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\mathrm{d}s = \frac{\gamma_2(q)}{\gamma(q)}.$$
That is to say, both informatization and industrialization are prosperous.

(ii) *If* $\pi_2(q) < 1 < \pi_1(q)$, *then*
$$\lim_{t\to+\infty} N_2(t) = 0,$$
and
$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s = \beta_1(q),\ a.s..$$
That is to say, informatization is prosperous, however, industrialization is stagnated.

(iii) *If* $\pi_1(q) < 1$, *then* $\lim_{t\to+\infty} N_i(t) = 0$ *a.s.,* $i = 1, 2$. *That is to say, both informatization and industrialization are stagnated.*

Proof. We deduce from Itô's formula that
$$\begin{aligned}\ln N_1(t) - \ln N_1(0) &= \beta_1(q)t - \rho_{1l}^{1-q}\rho_{1u}^{q}\int_0^t N_2(s)\mathrm{d}s \\ &\quad - \int_0^t N_1(s)\mathrm{d}s + \psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t),\end{aligned} \tag{13}$$
$$\begin{aligned}\ln N_2(t) - \ln N_2(0) &= \beta_2(q)t - \rho_{2l}^{1-q}\rho_{2u}^{q}\int_0^t N_1(s)\mathrm{d}s \\ &\quad - \int_0^t N_2(s)\mathrm{d}s + \psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t).\end{aligned} \tag{14}$$

Clearly,
$$\pi_1(q) = \eta_{1l}^{1-q}\eta_{1u}^{q}/\alpha_1 \geq \pi_2(q) = \theta_2(q)/\sigma_2(q).$$
One can deduce from (13)–(14)$\times\rho_{1l}^{1-q}\rho_{1u}^{q}$ that
$$\begin{aligned}&t^{-1}\ln\Big(N_1(t)/N_1\Big) - \rho_{1l}^{1-q}\rho_{1u}^{q}t^{-1}\ln\Big(N_2(t)/N_2\Big) \\ &= \theta_1(q) - \sigma_1(q) - \gamma(q)\overline{N_1(t)} \\ &\quad + t^{-1}\Big[\psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t) - \rho_{1l}^{1-q}\rho_{1u}^{q}\psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t)\Big],\end{aligned} \tag{15}$$
Similarly, one can deduce from (14)–(13)$\times\rho_{2l}^{1-q}\rho_{2u}^{q}$ that
$$\begin{aligned}&t^{-1}\ln\Big(N_2(t)/N_2\Big) - t^{-1}\rho_{2l}^{1-q}\rho_{2u}^{q}\ln\Big(N_1(t)/N_1\Big) \\ &= \theta_2(q) - \sigma_2(q) - \gamma(q)\overline{N_2(t)} \\ &+t^{-1}\Big[\psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t) - \rho_{2l}^{1-q}\rho_{2u}^{q}\psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t)\Big],\end{aligned} \tag{16}$$
In view of (13) and (14), for sufficiently large t,
$$\begin{aligned}&\frac{\ln(N_1(t)/N_1(0))}{t} \\ &\leq \beta_1(q) + \varepsilon - \overline{N_1(t)} - \rho_{1l}^{1-q}\rho_{1u}^{q}\overline{N_2}^{*} \\ &+\psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t)/t;\\ &\frac{\ln(N_2(t)/N_2(0))}{t} \\ &\leq \beta_2(q) + \varepsilon - \rho_{2l}^{1-q}\rho_{2u}^{q}\overline{N_1}^{*} - \overline{N_2(t)} \\ &+\psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t)/t.\end{aligned}$$

Define
$$\phi_1(q) = \beta_1(q) + \varepsilon - \rho_{1l}^{1-q}\rho_{1u}^{q}\overline{N_2}^{*};$$
$$\phi_2(q) = \beta_2(q) + \varepsilon - \rho_{2l}^{1-q}\rho_{2u}^{q}\overline{N_1}^{*}.$$
As a result,
$$\frac{\ln(N_1(t)/N_1(0))}{t} \leq \phi_1(q) - \overline{N_1(t)} + \psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t)/t; \tag{17}$$
$$\frac{\ln(N_2(t)/N_2(0))}{t} \leq \phi_2(q) - \overline{N_2(t)} + \psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t)/t. \tag{18}$$

(i) By (7), for arbitrarily $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \geq T$
$$\begin{aligned}&-\rho_{1l}^{1-q}\rho_{1u}^{q}t^{-1}\ln\Big(N_2(t)/N_2(0)\Big) \\ &\leq -\rho_{1l}^{1-q}\rho_{1u}^{q}\Big[t^{-1}\ln N_2\Big]^{*} + \varepsilon \leq \varepsilon.\end{aligned}$$
Substituting this inequality into (15) results in
$$\begin{aligned}&t^{-1}\ln\Big(N_1(t)/N_1\Big) \geq \theta_1(q) - \sigma_1(q) - \varepsilon - \gamma(q)\overline{N_1(t)} \\ &+t^{-1}\Big[\psi_{1l}^{1-q}\psi_{1u}^{q}W_1(t) - \rho_{1l}^{1-q}\rho_{1u}^{q}\psi_{2l}^{1-q}\psi_{2u}^{q}W_2(t)\Big].\end{aligned} \tag{19}$$
Notice that
$$\theta_1(q)/\sigma_1(q) \geq \theta_2(q)/\sigma_2(q) > 1,$$
then there is a sufficiently small ε such that
$$\theta_1(q) - \sigma_1(q) - \varepsilon > 0.$$
According to (II) in Lemma 3 and the arbitrariness of ε, we obtain
$$\overline{N_1}_{*} \geq \frac{\theta_1(q) - \sigma_1(q)}{\gamma(q)} = \frac{\gamma_1(q)}{\gamma(q)},\quad a.s.. \tag{20}$$
Therefore, $\phi_1(q) > 0$. In the same way, by (16),
$$\overline{N_2}_{*} \geq \frac{\gamma_2(q)}{\gamma(q)}. \tag{21}$$
Thus $\phi_2(q) > 0$. By (I) in Lemma 3, we get
$$\overline{N_1}^{*} \leq \phi_1(q),\quad \overline{N_2}^{*} \leq \phi_2(q).$$
That is to say,
$$\overline{N_1}^{*} + \rho_{1l}^{1-q}\rho_{1u}^{q}\overline{N_2}^{*} \leq \beta_1(q) + \varepsilon, \tag{22}$$
$$\rho_{2l}^{1-q}\rho_{2u}^{q}\overline{N_1}^{*} + \overline{N_2}^{*} \leq \beta_2(q) + \varepsilon,\ a.s.. \tag{23}$$
Consequently
$$\overline{N_1}^{*} \leq \gamma_1(q)/\gamma(q),$$
$$\overline{N_2}^{*} \leq \gamma_2(q)/\gamma(q),\ a.s..$$
Then one derives the required assertion.

(ii) Notice that
$$\theta_1(q)/\sigma_1(q) > 1,$$
hence (20) holds. Thus
$$\overline{N_1}_{*} > \gamma_1(q)/\gamma(q).$$
Hence $\phi_1(q) > 0$, and (22) holds. If $\omega \in \{\overline{N_2(\omega)}^{*} > 0\}$, by Lemma 3, one has
$$\overline{N_2(\omega)}^{*} \leq \phi_2 = \beta_2(q) + \varepsilon - \rho_{2l}^{1-q}\rho_{2u}^{q}\overline{N_1(\omega)}^{*}.$$

Substituting this inequality into (22) yields

$$0<\gamma\overline{N_2(\omega)}^*\le\beta_2(q)-\rho_{2l}^{1-q}\rho_{2u}^q\beta_1(q)+\varepsilon$$
$$=\theta_2(q)-\sigma_2(q)+\varepsilon.$$

By the arbitrariness of ε leads to

$$\theta_2(q)/\sigma_2(q)\ge 1.$$

This is a contradiction. Hence,

$$\mathcal{P}\{\omega:\overline{N_2}^*>0\}=0,$$

in other words,

$$\overline{N_2}^*=0,\ a.s..$$

When (22) is used in (18), we get

$$\begin{aligned}&\frac{\ln(N_2(t)/N_2(0))}{t}\\&\le\beta_2(q)+\varepsilon-\rho_{2l}^{1-q}\rho_{2u}^q\Big(\beta_1+\varepsilon-\rho_{1l}^{1-q}\rho_{1u}^q\overline{N_2}^*\Big)\\&-\overline{N_2(t)}+\psi_{2l}^{1-q}\psi_{2u}^qW_2(t)/t\\&=\theta_2(q)-\sigma_2(q)+\varepsilon(t)+\varepsilon-\rho_{2l}^{1-q}\rho_{2u}^q\varepsilon\\&+\psi_{2l}^{1-q}\psi_{2u}^qW_2(t)/t,\end{aligned}$$

where

$$\varepsilon(t)=\rho_{1l}^{1-q}\rho_{1u}^q\rho_{2l}^{1-q}\rho_{2u}^q\overline{N_2}^*-\overline{N_2(t)}.$$

We the deduce from $1>\theta_2(q)/\sigma_2(q)$ that $\overline{N_2}^*=0$. Hence $\varepsilon(t)\to 0$. According to Lemma 3,

$$\lim_{t\to+\infty}N_2(t)=0,\quad a.s..$$

By (13), for sufficiently large t,

$$t^{-1}\ln\frac{N_1(t)}{N_1(0)}\le\beta_1(q)+\varepsilon-\overline{N_1(t)}+\psi_{1l}^{1-q}\psi_{1u}^qW_1(t)/t,\ (24)$$

$$t^{-1}\ln\frac{N_1(t)}{N_1(0)}\ge\beta_1(q)-\varepsilon-\overline{N_1(t)}+\psi_{1l}^{1-q}\psi_{1u}^qW_1(t)/t,\ (25)$$

where $\varepsilon\in(0,\beta_1)$. Using (I) and (II) in Lemma 3 to (24) and (25) respectively, we have

$$\beta_1(q)-\varepsilon\le\overline{N_1}_*\le\overline{N_1}^*\le\beta_1+\varepsilon,\quad a.s..$$

We then deduce from the arbitrariness of ε that

$$\lim_{t\to+\infty}\overline{N_1(t)}=\beta_1(q),\ a.s.$$

(iii) If $\overline{N_1}^*>0$, then $\phi_1(q)>0$. Similar to the proof of (ii), we get

$$\lim_{t\to+\infty}N_2(t)=0,\ a.s..$$

If $\overline{N_1}^*=0$, by (18), for sufficiently large t,

$$\frac{\ln(N_2(t)/N_2(0))}{t}\le\beta_2+\varepsilon-\overline{N_2(t)}+\psi_{2l}^{1-q}\psi_{2u}^qW_2(t)/t.$$

Notice that

$$1>\theta_2(q)/\sigma_2(q)>\eta_{2l}^{1-q}\eta_{2u}^q/\rho_{2l}^{1-q}\rho_{2u}^q.$$

In light of Lemma 3, we obtain

$$\lim_{t\to+\infty}N_2(t)=0,\ a.s..$$

Hence

$$\lim_{t\to+\infty}N_2(t)=0,\ a.s..$$

By (17), for sufficiently large t,

$$\frac{\ln(N_1(t)/N_1(0))}{t}\le\beta_1+\varepsilon-\overline{N_1(t)}+\psi_{1l}^{1-q}\psi_{1u}^qW_1(t)/t.$$

We then deduce from $\eta_{1l}^{1-q}\eta_{1u}^q/\rho_{1l}^{1-q}\rho_{1u}^q<1$ and Lemma 3 that the required assertion holds. □

Theorem 2. *If $\beta_1(q)>0$ and $\beta_2(q)>0$, then system (4) is stochastically prosperous, that is to say, for any $\epsilon>0$, there are two positive constants β and χ such that*

$$\liminf_{t\to+\infty}\mathbf{P}\Big\{N_i(t)\ge\beta\Big\}\ge 1-\epsilon,\ i=1,2,$$

$$\liminf_{t\to+\infty}\mathbf{P}\Big\{N_i(t)\le\chi\Big\}\ge 1-\epsilon,\ i=1,2.$$

Proof. First of all, fix a positive constant θ such that

$$\beta_1(q)>0.5\theta\alpha_i(q),\ i=1,2.$$

Define

$$V_1(N)=\Big(1+N_1^{-1}\Big)^\theta+\Big(1+N_2^{-1}\Big)^\theta.$$

We then deduce from Itô's formula that

$$\begin{aligned}&\mathrm{d}V_1(N)\\&=\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\mathrm{d}(N_1^{-1})\\&\quad+0.5\theta(\theta-1)\Big(1+N_1^{-1}\Big)^{\theta-2}(\mathrm{d}(N_1^{-1}))^2\\&\quad+\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\mathrm{d}(N_2^{-1})\\&\quad+0.5\theta(\theta-1)\Big(1+N_2^{-1}\Big)^{\theta-2}(\mathrm{d}(N_2^{-1}))^2\\&=\theta(1+N_1^{-1})^{\theta-2}\Big\{(1+N_1^{-1})\Big[-N_1^{-1}\\&\quad\times\Big(\eta_{1l}^{1-q}\eta_{1u}^q-N_1+\rho_{1l}^{1-q}\rho_{1u}^qN_2\Big)+N_1^{-1}\alpha_1(q)\Big]\\&\quad+0.5(\theta-1)N_1^{-2}\alpha_1(q)\Big\}\mathrm{d}t\\&\quad+\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^qN_1^{-1}\mathrm{d}W_1(t)\\&\quad+\theta\Big(1+N_2^{-1}\Big)^{\theta-2}\Big\{(1+N_2^{-1})\Big[-N_2^{-1}\\&\quad\times\Big(\eta_{2l}^{1-q}\eta_{2u}^q+\rho_{2l}^{1-q}\rho_{2u}^qN_1-N_2\Big)+N_1^{-2}\alpha_2(q)\Big]\\&\quad+0.5(\theta-1)N_2^{-2}\alpha_2(q)\Big\}\mathrm{d}t\\&\quad+\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^qN_2^{-1}\mathrm{d}W_2(t).\end{aligned}$$

Accordingly,

$$\begin{aligned}
\mathrm{d}V_1(N) &= \theta\Big(1+N_1^{-1}\Big)^{\theta-2}\Big\{-\frac{1}{N_1^2} \\
&\quad\times\Big(\beta_1(q)-0.5\theta\alpha_1(q)\Big) \\
&\quad+\frac{1}{N_1}\Big(-\eta_{1l}^{1-q}\eta_{1u}^{q}+\alpha_1(q)\Big) \\
&\quad-\frac{N_2}{N_1}\Big[\rho_{1l}^{1-q}\rho_{1u}^{q}+\frac{\rho_{1l}^{1-q}\rho_{1u}^{q}}{N_1}\Big]\Big\}\mathrm{d}t \\
&\quad+\theta(1+N_2^{-1})^{\theta-2}\Big\{\frac{1}{N_2^2}\Big(0.5\theta\alpha_2(q)-\beta_2(q)\Big) \\
&\quad+\frac{1}{N_2}\Big(-\eta_{2l}^{1-q}\eta_{2u}^{q}+\alpha_2(q)\Big) \\
&\quad-\frac{N_1}{N_2}\Big[\rho_{2l}^{1-q}\rho_{2u}^{q}+\frac{\rho_{2l}^{1-q}\rho_{2u}^{q}}{N_2}\Big]\Big\}\mathrm{d}t \\
&\quad+\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t) \\
&\le \theta(1+N_1^{-1})^{\theta-2}\Big\{\frac{1}{N_1^2}\Big(0.5\theta\alpha_1(q)-\beta_1(q)\Big) \\
&\quad+\frac{1}{N_1}\Big(r_{11}+\alpha_1(q)\Big)\Big\}\mathrm{d}t \\
&\quad+\theta(1+N_2^{-1})^{\theta-2}\Big\{\frac{1}{N_2^2}\Big(0.5\theta\alpha_2(q)-\beta_2(q)\Big) \\
&\quad+\frac{1}{N_2}\Big(r_{21}+\alpha_2(q)\Big)\Big\}\mathrm{d}t \\
&\quad+\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t).
\end{aligned}$$

Fix a sufficiently small κ such that

$$0<\frac{\kappa}{\theta}<b_i-0.5\theta\alpha_i^2,\ \ i=1,2.$$

Define

$$V_2(N)=e^{\kappa t}V_1(N)=e^{\kappa t}\sum_{i=1}^{2}\Big(1+N_i^{-1}\Big)^{\theta}.$$

We deduce from Itô's formula that

$$\begin{aligned}
&\mathrm{d}V_2(N(t)) \\
&=\kappa e^{\kappa t}V_1(N)\mathrm{d}t+e^{\kappa t}\mathrm{d}V_1(N) \\
&\le \theta e^{\kappa t}(1+N_1^{-1})^{\theta-2}\Big\{\kappa(1+N_1^{-1})^2/\theta \\
&\quad-\frac{1}{N_1^2}\Big(\beta_1(q)-0.5\theta\alpha_1(q)\Big)+\frac{1}{N_1}\alpha_1(q)\Big\}\mathrm{d}t \\
&+\theta e^{\kappa t}(1+N_2^{-1})^{\theta-2}\Big\{\kappa(1+N_2^{-1})^2/\theta \\
&\quad-\frac{1}{N_2^2}\Big(\beta_2(q)-0.5\theta\alpha_2(q)\Big)+\frac{1}{N_2}\alpha_2(q)\Big\}\mathrm{d}t \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t).
\end{aligned}$$

That is to say,

$$\begin{aligned}
&\mathrm{d}V_2(N(t)) \\
&=\theta e^{\kappa t}(1+N_1^{-1})^{\theta-2}\Big\{-\frac{1}{N_1^2} \\
&\quad\times\Big(\beta_1(q)-0.5\theta\alpha_1(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_1}\Big(\alpha_1(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\}\mathrm{d}t \\
&\quad+\theta e^{\kappa t}(1+N_2^{-1})^{\theta-2}\Big\{-\frac{1}{N_2^2} \\
&\quad\times\Big(\beta_2(q)-0.5\theta\alpha_2(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_2}\Big(\alpha_2(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\}\mathrm{d}t \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t).
\end{aligned}$$

As a result,

$$\begin{aligned}
&\mathrm{d}V_2(N(t)) \\
&\le\theta e^{\kappa t}(1+N_1^{-1})^{\theta-2}\Big\{-\frac{1}{N_1^2} \\
&\quad\times\Big(\beta_1(q)-\varepsilon-0.5\theta\alpha_1(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_1}\Big(\alpha_1(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\}\mathrm{d}t \\
&\quad+\theta e^{\kappa t}(1+N_2^{-1})^{\theta-2}\Big\{-\frac{1}{N_2^2} \\
&\quad\times\Big(\beta_2(q)-\varepsilon-0.5\theta\alpha_2(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_2}\Big(\alpha_2(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\}\mathrm{d}t \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t) \\
&=:e^{\kappa t}J(N)\mathrm{d}t \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}W_1(t) \\
&\quad+\kappa e^{\kappa t}\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t),
\end{aligned}$$

where

$$\begin{aligned}
J(N) &= \theta(1+N_1^{-1})^{\theta-2} \\
&\quad\times\Big\{-\frac{1}{N_1^2}\Big(\beta_1(q)-0.5\theta\alpha_1(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_1}\Big(\alpha_1(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\} \\
&\quad+\theta(1+N_2^{-1})^{\theta-2}\Big\{-\frac{1}{N_2^2} \\
&\quad\times\Big(\beta_2(q)-0.5\theta\alpha_2(q)-\kappa/\theta\Big) \\
&\quad+\frac{1}{N_2}\Big(\alpha_2(q)+2\kappa/\theta\Big)+\kappa/\theta\Big\}.
\end{aligned}$$

One can see that $J(N)$ is upper bounded in $\mathbb{R}_+^2$, that is to say,

$$K_1:=\sup_{x\in R_+^2}J(N)<+\infty.$$

Accordingly,

$$\begin{aligned}&\mathrm{d}V_2(N(t))\\ &\le K_1 e^{\kappa t}\mathrm{d}t\\ &-\kappa e^{\kappa t}\theta\Big(1+N_1^{-1}\Big)^{\theta-1}\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{-1}\mathrm{d}B_1(t)\\ &+\kappa e^{\kappa t}\theta\Big(1+N_2^{-1}\Big)^{\theta-1}\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{-1}\mathrm{d}W_2(t).\end{aligned}$$

Integrating both sides and then taking expectations, we derive

$$\begin{aligned}&\mathbb{E}[V_2(N(t))]=e^{\kappa t}\mathbb{E}\Big[(1+N_1^{-1}(t))^{\theta}+(1+N_2^{-1}(t))^{\theta}\Big]\\ &\le(1+N_1^{-1}(0))^{\theta}+(1+N_2^{-1}(0))^{\theta}+\frac{K_1}{\kappa}e^{\kappa t}.\end{aligned}$$

Set $K=\dfrac{K_1}{\kappa}$, hence

$$\limsup_{t\to+\infty}\mathbb{E}[N_1^{-\theta}(t)]\le\limsup_{t\to+\infty}\sum_{i=1}^{2}\mathbb{E}\Big(1+N_i^{-1}(t)\Big)^{\theta}\le K;$$

$$\limsup_{t\to+\infty}\mathbb{E}[N_2^{-\theta}(t)]\le\limsup_{t\to+\infty}\sum_{i=1}^{2}\mathbb{E}\Big(1+N_i^{-1}(t)\Big)^{\theta}\le K.$$

For any $\varepsilon>0$, set $\beta=\varepsilon^{\frac{1}{\theta}}/K^{\frac{1}{\theta}}$. We then deduce from Chebyshev's inequality that

$$\begin{aligned}&\mathbf{P}\Big\{N_i(t)<\beta\Big\}=\mathbf{P}\Big\{N_i^{-\theta}(t)>\beta^{-\theta}\Big\}\\ &\le\frac{\mathbb{E}[N_i^{-\theta}(t)]}{\beta^{-\theta}}=\beta^{\theta}\mathbb{E}[N_i^{-\theta}(t)],\ i=1,2.\end{aligned}$$

That is to say,

$$\limsup_{t\to+\infty}\mathbf{P}\Big\{N_i(t)<\beta\Big\}\le\beta^{\theta}K=\varepsilon.$$

Accordingly,

$$\liminf_{t\to+\infty}\mathbf{P}\Big\{N_i(t)\ge\beta\Big\}\ge1-\varepsilon,\ i=1,2.$$

The proof of

$$\liminf_{t\to+\infty}\mathbf{P}\Big\{N_i(t)\le\chi\Big\}\ge1-\varepsilon,\ i=1,2.$$

is standard and thus is left out. □

IV. Stationary distribution

Definition 1. *([11]) If there exists a unique probability measure $\varsigma(\cdot)$ such that for any $N(0)\in\mathbb{R}_+^2$, the transition probability $p(t,N(0),\cdot)$ of $N(t)$ weakly converges to $\varsigma(\cdot)$ as $t\to+\infty$, then model (4) is said to be asymptotically stable in distribution (ASD).*

Lemma 4. *Model (4) is ASD.*

Proof. Let $N(t;N(0))$ and $N(t;\tilde N(0))$ represent two solutions of (4) with $N(0)\in\mathbb{R}_+^2$ and $\tilde N(0)\in\mathbb{R}_+^2$ respectively. Define

$$U(t)=\sum_{i=1}^{2}\Big|\ln N_i(t;N(0))-\ln N_i(t;\tilde N(0))\Big|.$$

We then deduce from Itô's formula that

$$\begin{aligned}&\mathrm{d}U(t)\\ &=\mathrm{sgn}\Big(N_1(t;N(0))-N_1(t;\tilde N(0))\Big)\\ &\quad\times\Big[-\Big(N_1(t;N(0))-N_1(t;\tilde N(0))\Big)\\ &\quad+\rho_{1l}^{1-q}\rho_{1u}^{q}\Big(N_2(t;N(0))-N_2(t;\tilde N(0))\Big)\Big]\mathrm{d}t\\ &\quad+\mathrm{sgn}\Big(N_2(t;N(0))-N_2(t;\tilde N(0))\Big)\\ &\quad\times\Big[\rho_{2l}^{1-q}\rho_{2u}^{q}\Big(N_1(t;N(0))-N_1(t;\tilde N(0))\Big)\\ &\quad-\Big(N_2(t;N(0))-N_2(t;\tilde N(0))\Big)\Big]\mathrm{d}t\\ &\le-\Big(1-\rho_{2l}^{1-q}\rho_{2u}^{q}\Big)\Big|N_1(t;N(0))-N_1(t;\tilde N(0))\Big|\mathrm{d}t\\ &\quad-\Big(1-\rho_{1l}^{1-q}\rho_{1u}^{q}\Big)\Big|N_2(t;N(0))-N_2(t;\tilde N(0))\Big|\mathrm{d}t.\end{aligned}$$

Hence

$$\begin{aligned}&0\le\mathbb{E}(U(t))\le U(0)\\ &-\Big(1-\rho_{2l}^{1-q}\rho_{2u}^{q}\Big)\int_0^t\mathbb{E}\Big|N_1(s;N(0))-N_1(s;\tilde N(0))\Big|\mathrm{d}s\\ &-\Big(1-\rho_{1l}^{1-q}\rho_{1u}^{q}\Big)\int_0^t\mathbb{E}\Big|N_2(s;N(0))-N_2(s;\tilde N(0))\Big|\mathrm{d}s.\end{aligned}$$

By $U(0)<+\infty$,

$$\begin{aligned}&\Big(1-\rho_{2l}^{1-q}\rho_{2u}^{q}\Big)\int_0^t\mathbb{E}\Big|N_1(s;N(0))-N_1(s;\tilde N(0))\Big|\mathrm{d}s\\ &+\Big(1-\rho_{1l}^{1-q}\rho_{1u}^{q}\Big)\int_0^t\mathbb{E}\Big|N_2(s;N(0))-N_2(s;\tilde N(0))\Big|\mathrm{d}s\\ &\le U(0)<+\infty.\end{aligned}$$

As a result,

$$\mathbb{E}\Big|N_i(t;N(0))-N_i(t;\tilde N(0))\Big|\in L^1[0,\infty),\ i=1,2.$$

We then deduce from (4) that

$$\begin{aligned}\mathbb{E}(N_1(t))&=N_1(0)+\int_0^t\Big[\eta_{1l}^{1-q}\eta_{1u}^{q}\mathbb{E}(N_1(s))\\ &\quad-\mathbb{E}(N_1^2(s))+\rho_{1l}^{1-q}\rho_{1u}^{q}\mathbb{E}(N_1(s)N_2(s))\Big]\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(N_2(t))&=N_2(0)+\int_0^t\Big[\eta_{2l}^{1-q}\eta_{2u}^{q}\mathbb{E}(N_2(s))\\ &\quad-\mathbb{E}(N_2^2(s))+\rho_{2l}^{1-q}\rho_{2u}^{q}\mathbb{E}(N_1(s)N_2(s))\Big]\mathrm{d}s.\end{aligned}$$

Thereby, $\mathbb{E}(N_1(t))$ and $\mathbb{E}(N_2(t))$ are continuously differentiable. In light of (5),

$$\begin{aligned}&\frac{\mathrm{d}\mathbb{E}(N_1(t))}{\mathrm{d}t}=\eta_{1l}^{1-q}\eta_{1u}^{q}\mathbb{E}(N_1(t))\\ &-\mathbb{E}(N_1^2(t))+\rho_{1l}^{1-q}\rho_{1u}^{q}\mathbb{E}\Big(N_1(t)N_2(t)\Big)\\ &\le\eta_{1l}^{1-q}\eta_{1u}^{q}\mathbb{E}(N_1(t))+\frac{\rho_{1l}^{1-q}\rho_{1u}^{q}}{2}\mathbb{E}\Big(N_1^2(t)+N_2^2(t)\Big)\\ &\le K_1,\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}\mathbb{E}(N_2(t))}{\mathrm{d}t}=\eta_{2l}^{1-q}\eta_{2u}^{q}\mathbb{E}(N_2(t))\\&-\mathbb{E}(N_2^2(t))+\rho_{2l}^{1-q}\rho_{2u}^{q}\mathbb{E}\bigg(N_1(t)N_2(t)\bigg)\\&\le\eta_{2l}^{1-q}\eta_{2u}^{q}\mathbb{E}(N_2(t))+\frac{\rho_{2l}^{1-q}\rho_{2u}^{q}}{2}\mathbb{E}\bigg(N_1^2(t)+N_2^2(t)\bigg)\\&\le K_1,\end{aligned}$$

where $K_1>0$ is a constant. As a result, $\mathbb{E}(N_1(t))$ and $\mathbb{E}(N_2(t))$ are uniformly continuous. We then deduce from Barbalat's result [2] that

$$\lim_{t\to+\infty}\mathbb{E}\bigg|N_i(t;N(0))-N_i(t;\tilde{N}(0))\bigg|=0,\ i=1,2. \quad (26)$$

Let $P(t,N(0),\mathscr{A})$ represent the probability of $\{N(t;N(0))\in\mathscr{A}\}$ with $N(0)\in\mathbb{R}_+^2$. According to (5) and Chebyshev's inequality, the family of $\{p(t,N(0),\mathrm{d}x)\}$ is tight ([11]). Let $\mathscr{P}(\mathbb{R}_+^2)$ be all the probability measures on $\mathbb{R}_+^2$. For $P_1,\ P_2\in\mathscr{P}$, define

$$\begin{aligned}\mathrm{d}_\Xi(P_1,P_2)\ &=\sup_{\xi\in\Xi}\bigg|\int_{\mathbb{R}_+^2}\xi(N)P_1(\mathrm{d}N)\\&\quad-\int_{\mathbb{R}_+^2}\xi(N)P_2(\mathrm{d}N)\bigg|,\end{aligned}$$

where

$$\Xi=\bigg\{\xi:\mathbb{R}_+^2\to\mathbb{R}\bigg||\xi(x)-\xi(y)|\le||x-y||,|\xi(\cdot)|\le1\bigg\}.$$

For any $\xi\in\Xi$ and $t,s>0$, we get

$$\begin{aligned}&\bigg|\mathbb{E}\xi(N(t+s;N(0)))-\mathbb{E}\xi(N(t;N(0)))\bigg|\\&=\bigg|\mathbb{E}\bigg[\mathbb{E}\bigg(\xi\bigg(N(t+s;N(0))\bigg)\bigg|\mathcal{F}_s\bigg)\bigg]\\&\quad-\mathbb{E}\xi\bigg(N(t;N(0))\bigg)\bigg|\\&=\bigg|\int_{\mathbb{R}_+^2}\mathbb{E}\xi\bigg(N(t;\tilde{N}(0))\bigg)p\bigg(s,N(0),\mathrm{d}\tilde{N}(0)\bigg)\\&\quad-\mathbb{E}\xi\bigg(N(t;N(0))\bigg)\bigg|\\&\le\int_{\mathbb{R}_+^2}\bigg|\mathbb{E}\xi\bigg(N(t;\tilde{N}(0))\bigg)\\&\quad-\mathbb{E}\xi\bigg(N(t;N(0))\bigg)\bigg|p\bigg(s,N(0),\mathrm{d}\tilde{N}(0)\bigg).\end{aligned} \quad (27)$$

According to (26), there is a $T>0$ such that for $t\ge T$,

$$\begin{aligned}&\bigg|\mathbb{E}\xi(N(t;\tilde{N}(0)))-\mathbb{E}\xi(N(t;N(0)))\bigg|\\&\le\mathbb{E}\bigg|\xi(N(t;\tilde{N}(0)))-\xi(N(t;N(0)))\bigg|\\&\le\mathbb{E}\bigg|N(t;\tilde{N}(0))-N(t;N(0))\bigg|\le\varepsilon.\end{aligned} \quad (28)$$

When (28) is used in (27), one gets

$$\bigg|\mathbb{E}\xi(N(t+s;N(0)))-\mathbb{E}\xi(N(t;N(0)))\bigg|\le\varepsilon,\forall t\ge T,s>0.$$

We then deduce from the arbitrariness of ξ that $\forall\ t\ge T,\ s>0$,

$$\sup_{\xi\in\Xi}\bigg|\mathbb{E}\xi(N(t+s;N(0)))-\mathbb{E}\xi(N(t;N(0)))\bigg|\le\varepsilon.$$

Consequently,

$$\mathrm{d}_\Xi\bigg(p(t+s,N(0),\cdot),p(t,N(0),\cdot)\bigg)\le\varepsilon,\forall\ t\ge T,s>0.$$

For arbitrary $N(0)\in\mathbb{R}_+^2$, $\{p(t,N(0),\cdot):t\ge0\}$ is Cauchy in $\mathscr{P}$. As a result, $\{p(t,0.1,\cdot):t\ge0\}$ is Cauchy in $\mathscr{P}$. Thereby, there is a unique $\varsigma(\cdot)\in\mathscr{P}(\mathbb{R}_+^2))$ such that

$$\lim_{t\to+\infty}\mathrm{d}_\Xi\bigg(p(t,0.1,\cdot),\varsigma(\cdot)\bigg)=0.$$

In light of (26),

$$\lim_{t\to+\infty}\mathrm{d}_\Xi\bigg(p(t,N(0),\cdot),p(t,0.1,\cdot)\bigg)=0.$$

Therefore,

$$\begin{aligned}&\lim_{t\to+\infty}\mathrm{d}_\Xi\bigg(p(t,N(0),\cdot),\varsigma(\cdot)\bigg)\\&\le\lim_{t\to+\infty}\mathrm{d}_\Xi\bigg(p(t,N(0),\cdot),p(t,0.1,\cdot)\bigg)\\&+\lim_{t\to+\infty}\mathrm{d}_\Xi\bigg(p(t,0.1,\cdot),\varsigma(\cdot)\bigg)=0.\end{aligned}$$

□

Theorem 3. *For model (4),*

(i') *If $\pi_2(q)>1$, then both N_1 and N_2 are prosperous, and there exists a unique $\varsigma_1(\cdot)\in\mathscr{P}(\mathbb{R}_+^2)$ which is ergodic such that*

$$\lim_{t\to+\infty}t^{-1}\int_0^t N_1(s)\mathrm{d}s=\int_{\mathbb{R}_+^2}x_1\varsigma_1(\mathrm{d}x_1,\mathrm{d}x_2)=\frac{\gamma_1(q)}{\gamma(q)},$$

$$\lim_{t\to+\infty}t^{-1}\int_0^t N_2(s)\mathrm{d}s=\int_{\mathbb{R}_+^2}x_2\varsigma_1(\mathrm{d}x_1,\mathrm{d}x_2)=\frac{\gamma_2(q)}{\gamma(q)}.$$

(ii') *If $\pi_2(q)<1<\pi_1(q)$, then N_2 is stagnated: $\lim\limits_{t\to+\infty}N_2(t)=0$, a.s., and there is a unique $\varsigma_2(\cdot)\in\mathscr{P}(\mathbb{R}_+)$ which is ergodic such that*

$$\lim_{t\to+\infty}t^{-1}\int_0^t N_1(s)\mathrm{d}s=\int_{\mathbb{R}_+^2}x_1\varsigma_2(\mathrm{d}x_1)=\beta_1(q),a.s.$$

(iii') *If $\pi_1(q)<1$, then both N_1 and N_2 are stagnated a.s..*

Proof. (i'). By Lemma 4, model (4) possesses a unique invariant measure $\varsigma(\cdot)$. In light of Corollary 3.4.3 in [18], $\varsigma(\cdot)$ is strong mixing. By Theorem 3.2.6 in [18], $\varsigma(\cdot)$ is ergodic. We then deduce from (3.3.2) in [18] that

$$\lim_{t\to+\infty}t^{-1}\int_0^t N_i(s)\mathrm{d}s=\int_{\mathbb{R}_+^2}x_i\varsigma(\mathrm{d}x_1,\mathrm{d}x_2),\ i=1,2.$$

According to Theorem 1, the required assertion follows. The proof of (ii') is analogous to that in (i') and hence is left out. (iii) in Lemma 1 means (iii') holds. □

V. NUMERICAL SIMULATIONS

Now let us take advantage of Milstein's method and Monte Carlo's approach (see e.g. [5]) to reflect the theoretical findings. Pay attention to the discretization equation:

$$\begin{cases} N_1^{(k+1)} & = N_1^{(k)} + N_1^{(k)}\Big[\eta_{1l}^{1-q}\eta_{1u}^{q} - N_1^{(k)} \\ & +\rho_{1l}^{1-q}\rho_{1u}^{q}N_2^{(k)}\Big]\Delta t + N_1^{(k)}\psi_{1l}^{1-q}\psi_{1u}^{q}\xi_1^{(k)} \\ & +0.5\psi_{1l}^{1-q}\psi_{1u}^{q}N_1^{(k)}\Big((\xi_1^{(k)})^2 - 1\Big), \\ N_2^{(k+1)} & = N_2^{(k)} + N_2^{(k)}\Big[\eta_{2l}^{1-q}\eta_{2u}^{q} + \rho_{2l}^{1-q}\rho_{2u}^{q}N_1^{(k)} \\ & -N_2^{(k)}\Big]\Delta t + N_2^{(k)}\psi_{2l}^{1-q}\psi_{2u}^{q}\xi_2^{(k)} \\ & +0.5\psi_{2l}^{1-q}\psi_{2u}^{q}N_2^{(k)}\Big((\xi_2^{(k)})^2 - 1\Big), \end{cases}$$

where $\xi_1^{(k)}$ and $\xi_2^{(k)}$, $k = 1, 2, ...$, are random variables which follow the Gaussian distribution.

In the following figures, we choose

$$\eta_{1l} = 0.05,\ \eta_{1u} = 0.12,\ \eta_{2l} = 0.03,\ \eta_{2u} = 0.07,$$

$$\rho_{1l} = 0.18,\ \rho_{1u} = 0.24,\ \rho_{2l} = 0.28,\ \rho_{2u} = 0.32,\ q = 0.5.$$

Then

$$\gamma(q) = 0.9378,\ \theta_1(q) = 0.0870,\ \theta_2(q) = 0.0690.$$

First, let us change the values of ψ_{1l}, ψ_{1u} and ψ_{2l}, ψ_{2u}.

(a) Fig.1 chooses $\psi_{1l} = \psi_{1u} = \psi_{2l} = \psi_{2u} = 0$. According to [22], the positive equilibrium

$$N^* = \left(\frac{\theta_1}{\gamma}, \frac{\theta_2}{\gamma}\right) = (0.0928, 0.0736).$$

is globally asymptotically stable, see Fig.1.

(b) Fig.2 and Fig.3 are with $\psi_{1l} = \psi_{2l} = 0.1$, $\psi_{1u} = \psi_{2u} = 0.6$, hence

$$\alpha_1(q) = \psi_{2l}^{2(1-q)}\psi_{2u}^{2q}/2 = 0.03,$$

$$\alpha_2(q) = \psi_{2l}^{2(1-q)}\psi_{2u}^{2q}/2 = 0.03,$$

$$\sigma_1(q) = 0.0362,\ \sigma_2(q) = 0.0390.$$

Thus $\theta_2(q) > \sigma_2(q)$. In light of (i') in Theorem 3,

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s = \frac{\theta_1(q) - \sigma_1(q)}{\gamma(q)} = 0.0542,$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\mathrm{d}s = \frac{\theta_2(q) - \sigma_2(q)}{\gamma(q)} = 0.0320.$$

see Fig.2 and Fig.3.

(c) Fig.4 and Fig.5 are with $\psi_{1l} = \psi_{2l} = 0.2$, $\psi_{1u} = 0.3$, $\psi_{2u} = 0.8$. Then

$$\psi_{1l}^{2(1-q)}\psi_{1u}^{2q}/2 = 0.03,\ \psi_{2l}^{2(1-q)}\psi_{2u}^{2q}/2 = 0.08.$$

As a result,

$$\eta_{1l}^{2(1-q)}\eta_{1u}^{q} > \psi_{1l}^{2(1-q)}\psi_{1u}^{2q}/2,\ \theta_2(q) < \sigma_2(q).$$

In light of (ii') in Theorem 3, N_2 is stagnated and

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s = \beta_1(q) = 0.0475.$$

See Fig.4 and Fig.5.

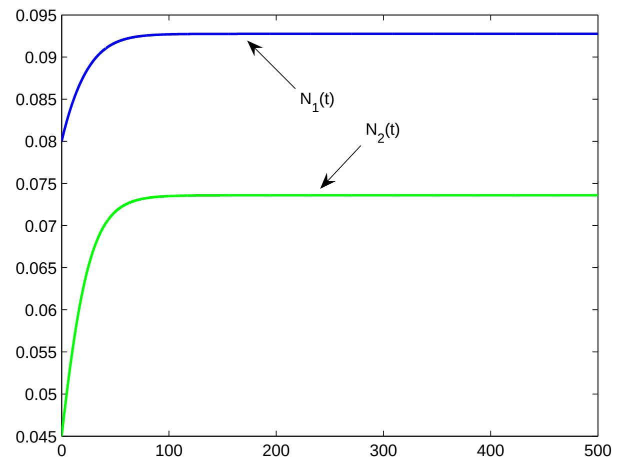

Fig. 1: Solutions of (4) with $\psi_{1l} = \psi_{1u} = \psi_{2l} = \psi_{2u} = 0$.

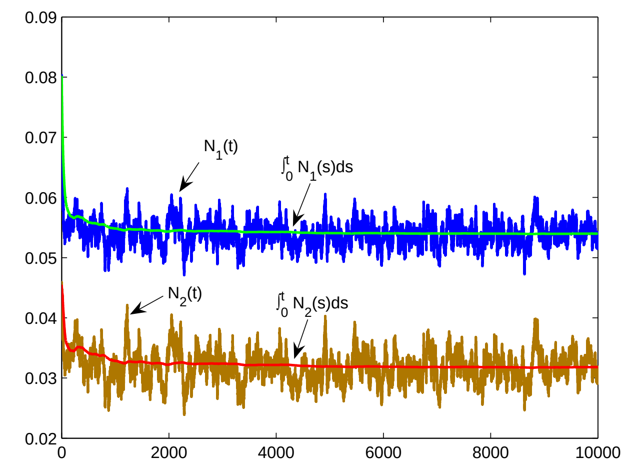

Fig. 2: Solutions of (4) with $\psi_{1l} = \psi_{2l} = 0.1$, $\psi_{1u} = \psi_{2u} = 0.6$.

(d) Fig.6 is with $\psi_{1l} = \psi_{2l} = 0.4$, $\psi_{1u} = 0.55$, $\psi_{2u} = 0.4$. Then

$$\psi_{1l}^{1-q}\psi_{1u}^{2q}/2 = 0.11,\ \psi_{2l}^{1-q}\psi_{2u}^{2q}/2 = 0.08,$$

and

$$\eta_{1l}^{1-q}\eta_{1u}^{q} > \psi_{1l}^{1-q}\psi_{1u}^{2q}/2.$$

In view of (iii') in Theorem 3, both N_1 and N_2 are stagnated. See Fig.4.

Comparing Fig.1 with Fig.2-Fig.5, one can see that the stochastic perturbations can change the properties of the model greatly.

Next, we set $q = 0.9$, and other parameters are the same with those in Fig.1. In this case, it is easy to check that

$$\eta_{1l}^{1-q}\eta_{1u}^{q} > \psi_{1l}^{1-q}\psi_{1u}^{2q}/2.$$

In view of (iii') in Theorem 3, both N_1 and N_2 are stagnated. See Fig.7. Comparing Fig.2 with Fig.7, one can observe that the imprecise parameters can change the properties of the model greatly.

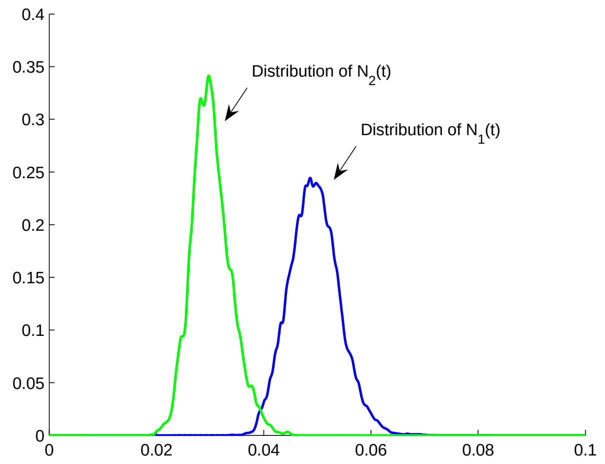

Fig. 3: Distribution of (4) with $\psi_{1l} = \psi_{2l} = 0.1$, $\psi_{1u} = \psi_{2u} = 0.6$.

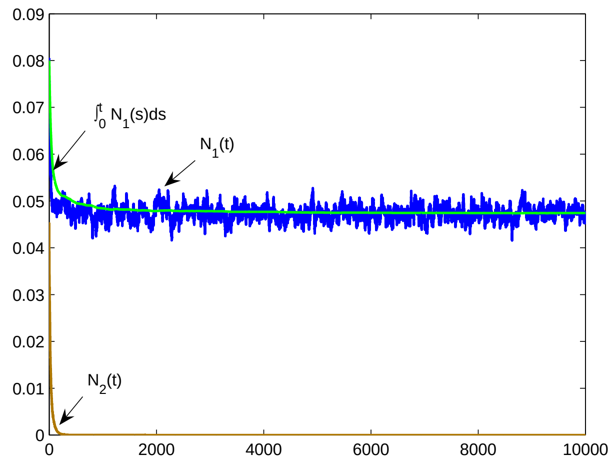

Fig. 4: Solutions of (4) with $\psi_{1l} = \psi_{2l} = 0.2$, $\psi_{1u} = 0.3$, $\psi_{2u} = 0.8$.

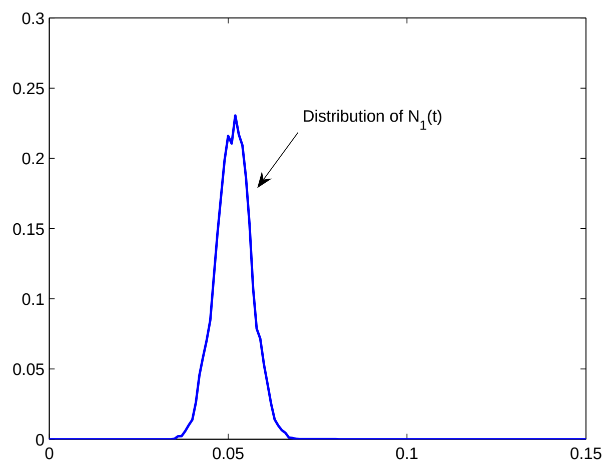

Fig. 5: Distribution of (4) with $\psi_{1l} = \psi_{2l} = 0.2$, $\psi_{1u} = 0.3$, $\psi_{2u} = 0.8$.

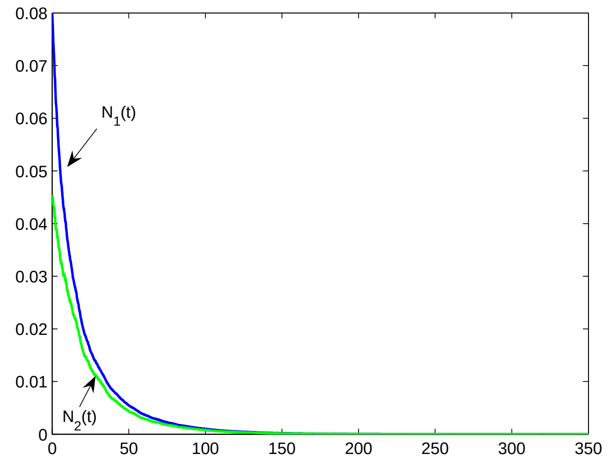

Fig. 6: Solutions of (4) with $\psi_{1l} = \psi_{2l} = 0.4$, $\psi_{1u} = 0.55$, $\psi_{2u} = 0.4$.

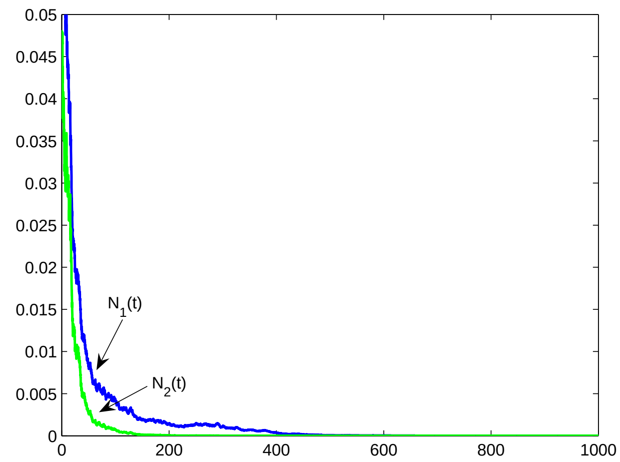

Fig. 7: Solutions of (4) for $q = 0.9$. Other parameters are the same with those in Fig.1(b).

VI. Conclusions

In this letter, we propose and investigate a fuzzy stochastic model which describes MIII. Sharp sufficient criteria for stagnation and prosperity of industrialization and informatization are obtained. Some critical functions of imprecise parameters and stochastic perturbations on MIII are provided and numerically illustrated.

In this letter, we only test the influences of imprecise parameters and white noise, one can examine the influence of time delay ([8], [24]). In addition, this paper considers the differential equation models, it is interesting to dissect the discrete models ([26], [33]).

References

[1] X. Bai, and L. Lei, "A dynamic space-time analysis on the level of integration of informatization and industrialization in China," *Economic Geography*, vol. 34, pp. 52-57, 2014.
[2] I. Barbalat, "Systems dequations differentielles d'osci d'oscillations nonlineaires," *Revue Roumaine de Mathematiques Pures et Appliquees*, vol. 4, no. 2, pp. 267–270, 1959.
[3] L. Becchetti, D. Bedoya, and L. Paganetto. "ICT investment, productivity and efficiency: evidence at firm level using a stochastic frontier approach," *Journal of Productivity Analysis*, vol. 20, pp. 143-167, 2003.

[4] A.S. Bharadwaj, “A resource-based perspective on information technology capability and firm performance: an empirical investigation,” *Mis Quarterly*, vol. 24, pp. 169-196, 2000.
[5] N. Bruti-Liberati, and E. Platen, “Monte Carlo simulation for stochastic differential equations,” *Encyclopedia of Quantitative Finance*, vol. (2010), pp. 1-8, 2010.
[6] L. Chen, *The Role of Informatization in Industrialization*. Wuhan: Huazhong University of Science and Technology, 2011.
[7] C. Du, and Z. Yang, “Research on evaluation of convergence between industrialization and informatization and its development level in China,” *Journal of China University of Geosciences (Social Sciences Edition)*, vol. 15, pp. 84-97, 2015.
[8] J. Ding, T. Zhao, Z. Liu, and Q. Guo, “Stability and bifurcation analysis of a delayed worm propagation model in mobile internet,” *IAENG International Journal of Computer Science*, vol. 47, no.3, pp. 533-539, 2020.
[9] C. Feng, *A Study on the Basic Theory and Solution of integration of Informatization and Industrialization*. Wuhan: Wuhan University, 2010.
[10] J. Jin, “Theoretical system of integration of informatization and industrialization,” *Informatization Construction*, vol. 4, pp. 9-12, 2009.
[11] M. Liu, and M. Fan, “Stability in distribution of a three-species stochastic cascade predator-prey system with time delays,” *IMA J. Appl. Math.* , vol. 82, pp. 396-423, 2017.
[12] M. Liu, and K. Wang, “Stochastic Lotka-Volterra systems with Lévy noise,” *J. Math. Anal. Appl.* , vol. 410, pp. 750-763, 2014.
[13] M. Liu, K. Wang, and Q. Wu, “Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle,” *Bull. Math. Biol*, vol. 73, pp. 1969-2012, 2011.
[14] H. Lu, Y. Feng, and S. Qu, “On coordinative development of industrialization and informatization,” *China Soft Science*, vol. 10, pp. 27-31, 2003.
[15] X.R. Mao, and C. Yuan, *Stochastic Differential Equations with Markovian Switching*. London: Imperial College Press, 2006.
[16] W. Miao, *Transformation of development mode: industrial restructuring and upgrading*. Beijing: Study Times, 2011.
[17] D. Pal, G.S. Mahapatra, “Dynamic behavior of a predator-prey system of combined harvesting with interval-valued rate parameters,” *Nonlinear Dyn.*, vol. 83, pp. 2113-2123, 2016.
[18] D. Prato, and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge: Cambridge University Press, 1996.
[19] S. Sharma and G.P. Samanta, “Optimal harvesting of a two species competition model with imprecise biological parameters,” *Nonlinear Dyn.*, vol. 77, pp. 1101-1119, 2014.
[20] Z. Sun, and H. Qin, “The optimal estimation of the Turan-type inequality for the Gamma function and its numerical method,” *Engineering Letters*, vol. 28, no.4, pp. 1075-1080, 2020.
[21] H. Tong, *Research on Evaluation System and Mechanism Analysis of Convergence of Informatization and Industrialization*. Beijing: Beijing University of Posts and Telecommunications, 2012.
[22] J. Wang, *The Mechanism and Performance of Integration of Informatization and Industrialization*. Tianjin: Nankai University, 2012.
[23] X. Wang, and X. Du, “The fuzzy evaluation of the measurement of integration of industrialization and informatization in China,” *Journal of Lanzhou University (Social Sciences)*, vol. 42, pp. 88-97, 2014.
[24] C. Wang, L. Jia, L. Li, and W. Wei, “Global stability in a delayed ratio-dependent predator-prey system with feedback controls,” *IAENG International Journal of Applied Mathematics*, vol. 50, no.3, pp. 690-698, 2020.
[25] Y. Wang, and H. Qin, “Spatial pattern and change mechanism analysis on the coupling and coordinating degree of regional informatization and new industrialization in China,” *Economic Geography*, vol. 34, pp. 93-100, 2014.
[26] C. Wei, X. Li, X. Zhang, and Z. Liu, “Mean square asymptotic analysis of discretely observed hybrid stochastic systems by feedback control,” *Engineering Letters*, vol. 28, no.3, pp. 880-886, 2020
[27] J. Wu, “The road to industrialization,” *Management World*, vol. 8, pp. 1-7, 2006.
[28] K. Xie, L. Li, and A. Tan, “Integration of informatization and industrialization, technical efficiency and convergence,” *Management Review*, vol. 21, pp. 3-12, 2009.
[29] K. Xie, J. Xiao, and X. Zhou, “Quality of convergence between industrialization and informatization in China,” *Economic Research Journal*, vol. 1, pp. 4-16, 2012.
[30] C. Yan, J. Ma, and D. Li, “On the mechanism of integration of informatization and industrialization based on dynamics of stochastic model,” *J. Nonlinear Sci. Appl.*, vol. 10, pp. 6311-6323, 2017.
[31] L. Yu, Y. Pan, and Y. Wu, “Study on relationship between industrialization and informatization,” *China Soft Science*, vol. 1, pp. 34-40, 2009.
[32] X. Zhao, *Research on the Interaction Mechanism between Informatization and Industrialization*. Beijing: Beijing University of Posts and Telecommunications, 2015.
[33] Z. Zhu, F. Chen, L. Lai, and Z. Li, “Dynamic behaviors of a discrete May type cooperative system incorporating Michaelis-Menten type harvesting,” *IAENG International Journal of Applied Mathematics*, vol. 50, no.3, pp. 458-467, 2020.
[34] X. Zhu, X. Xiao, T. Tjahjadi, Z. Wu, and J. Tang, “Image enhancement using fuzzy intensity measure and adaptive clipping histogram equalization,” *IAENG International Journal of Computer Science*, vol. 46, no.3, pp. 395-408, 2019.
[35] M. Zulkifly, A. Wahab, and R. Zakaria, “B-Spline curve interpolation model by using intuitionistic fuzzy approach,” *IAENG International Journal of Applied Mathematics*, vol. 50, no.4, pp. 760-766, 2020.