The General Dual Log Radial Bodies and Related Star Duality

Tian Li and Weidong Wang*

Abstract—**Wang and Liu proposed the definition of log radial sum and researched the dual log-Brunn-Minkowski theory. Motivated by the log radial sum of star bodies, we define the general log radial bodies and explore some related properties. For this notion and its the star dual form, we prove two extremum inequalities for dual quermassintegrals. A star dual log-Minkowski inequality is established by us.**

Index Terms—**Log radial sum, general log radial bodies, extremal value, dual quermassintegrals, star dual log-Minkowski inequality.**

I. Introduction

The dual Brunn-Minkowski theory is crucial and significant in geometry of star bodies (see[3], [20], [21], [22], [28], [51]). The fundamental dual Brunn-Minkowski inequality narrates that for two star bodies(about the origin) M and N in n-dimensional Euclidean space $\mathbf{R}^n$, the volume of the bodies and radial sum $M\widetilde{+}N$ are related by

$$V(M\widetilde{+}N)^{\frac{1}{n}} \leq V(M)^{\frac{1}{n}} + V(N)^{\frac{1}{n}},$$

with equality if and only if M and N are dilates. Here $V(Q)$ is the n-dimensional volume of a body Q.

Throughout the article, the set of all star bodies in the Euclidean space $\mathbf{R}^n$ is denoted by $\mathcal{S}^n$. For the set of all star bodies (with respect to the origin), we use a notation $\mathcal{S}^n_o$ to denote it. If star bodies are origin-symmetric in $\mathbf{R}^n$, then the set of such star bodies is denoted by $\mathcal{S}^n_{os}$. The unit sphere in $\mathbf{R}^n$ will be written as S^{n-1}.

The radial function for a compact star-shaped (with respect to the origin) M in $\mathbf{R}^n$ is defined by ([28])

$$\rho(M, z) = \max\{c \geq 0 : c \cdot z \in M\}, \quad z \in \mathbf{R}^n \setminus \{0\}.$$

If $\rho(M, \cdot)$ is positive and continuous, then M will be called a star body (with respect to the origin). If $\rho(M, w)/\rho(N, w)$ is independent of $w \in S^{n-1}$, then two bodies M and N are said to be dilated of each other.

By the definition of radial function, for $\phi \in GL(n)$ ([28]), it follows that

$$\rho(\phi M, w) = \rho(M, \phi^{-1} w), \quad w \in S^{n-1}. \tag{1.1}$$

Here $GL(n)$ is the group of general nonsingular (linear) transformations and ϕ^{-1} is the reverse of ϕ.

Manuscript received March 14, 2020; revised February 25, 2021. This work is supported by the Natural Science Foundation of China (No.11371224) and the Research Foundation of Qinghai Normal University (No.2019zr002).

Tian Li is a Lecturer in the College of Mathematics and Statistics, Qinghai Normal University, Xining, 810000, Qinghai, China (e-mail: litian_0920@163.com).

Weidong Wang* is an associate Professor of the Department of Mathematics, China Three Gorges University, Yichang, 443002, Hubei, China (corresponding author, e-mail: wangwd722@163.com).

Recently, Wang and Liu (see[35]) introduced the dual log-Brunn-Minkowski theory which is interesting and meaningful. They gave the notion of log radial sum firstly. Assume $M, N \in \mathcal{S}^n$ and $\lambda \in [0, 1]$, then the log radial sum, $(1-\lambda) \cdot M \widetilde{+}_0 \lambda \cdot N$, is defined by

$$\rho((1-\lambda) \cdot M \widetilde{+}_0 \lambda \cdot N, w) = \rho(M, w)^{1-\lambda} \rho(N, w)^{\lambda}, \tag{1.2}$$

$w \in S^{n-1}$. In particular, $(1-\lambda) \cdot M \widetilde{+}_0 \lambda \cdot N = M$ when $\lambda = 0$; $(1-\lambda) \cdot M \widetilde{+}_0 \lambda \cdot N = N$ when $\lambda = 1$.

In (1.2), if $N = -M$ and $\lambda = \frac{1}{2}$, then we obtain

$$\rho\left(\frac{1}{2} \cdot M \widetilde{+}_0 \frac{1}{2} \cdot (-M), w\right) = \rho(M, w)^{\frac{1}{2}} \rho(-M, w)^{\frac{1}{2}}. \tag{1.3}$$

At this point, we call $\widetilde{\triangle}_0 M = \frac{1}{2} \cdot M \widetilde{+}_0 \frac{1}{2} \cdot (-M)$ as a log radial body of $M \in \mathcal{S}^n$. Apparently, $\widetilde{\triangle}_0 M \in \mathcal{S}^n_{os}$.

Moreover, according to above notion, Wang and Liu ([35]) established the dual log-Brunn-Minkowski inequality as follows.

Theorem 1.A. *Suppose $M_1, M_2 \in \mathcal{S}^n$ and $\lambda \in [0, 1]$, then for $0 \leq i < n$,*

$$\widetilde{W}_i\left((1-\lambda) \cdot M_1 \widetilde{+}_0 \lambda \cdot M_2\right) \leq \widetilde{W}_i(M_1)^{1-\lambda} \widetilde{W}_i(M_2)^{\lambda}.$$

When $0 < \lambda < 1$, the equality holds if and only if M_1 and M_2 are dilated of each other. Here $\widetilde{W}_i(Q)$ is the dual quermassintegrals of the star body Q.

In fact, the i of Theorem 1.A can be taken as arbitrary real number that $i \neq n$.

Besides, the beautiful and powerful dual log-Minkowski inequality was deduced by Wang and Liu (see[35]).

Theorem 1.B. *Suppose $M_1, M_2 \in \mathcal{S}^n$, then for $0 \leq i < n$,*

$$\int_{S^{n-1}} \log\left(\frac{\rho(M_2, w)}{\rho(M_1, w)}\right) d\widetilde{V}_{i, M_1}(w) \leq \frac{1}{n-i} \log \frac{\widetilde{W}_i(M_2)}{\widetilde{W}_i(M_1)},$$

with equality if and only if M_1 and M_2 are dilates. Here $\widetilde{V}_{i,Q}(\cdot)$ denotes the dual mixed cone-volume probability measure of $Q \in \mathcal{S}^n$ (see Section II for a precise definition). And the i can also be taken as the real number $i < n$, actually.

Now, let $M \in \mathcal{S}^n_o$ and $\iota \in [-1, 1]$, then we define general log radial bodies as

$$\rho(\widetilde{\triangle}^{\iota}_0 M, w) = \rho(M, w)^{g_1(\iota)} \rho(-M, w)^{g_2(\iota)}. \tag{1.4}$$

Here

$$g_1(\iota) = \frac{(1+\iota)^2}{2(1+\iota^2)}, \qquad g_2(\iota) = \frac{(1-\iota)^2}{2(1+\iota^2)}. \tag{1.5}$$

By (1.5), one can show that

$$g_1(\iota) + g_2(\iota) = 1, \tag{1.6}$$

$$g_1(-\iota) = g_2(\iota), \qquad g_2(-\iota) = g_1(\iota). \tag{1.7}$$

According to (1.4), it easily yields

$$\widetilde{\triangle}_0^{\iota}M = g_1(\iota)\cdot M\widetilde{+}_0 g_2(\iota)\cdot(-M). \tag{1.8}$$

Using (1.3), (1.5) and (1.8), $\widetilde{\triangle}_0^{0}M = \widetilde{\triangle}_0 M$ when $\iota = 0$; $\widetilde{\triangle}_0^{+1}M = M$ and $\widetilde{\triangle}_0^{-1}M = -M$ when $\iota = \pm 1$, respectively.

The main purpose of this article is to expose extremal values for dual quermassintegrals of general log radial bodies and star duality of general log radial bodies. Further, we will prove a star dual log-Minkowski inequality.

An extremum inequality for dual quermassintegrals of general log radial bodies is established.

Theorem 1.1. *If $M \in \mathcal{S}_o^n$, $\iota \in [-1,1]$, then for arbitrary real number $i \neq n$, we have*

$$\widetilde{W}_i(\widetilde{\triangle}_0 M) \le \widetilde{W}_i(\widetilde{\triangle}_0^{\iota}M) \le \widetilde{W}_i(M). \tag{1.9}$$

If $M \notin \mathcal{S}_{os}^n$, then $\widetilde{W}_i(\widetilde{\triangle}_0 M) = \widetilde{W}_i(\widetilde{\triangle}_0^{\iota}M)$ if and only if $\iota = 0$; $\widetilde{W}_i(\widetilde{\triangle}_0^{\iota}M) = \widetilde{W}_i(M)$ if and only if $\iota = \pm 1$.

In 1999, Moszyńska (see[24]) introduced the star duality of star bodies. Let $M \in \mathcal{S}_o^n$ and i be the inversion of $\mathbf{R}^n\backslash\{0\}$ with respect to S^{n-1}:

$$i(x) := \frac{x}{\|x\|^2}.$$

Then define the star duality M° of M by

$$M^\circ = cl(\mathbf{R}^n \setminus \{i(M)\}).$$

Meanwhile, Moszyńska (see[24]) proved that for $M \in \mathcal{S}_o^n$ and any $w \in S^{n-1}$, the star dual M° satisfies

$$\rho(M^\circ, w) = \frac{1}{\rho(M,w)}. \tag{1.10}$$

With the emergence of this notion, a number of characterizations and inequalities were established about star bodies (see[15], [16], [17], [25], [47], [54]). Especially, $(-M)^\circ = -M^\circ$.

Next, according to the star duality of star bodies, we obtain the extremum of star dual form for general log radial bodies.

Theorem 1.2. *If $M \in \mathcal{S}_o^n$, $\iota \in [-1,1]$, then for arbitrary real number $i \neq n$, we have*

$$\widetilde{W}_i(\widetilde{\triangle}_0^{\circ}M) \le \widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ}M) \le \widetilde{W}_i(M^\circ). \tag{1.11}$$

If $M \notin \mathcal{S}_{os}^n$, then $\widetilde{W}_i(\widetilde{\triangle}_0^{\circ}M) = \widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ}M)$ if and only if $\iota = 0$; $\widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ}M) = \widetilde{W}_i(M^\circ)$ if and only if $\iota = \pm 1$.

Finally, the star dual form of Theorem 1.B is also established.

Theorem 1.3. *Suppose $M_1, M_2 \in \mathcal{S}_o^n$, then for every real number $i < n$, we have*

$$\int_{S^{n-1}} \log\left(\frac{\rho(M_2^\circ, w)}{\rho(M_1^\circ, w)}\right) d\widetilde{V}_{i,M_2}(w) \le \frac{1}{n-i}\log\frac{\widetilde{W}_i(M_1)}{\widetilde{W}_i(M_2)}, \tag{1.12}$$

with equality if and only if M_1 and M_2 are dilated of each other.

For more researches of asymmetric Brunn-Minkowski theory, please see articles [1], [2], [4], [5], [6], [7], [8], [10], [11], [12], [13], [14], [18], [19], [26], [27], [29], [30], [31], [32], [33], [34], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [48], [49], [50], [52], [53].

In this article, Section IV is dedicated to prove the Theorems 1.1-1.2. In last Section, we will give the proof of Theorem 1.3. In Section III, some properties of general log radial bodies will be obtained as well.

II. Preliminaries

A. The Radial Combination

Let $M, N \in \mathcal{S}_o^n$ and $a, b \ge 0$ ($a + b \neq 0$), then the radial combination, $a\cdot M\widetilde{+}b\cdot N \in \mathcal{S}_o^n$, of M and N satisfies ([20])

$$\rho(a\cdot M\widetilde{+}b\cdot N, \cdot) = a\rho(M,\cdot) + b\rho(N,\cdot).$$

Here '$\widetilde{+}$' denotes the radial addition and '$\cdot$' denotes the radial scalar multiplication.

B. Dual Quermassintegrals and Dual Mixed Quermassintegrals

The dual quermassintegrals $\widetilde{W}_i(M)$ of $M \in \mathcal{S}_o^n$ is defined by (see[23])

$$\widetilde{W}_i(M) = \frac{1}{n}\int_{S^{n-1}} \rho(M,w)^{n-i}dS(w), \quad \forall\ w \in S^{n-1}.$$

Here i is an arbitrary real number and S is the Lebesgue measure on S^{n-1}. Particularly, $\widetilde{W}_0(M) = V(M)$.

For $M, N \in \mathcal{S}_o^n$ and every real number i, the dual mixed quermassintegrals $\widetilde{W}_i(M,N)$ is defined as (see[23])

$$\widetilde{W}_i(M,N) = \frac{1}{n}\int_{S^{n-1}} \rho(M,w)^{n-i-1}\rho(N,w)dS(w),$$

$\forall\ w \in S^{n-1}$.

By the Hölder's integral inequality, a Minkowski inequality for dual mixed quermassintegrals was obtained easily.

Theorem 2.A. *Suppose $M, N \in \mathcal{S}_o^n$ and the real number $i < n-1$, then*

$$\widetilde{W}_i(M,N)^{n-i} \le \widetilde{W}_i(M)^{n-i-1}\widetilde{W}_i(N), \tag{2.1}$$

with equality if and only if M and N are dilates. When $i = n-1$, the (2.1) is identical.

Besides, the following Blaschke-Santaló inequality (see[17]) for dual quermassintegrals of star bodies is significant.

Theorem 2.B. *Suppose $M \in \mathcal{S}_o^n$ and $i \in \mathbf{R}$, then*

$$\widetilde{W}_i(M)\widetilde{W}_i(M^\circ) \ge \kappa_n^2, \tag{2.2}$$

with equality if and only if M is a centered ball. Here κ_n is the n- dimensional volume of the standard unit ball B in $\mathbf{R}^n$.

C. The Jessen's Inequality and Dual Mixed Cone-Volume Probability Measure

Let μ is a probability measure on a space Ω and $h : \Omega \to D \subset \mathbf{R}$ is a μ-intergrable function, where D is a possibly infinite interval. Jessen's inequality can be stated that if $\varphi : D \to \mathbf{R}$ is a concave function, then

$$\int_\Omega \varphi(h(z))d\mu(z) \le \varphi\Big(\int_\Omega h(z)d\mu(z)\Big). \tag{2.3}$$

When φ is strictly concave, the equality holds if and only if $h(z)$ is a constant for μ-almost every $z \in \Omega$ (see[9]).

If $M \in \mathcal{S}^n$, then the dual mixed cone-volume probability measure shall be written (see[35])

$$\widetilde{V}_{i,M}(\cdot) = \frac{\rho(M,\cdot)^{n-i} dS(\cdot)}{n\widetilde{W}_i(M)}. \tag{2.4}$$

Specially, the measure $\widetilde{V}_{0,M}(\cdot) = \widetilde{V}_M(\cdot)$ denotes the dual cone-volume probability measure while $i = 0$.

III. Properties of General log radial bodies

The following properties for general log radial bodies will be established.

Theorem 3.1. *If $M \in \mathcal{S}_o^n$, $\iota \in [-1,1]$, then for $\phi \in GL(n)$, we have*

$$\widetilde{\triangle}_0^{\iota}(\phi M) = \phi \widetilde{\triangle}_0^{\iota} M.$$

Proof. Combining with (1.1) and (1.4), it yields that for all $w \in S^{n-1}$, we obtain

$$\begin{aligned}\rho(\widetilde{\triangle}_0^{\iota}(\phi M), w) &= \rho(\phi M, w)^{g_1(\iota)} \rho(-\phi M, w)^{g_2(\iota)} \\ &= \rho(M, \phi^{-1} w)^{g_1(\iota)} \rho(-M, \phi^{-1} w)^{g_2(\iota)} \\ &= \rho(\widetilde{\triangle}_0^{\iota} M, \phi^{-1} w) \\ &= \rho(\phi \widetilde{\triangle}_0^{\iota} M, w).\end{aligned}$$

Therefore, $\widetilde{\triangle}_0^{\iota}(\phi M) = \phi \widetilde{\triangle}_0^{\iota} M$.

Theorem 3.2. *Let $M \in \mathcal{S}_o^n$ and $\iota \in [-1,1]$. If $\iota \neq 0$, then*

$$\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M \Longleftrightarrow M \in \mathcal{S}_{os}^n.$$

Proof. Since

$$\rho(\widetilde{\triangle}_0^{\iota} M, w) = \rho(M, w)^{g_1(\iota)} \rho(-M, w)^{g_2(\iota)}. \tag{3.1}$$

According to (1.4) and (1.7), we obtain that for any $w \in S^{n-1}$,

$$\begin{aligned}\rho(\widetilde{\triangle}_0^{-\iota} M, w) &= \rho(M, w)^{g_1(-\iota)} \rho(-M, w)^{g_2(-\iota)} \\ &= \rho(M, w)^{g_2(\iota)} \rho(-M, w)^{g_1(\iota)}. \end{aligned} \tag{3.2}$$

If $M \in \mathcal{S}_{os}^n$, i.e. $M = -M$, then from (3.1) and (3.2), we have

$$\rho(\widetilde{\triangle}_0^{\iota} M, w) = \rho(\widetilde{\triangle}_0^{-\iota} M, w).$$

So $\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M$.

On the contrary, if $\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M$, then

$$\rho(\widetilde{\triangle}_0^{\iota} M, w) = \rho(\widetilde{\triangle}_0^{-\iota} M, w).$$

Using (3.1) and (3.2), we see

$$\rho(M, w)^{g_1(\iota)} \rho(-M, w)^{g_2(\iota)} = \rho(M, w)^{g_2(\iota)} \rho(-M, w)^{g_1(\iota)}.$$

Then

$$\left(\frac{\rho(M, w)}{\rho(-M, w)}\right)^{g_1(\iota)} \left(\frac{\rho(-M, w)}{\rho(M, w)}\right)^{g_2(\iota)} = 1.$$

Namely,

$$\left(\frac{\rho(M, w)}{\rho(-M, w)}\right)^{g_1(\iota) - g_2(\iota)} = 1.$$

From $\iota \neq 0$, it yields $g_1(\iota) - g_2(\iota) \neq 0$. Hence $\rho(M, w) = \rho(-M, w)$, i.e. $M \in \mathcal{S}_{os}^n$.

By Theorem 3.2, the following corollary is hold.

Corollary 3.1. *Let $M \in \mathcal{S}_o^n$ and $\iota \in [-1,1]$. If M is not origin-symmetric, then*

$$\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M \Longleftrightarrow \iota = 0.$$

Theorem 3.3. *If $M \in \mathcal{S}_o^n$ and $\iota \in [-1,1]$, then*

$$\widetilde{\triangle}_0^{-\iota} M = \widetilde{\triangle}_0^{\iota}(-M) = -\widetilde{\triangle}_0^{\iota} M. \tag{3.3}$$

Proof. Using (3.2) and (1.4), for every $w \in S^{n-1}$, we have

$$\begin{aligned}\rho(\widetilde{\triangle}_0^{-\iota} M, w) &= \rho(M, w)^{g_2(\iota)} \rho(-M, w)^{g_1(\iota)} \\ &= \rho(-M, w)^{g_1(\iota)} \rho(-(-M), w)^{g_2(\iota)} \\ &= \rho(\widetilde{\triangle}_0^{\iota}(-M), w).\end{aligned}$$

So $\widetilde{\triangle}_0^{-\iota} M = \widetilde{\triangle}_0^{\iota}(-M)$.

Furthermore, from the definition (1.4), it yields that for every $w \in S^{n-1}$,

$$\begin{aligned}\rho(-\widetilde{\triangle}_0^{\iota} M, w) &= \rho(\widetilde{\triangle}_0^{\iota} M, -w) \\ &= \rho(M, -w)^{g_1(\iota)} \rho(-M, -w)^{g_2(\iota)} \\ &= \rho(-M, w)^{g_1(\iota)} \rho(-(-M), w)^{g_2(\iota)} \\ &= \rho(\widetilde{\triangle}_0^{\iota}(-M), w).\end{aligned}$$

Hence $-\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{\iota}(-M)$. This gives (3.3).

Theorem 3.4. *If $M \in \mathcal{S}_{os}^n$ and $\iota \in [-1,1]$, then*

$$\widetilde{\triangle}_0^{\iota} M = M. \tag{3.4}$$

Proof. Since $M \in \mathcal{S}_{os}^n$, namely $M = -M$. By the definition (1.4) and (1.6), we see that for every $w \in S^{n-1}$,

$$\rho(\widetilde{\triangle}_0^{\iota} M, w) = \rho(M, w)^{g_1(\iota)} \rho(-M, w)^{g_2(\iota)} = \rho(M, w).$$

This yields (3.4).

Theorem 3.4 immediately implies the following corollary.

Corollary 3.2. *If $M_1, M_2 \in \mathcal{S}_{os}^n$ and $\iota \in [-1,1]$, then*

$$\widetilde{\triangle}_0^{\iota} M_1 = \widetilde{\triangle}_0^{\iota} M_2 \Longleftrightarrow M_1 = M_2 \ .$$

IV. The Extremum Inequalities of General Log Radial Bodies

In this section, we will prove Theorems 1.1-1.2.

Proof of Theorem 1.1. According to the Theorem 1.A and (1.6), we infer that for all $\iota \in [-1,1]$,

$$\begin{aligned}\widetilde{W}_i(\widetilde{\triangle}_0^{\iota} M) &= \widetilde{W}_i\Big(g_1(\iota) \cdot M \widetilde{+}_0 g_2(\iota) \cdot (-M)\Big) \\ &\leq \widetilde{W}_i(M)^{g_1(\iota)} \widetilde{W}_i(-M)^{g_2(\iota)} \\ &= \widetilde{W}_i(M). \end{aligned} \tag{4.1}$$

This yields the right hand side inequality of (1.9).

Apparently, (4.1) is an equation when $\iota = \pm 1$. So if $\iota \neq \pm 1$, then by the equality condition of Theorem 1.A, we know that the equality holds in (4.1) if and only if M and $-M$ are dilated of each other. It shows $M = -M$, thereby $M \in \mathcal{S}_{os}^n$.

Therefore, in the right hand side inequality of (1.9), the equality holds if and only if $M \in \mathcal{S}^n_{os}$ or $\iota = \pm 1$. This implies that if $M \notin \mathcal{S}^n_{os}$, then the equality holds in the right hand side inequality of (1.9) if and only if $\iota = \pm 1$.

Next, we complete the proof of left hand side inequality in (1.9). Combining with (1.4), (3.2) and (1.6), we obtain

$$\begin{aligned}
&\rho(\widetilde{\triangle}_0^{\iota} M, w)^{\frac{1}{2}} \rho(\widetilde{\triangle}_0^{-\iota} M, w)^{\frac{1}{2}} \\
&= \Big[\rho(M,w)^{g_1(\iota)} \rho(-M,w)^{g_2(\iota)}\Big]^{\frac{1}{2}} \\
&\cdot \Big[\rho(M,w)^{g_2(\iota)} \rho(-M,w)^{g_1(\iota)}\Big]^{\frac{1}{2}} \\
&= \rho(M,w)^{\frac{1}{2}} \rho(-M,w)^{\frac{1}{2}} = \rho(\widetilde{\triangle}_0 M, w)
\end{aligned}$$

for arbitrary $w \in S^{n-1}$. Thus,

$$\widetilde{\triangle}_0 M = \frac{1}{2} \cdot \widetilde{\triangle}_0^{\iota} M \widetilde{+}_0 \frac{1}{2} \cdot \widetilde{\triangle}_0^{-\iota} M. \tag{4.2}$$

From the Theorem 1.A and (3.3), it yields

$$\begin{aligned}
\widetilde{W}_i(\widetilde{\triangle}_0 M) &\le \widetilde{W}_i\Big(\widetilde{\triangle}_0^{\iota} M\Big)^{\frac{1}{2}} \widetilde{W}_i\Big(\widetilde{\triangle}_0^{-\iota} M\Big)^{\frac{1}{2}} \\
&= \widetilde{W}_i\Big(\widetilde{\triangle}_0^{\iota} M\Big)^{\frac{1}{2}} \widetilde{W}_i\Big(-\widetilde{\triangle}_0^{\iota} M\Big)^{\frac{1}{2}} \\
&= \widetilde{W}_i(\widetilde{\triangle}_0^{\iota} M).
\end{aligned} \tag{4.3}$$

This gives the left hand side inequality of (1.9).

Clearly, (4.3) is an identity when $\iota = 0$. Hence if $\iota \neq 0$, then by the condition of equality in Theorem 1.A, the equality holds in (4.3) if and only if $\widetilde{\triangle}_0^{\iota} M$ and $\widetilde{\triangle}_0^{-\iota} M$ are dilates. This means $\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M$. So from Corollary 3.1, one can verify that if $M \notin \mathcal{S}^n_{os}$, then the equality holds in the left hand side inequality of (1.9) if and only if $\iota = 0$.

By Theorem 1.1 and (2.2), we can easily deduce the following conclusion.

Corollary 4.1. *If $M \in \mathcal{S}^n_o$ and $\iota \in [-1, 1]$, then for real number $i \neq n$, we have*

$$\widetilde{W}_i(M)\widetilde{W}_i(\widetilde{\triangle}_0^{\circ} M) \ge \kappa_n^2; \tag{4.4}$$

and

$$\widetilde{W}_i(M)\widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ} M) \ge \kappa_n^2. \tag{4.5}$$

In (4.4), the equality holds if and only if $\widetilde{\triangle}_0 M$ is a centered ball; in (4.5), the equality holds if and only if $\widetilde{\triangle}_0^{\iota} M$ is a centered ball.

For the proof of Theorem 1.2, the following lemma is essential.

Lemma 4.1. *Suppose $M_1, M_2 \in \mathcal{S}^n_o$ and $\lambda \in [0, 1]$, then for the real number $i \neq n$,*

$$\widetilde{W}_i\Big[\Big((1-\lambda)\cdot M_1 \widetilde{+}_0 \lambda \cdot M_2\Big)^{\circ}\Big] \le \widetilde{W}_i(M_1^{\circ})^{1-\lambda}\widetilde{W}_i(M_2^{\circ})^{\lambda}. \tag{4.6}$$

When $0 < \lambda < 1$, the equality holds if and only if M_1 and M_2 are dilates.

Proof. Note that $\frac{1}{1-\lambda} > 1$ when $\lambda \in (0, 1)$. By (1.10), (1.2) and the Hölder's integral inequality, we have that for arbitrary $w \in S^{n-1}$,

$$\begin{aligned}
&\widetilde{W}_i\Big[\Big((1-\lambda)\cdot M_1 \widetilde{+}_0 \lambda \cdot M_2\Big)^{\circ}\Big] \\
&= \frac{1}{n}\int_{S^{n-1}} \rho\Big[\Big((1-\lambda)\cdot M_1 \widetilde{+}_0 \lambda \cdot M_2\Big)^{\circ}, w\Big]^{n-i} dS(w) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{\rho((1-\lambda)\cdot M_1 \widetilde{+}_0 \lambda \cdot M_2, w)^{n-i}} dS(w) \\
&= \frac{1}{n}\int_{S^{n-1}} \frac{1}{\rho(M_1,w)^{(1-\lambda)(n-i)}\rho(M_2,w)^{\lambda(n-i)}} dS(w) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(M_1^{\circ},w)^{(1-\lambda)(n-i)}\rho(M_2^{\circ},w)^{\lambda(n-i)} dS(w) \\
&\le \Big[\frac{1}{n}\int_{S^{n-1}} \rho(M_1^{\circ},w)^{(1-\lambda)(n-i)\cdot\frac{1}{1-\lambda}} dS(w)\Big]^{1-\lambda} \\
&\quad \cdot \Big[\frac{1}{n}\int_{S^{n-1}} \rho(M_2^{\circ},w)^{\lambda(n-i)\cdot\frac{1}{\lambda}} dS(w)\Big]^{\lambda} \\
&= \widetilde{W}_i(M_1^{\circ})^{1-\lambda}\widetilde{W}_i(M_2^{\circ})^{\lambda}.
\end{aligned}$$

Evidently, (4.6) is an identity when $\lambda = 0$ (or $\lambda = 1$). If $\lambda \in (0, 1)$, then according to the equality condition of Hölder's integral inequality, the equality holds in (4.6) if and only if there exists a constant C such that

$$\frac{\rho(M_1^{\circ},w)^{n-i}}{\rho(M_2^{\circ},w)^{n-i}} = C.$$

Namely, M_1° and M_2° are dilates. This means that the equality holds in (4.6) if and only if M_1 and M_2 are dilates.

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, on the one hand, from (4.6) and (1.6), we have

$$\begin{aligned}
\widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ} M) &= \widetilde{W}_i\Big[\Big(g_1(\iota)\cdot M \widetilde{+}_0 g_2(\iota)\cdot(-M)\Big)^{\circ}\Big] \\
&\le \widetilde{W}_i(M^{\circ})^{g_1(\iota)}\widetilde{W}_i(-M^{\circ})^{g_2(\iota)} \\
&= \widetilde{W}_i(M^{\circ}).
\end{aligned} \tag{4.7}$$

This obtains the right hand side inequality of (1.11).

If $\iota \neq \pm 1$, then from the equality condition of (4.6), the equality holds in (4.7) if and only if M and $-M$ are dilated of each other. It means $M = -M$, thereby $M \in \mathcal{S}^n_{os}$.

Therefore, in the right hand side inequality of (1.11), the equality holds if and only if $M \in \mathcal{S}^n_{os}$ or $\iota = \pm 1$. This implies that if $M \notin \mathcal{S}^n_{os}$, then in the right hand side inequality of (1.11), the equality holds if and only if $\iota = \pm 1$.

The other side of the coin, from (4.2), (4.6) and (3.3), we obtain

$$\begin{aligned}
\widetilde{W}_i(\widetilde{\triangle}_0^{\circ} M) &= \widetilde{W}_i\Big[\Big(\frac{1}{2}\cdot \widetilde{\triangle}_0^{\iota} M \widetilde{+}_0 \frac{1}{2}\cdot \widetilde{\triangle}_0^{-\iota} M\Big)^{\circ}\Big] \\
&\le \widetilde{W}_i\Big(\widetilde{\triangle}_0^{\iota,\circ} M\Big)^{\frac{1}{2}} \widetilde{W}_i\Big(\widetilde{\triangle}_0^{-\iota,\circ} M\Big)^{\frac{1}{2}} \\
&= \widetilde{W}_i\Big(\widetilde{\triangle}_0^{\iota,\circ} M\Big)^{\frac{1}{2}} \widetilde{W}_i\Big(-\widetilde{\triangle}_0^{\iota,\circ} M\Big)^{\frac{1}{2}} \\
&= \widetilde{W}_i(\widetilde{\triangle}_0^{\iota,\circ} M).
\end{aligned} \tag{4.8}$$

The left hand side inequality of (1.11) is proved.

If $\iota \neq 0$, then by Lemma 4.1, the equality can be obtained in (4.8) if and only if $\widetilde{\triangle}_0^{\iota} M$ and $\widetilde{\triangle}_0^{-\iota} M$ are dilated of each other. It will show $\widetilde{\triangle}_0^{\iota} M = \widetilde{\triangle}_0^{-\iota} M$. So, the Corollary 3.1 tell us that if $M \notin \mathcal{S}_{os}^n$, then in the left hand side inequality of (1.11), the equality holds if and only if $\iota = 0$.

V. The Star Dual Form of The Dual Log-Minkowski Inequality

In this section, we shall prove the star dual form of the dual log-Minkowski inequality.

Proof of Theorem 1.3. Notice that the logarithmic function $\log(\cdot)$ is concave and increasing on $(0, +\infty)$. So by (2.4), the Jessen's inequality (2.3) and (2.1), we obtain

$$\begin{aligned}
&\int_{S^{n-1}} \log\left(\frac{\rho(M_2^\circ, w)}{\rho(M_1^\circ, w)}\right) d\widetilde{V}_{i,M_2}(w) \\
&= \int_{S^{n-1}} \log\left(\frac{\rho(M_2^\circ, w)}{\rho(M_1^\circ, w)}\right) \cdot \frac{\rho(M_2, w)^{n-i}}{n\widetilde{W}_i(M_2)} dS(w) \\
&= \frac{1}{n\widetilde{W}_i(M_2)} \int_{S^{n-1}} \log\left(\frac{\rho(M_2^\circ, w)}{\rho(M_1^\circ, w)}\right) \cdot \rho(M_2, w)^{n-i} dS(w) \\
&\leq \log\left(\frac{1}{n\widetilde{W}_i(M_2)} \int_{S^{n-1}} \frac{\rho(M_1, w)}{\rho(M_2, w)} \cdot \rho(M_2, w)^{n-i} dS(w)\right) \\
&= \log\left(\frac{\widetilde{W}_i(M_2, M_1)}{\widetilde{W}_i(M_2)}\right) \\
&\leq \log\left(\frac{\widetilde{W}_i(M_2)^{\frac{n-i-1}{n-i}} \widetilde{W}_i(M_1)^{\frac{1}{n-i}}}{\widetilde{W}_i(M_2)}\right) \\
&= \log\left(\frac{\widetilde{W}_i(M_1)}{\widetilde{W}_i(M_2)}\right)^{\frac{1}{n-i}}.
\end{aligned}$$

This yields the inequality (1.12).

Because $\log(\cdot)$ is strictly increasing, by the equality condition of Jessen's inequality (2.3) and the Minkowski inequality (2.1), we find that the equality holds in (1.12) if and only if M_1 and M_2 are dilates.

Acknowledgment

We are greatly indebted to the referees for very valuable comments and suggestions, which improved the paper greatly.

References

[1] Y. B. Feng and W. D. Wang, "General L_p-harmonic Blaschke bodies," *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, vol. 124, no. 1, pp. 109–119, 2014.
[2] Y. B. Feng, W. D. Wang and F. H. Lu, "Some inequalities on general L_p-centroid bodies," *Mathematical Inequalities & Applications*, vol. 18, no. 1, pp. 39–49, 2015.
[3] R. J. Gardner, "Geometric Tomography," *2nd edn, Cambridge University Press, Cambridge*, 2006.
[4] C. Haberl, "L_p-intersection bodies," *Advances in Mathematics*, vol. 217, no. 6, pp. 2599–2624, 2008.
[5] C. Haberl and M. Ludwig, "A characterization of L_p intersection bodies," *International Mathematics Research Notices*, vol. 2006, pp. 1–30, 2006.
[6] C. Haberl and F. E. Schuster, "General L_p affine isoperimetric inequalities," *Journal of Differential Geometry*, vol. 83, no. 1, pp. 1–26, 2009.
[7] C. Haberl and F. E. Schuster, "Asymmetric affine L_p Sobolev inequalities," *Journal of Functional Analysis*, vol. 257, no. 3, pp. 641–658, 2009.
[8] C. Haberl, F. E. Schuster and J. Xiao, "An asymmetric affine Pólya-Szegö principle," *Mathematische Annalen*, vol. 352, no. 3, pp. 517–542, 2012.
[9] G. H. Hardy, J. E. Littlewood and G. Pólya, " Inequalities," *Reprint of the 1952 edition, Cambridge mathematical Library, Cambridge University Press, Cambridge*, 1988.
[10] C. Li and W. D. Wang, "On the Shephard type problems for general L_p-projection bodies," *IAENG International Journal of Applied Mathematics*, vol. 49, no. 1, pp. 122–126, 2019.
[11] T. Li and W. D. Wang, "Some inequalities for asymmetric L_p-mean zonoids," *Quaestiones Mathematicae*, vol. 42, no. 5, pp. 631–649, 2019.
[12] T. Li, W. D. Wang and Y. P. Mao, "General Blaschke bodies and the asymmetric negative solutions of Shephard problem," *Mathematics*, vol. 7, no. 7, pp. 1–10, 2019.
[13] Z. F. Li and W. D. Wang, "General L_p-mixed chord integrals of star Bodies," *Journal of Inequalities and Applications*, vol. 2016, no. 1, pp. 1–12, 2016.
[14] Z. F. Li and W. D. Wang, "Inequalities on asymmetric L_p-harmonic radial bodies," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 3612–3618, 2017.
[15] Z. F. Li and W. D. Wang, "Star dual of radial Blaschke-Minkowski homomorphism," *Wuhan University Journal of Natural Sciences*, vol. 23, no. 4, pp. 295–300, 2018.
[16] L. J. Liu, W. Wang and B. W. He, "Brunn-Minkowski inequalities for star duals of intersection bodies and two additions," *Journal of Shanghai University (English Edition)*, vol. 14, no. 3, pp. 201–205, 2010.
[17] F. H. Lu, W. H. Mao and G. S. Leng, "On star duality of mixed intersection bodies," *Journal of Inequalities and Applications*, vol. 2007, no. 1, pp. 1–12, 2007.
[18] M. Ludwig, "Projection bodies and valuations," *Advances in Mathematics*, vol. 172, no. 2, pp. 158–168, 2002.
[19] M. Ludwig, "Minkowski valuations," *Transactions of the American Mathematical*, vol. 357, no. 10, pp. 4191–4213, 2005.
[20] E. Lutwak, "Dual mixed volumes," *Pacific Journal of Mathematics*, vol. 58, no. 2, pp. 531–538, 1975.
[21] E. Lutwak, "Intersection bodies and dual mixed volumes," *Advances in Mathematics*, vol. 71, no. 2, pp. 232–261, 1988.
[22] E. Lutwak, "Centroid bodies and dual mixed volumes," *Proceedings of the London Mathematical Society*, vol. 60, no. 2, pp. 365–391, 1990.
[23] E. Lutwak, "The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.
[24] M. Moszyńska, "Quotient star bodies, intersection bodies, and star duality," *Journal of Mathematical Analysis and Applications*, vol. 232, no. 1, pp. 45–60, 1999.
[25] M. Moszyńska, "Selected topics in convex geometry," *Boston: Birkhauser*, 2006.
[26] Y. N. Pei and W. D. Wang, "Shephard type problems for general L_p-centroid bodies," *Journal of Inequalities and Applications*, vol. 2015, no. 1, pp. 1–13, 2015.
[27] Y. N. Pei and W. D. Wang, "A type of Busemann-Petty problems for general L_p-intersection bodies," *Wuhan University Journal of Natural Sciences*, vol. 20, no. 6, pp. 471–475, 2015.
[28] R. Schneider, "Convex Bodies: The Brunn-Minkowski Theory," *2nd edn, Cambridge University Press, Cambridge*, 2014.
[29] F. E. Schuster and M. Weberndorfer, "Volume inequalities for asymmetric Wulff shapes," *Journal of Differential Geometry*, vol. 92, no. 1, pp. 263–283, 2012.
[30] Z. H. Shen, Y. N. Li and W. D. Wang, "L_p-dual geominimal surface areas for the general L_p-intersection bodies," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 3519–3529, 2017.
[31] W. Shi and W. D. Wang, "Extremum values of asymmetric L_p-difference bodies for quermassintegrals and dual quermassintegrals," *Wuhan University Journal of Natural Sciences*, vol. 23, no. 4, pp. 283–288, 2018.
[32] W. Shi, W. D. Wang and T. Y. Ma, "Orlicz difference bodies," *Open Mathematics*, vol. 16, no. 1, pp. 1187–1195, 2018.
[33] X. Y. Wan and W. D. Wang, "Petty projection inequalities for the general L_p-mixed projection bodies," *Wuhan University Journal of Natural Sciences*, vol. 17, no. 3, pp. 190–194, 2012.
[34] J. Y. Wang and W. D. Wang, "General L_p-dual Blaschke bodies and the applications," *Journal of Inequalities and Applications*, vol. 2015, no. 1, pp. 1–11, 2015.
[35] W. Wang and L. J. Liu, "The dual log-Brunn-Minkowski inequalities," *Taiwanese Journal of Mathematics*, vol. 20, no. 4, pp. 909–919, 2016.
[36] W. D. Wang and Y. B. Feng, "A general L_p-version of Petty's affine projection inequality," *Taiwanese Journal of Mathematics*, vol. 17, no. 2, pp. 517–528, 2013.
[37] W. D. Wang and T. Li, "Volume extremals of general L_p-centroid bodies," *Journal of Mathematical Inequalities*, vol. 11, no. 1, pp. 193–207, 2017.

[38] W. D. Wang and Y. N. Li, "Busemann-Petty problems for general L_p-intersection bodies," *Acta Mathematica Sinica (English Series)*, vol. 31, no. 5, pp. 777–786, 2015.
[39] W. D. Wang and Y. N. Li, "General L_p-intersection bodies," *Taiwanese Journal of Mathematics*, vol. 19, no. 4, pp. 1247–1259, 2015.
[40] W. D. Wang and T. Y. Ma, "Asymmetric L_p-difference bodies," *Proceedings of the American Mathematical Society*, vol. 142, no. 7, pp. 2517–2527, 2014.
[41] W. D. Wang and X. Y. Wan, "Shephard type problems for general L_p-projection bodies," *Taiwanese Journal of Mathematics*, vol. 16, no. 5, pp. 1749–1762, 2012.
[42] W. D. Wang and J. Y. Wang, "Extremum of geometric functionals involving general L_p-projection bodies," *Journal of Inequalities and Applications*, vol. 2016, no. 1, pp. 1–16, 2016.
[43] M. Weberndorfer, "Shadow systems of asymmetric L_p zonotopes," *Advances in Mathematics*, vol. 240, no. 1, pp. 613–635, 2013.
[44] B. Wei and W. D. Wang, "Some inequalities for general L_p-harmonic Blaschke bodies," *Journal of Mathematical Inequalities*, vol. 10, no. 1, pp. 63–73, 2016.
[45] L. Yan and W. D. Wang, "The general L_p-mixed brightness integrals," *Journal of Inequalities and Applications*, vol. 2015, no. 1, pp. 1–11, 2015.
[46] L. M. Yu, Y. Y. Zhang and W. D. Wang, "Cyclic Brunn-Minkowski inequalities for general width and chord-integrals," *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–9, 2019.
[47] J. Yuan, H. W. Zhu and G. S. Leng, "Inequalities for star duals of intersection bodies," *Journal of the Korean Mathematical Society*, vol. 44, no. 2, pp. 297–306, 2007.
[48] J. Zhang and W. D. Wang, "The Shephard type problems for general L_p-centroid bodies," *Communications in Mathematical Research*, vol. 35, no. 1, pp. 27–34, 2019.
[49] J. Zhang and W. D. Wang, "L_p-dual affine surfac areas for the genaral L_p-centroid bodies," *Journal of Mathematical Inequalities*, vol. 13, no. 3, pp. 655–666, 2019.
[50] J. Zhang and W. D. Wang, "L_p-dual affine surface areas for the general L_p-intersection bodies," *Filomat*, vol. 33, no. 14, pp. 4421–4428, 2019.
[51] G. Y. Zhang, "Centroid bodies and dual mixed volumes," *Transactions of the American Mathematical Society*, vol. 345, no. 2, pp. 777–801, 1994.
[52] P. Zhang, X. H. Zhang and W. D. Wang, "The general L_p-dual mixed brightness integrals," *IAENG International Journal of Applied Mathematics*, vol. 47, no. 2, pp. 138–142, 2017.
[53] Y. P. Zhou, "General L_p-mixed width-integral of convex bodies and related inequalities," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 4372–4380, 2017.
[54] X. Y. Zhu and Y. J. Shen, "The dual Brunn-Minkowski inequalities for star dual of mixed intersection bodies," *Acta Mathematica Sinica (English Series)*, vol. 30, no. 3, pp. 739–746, 2010.

Tian Li was born in Tianshui, Gansu, China in 1993. She received her Bachelor degree in Mathematics from Hexi University in Zhangye, China in 2015 and obtained Master degree in Mathematics, China Three Gorges University, Yichang, China in 2018. Now, she works in College of Mathematics and Statistics, Qinghai Normal University.

She is currently a full-time teacher of College of Mathematics and Statistics at Qinghai Normal University. Her research interest is convex geometry analysis. Email: litian_0920@163.com.

Weidong Wang is a professor in department of Mathematics and Three Gorges Mathematical Research Center, China Three Gorges University. His research interest is convex geometry analysis. Email: wangwd722@163.com.