The General Dual Log Radial Bodies and Related Star Duality

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Abstract—Wang and Liu proposed the definition of log radial sum and researched the dual log-Brunn-Minkowski theory. Motivated by the log radial sum of star bodies, we define the general log radial bodies and explore some related properties. For this notion and its the star dual form, we prove two extremum inequalities for dual quermassintegrals. A star dual log-Minkowski inequality is established by us.

Index Terms—Log radial sum, general log radial bodies, extremal value, dual quermassintegrals, star dual log-Minkowski inequality.

I. INTRODUCTION

The dual Brunn-Minkowski theory is crucial and significant in geometry of star bodies (see[3], [20], [21], [22], [28], [51]). The fundamental dual Brunn-Minkowski inequality narrates that for two star bodies (about the origin) and , the volume of the bodies and radial sum is related by

with equality if and only if and are dilates. Here is the -dimensional volume of a body .

Throughout the article, the set of all star bodies in the Euclidean space is denoted by . For the set of all star bodies (with respect to the origin), we use a notation to denote it. If star bodies are origin-symmetric in , then the set of such star bodies is denoted by . The unit sphere in will be written as .

The radial function for a compact star-shaped (with respect to the origin) in is defined by (28).

If is positive and continuous, then will be called a star body (with respect to the origin). If is independent of when , then two bodies and are said to be dilated of each other.

By the definition of radial function, for , it follows that

Here is the group of general nonsingular (linear) transformations and is the reverse of .

Recently, Wang and Liu (see[35]) introduced the dual log-Brunn-Minkowski theory which is interesting and meaningful. They gave the notion of log radial sum firstly. Assume and , then the log radial sum, is defined by

with equality if and only if .

In (1.2), if , then we obtain

At this point, we call as a log radial body of . Apparently, .

Moreover, according to above notion, Wang and Liu ([35]) established the dual log-Brunn-Minkowski inequality as follows.

Theorem 1.A. Suppose , and , then for ,

When , the equality holds if and only if and are dilates. Here is the dual quermassintegrals of the star body .

In fact, the of Theorem 1.A can be taken as arbitrary real number that .

Besides, the beautiful and powerful dual log-Minkowski inequality was deduced by Wang and Liu (see[35]).

Theorem 1.B. Suppose , and , then for ,

with equality if and only if and are dilates. Here denotes the dual mixed cone-volume probability measure of .

The and can also be taken as the real number when .

Now, let , and , then we define general log radial bodies as

Here

By (1.5), one can show that

(1.6)

(1.7)
According to (1.4), it easily yields
\[ \Delta^\ell_0 M = g_1(\ell) \cdot M + g_2(\ell) \cdot (-M). \]  
(1.8)
Using (1.3), (1.5) and (1.8), \( \Delta^\ell_0 M = \Delta^\ell_0 M \) when \( \ell = 0 \), \( \Delta^{\ell+1}_0 M = M \) and \( \Delta^{\ell-1}_0 M = -M \) when \( \ell = \pm 1 \), respectively.

The main purpose of this article is to expose extremal values for dual quermassintegrals of general log radial bodies and star duality of general log radial bodies. Further, we will prove a star dual log-Minkowski inequality.

An extremum inequality for dual quermassintegrals of general log radial bodies is established.

**Theorem 1.** If \( M \in S^n_0 \), \( i \in [-1, 1] \), then for arbitrary real number \( i \neq n \), we have
\[ \bar{W}_i(\Delta^\ell_0 M) \leq \bar{W}_i(\Delta^\ell_0 M) \leq \bar{W}_i(M). \]  
(1.9)
If \( M \notin S^n_0 \), then \( \bar{W}_i(\Delta^\ell_0 M) = \bar{W}_i(\Delta^\ell_0 M) \) if and only if \( \ell = 0 \); \( \bar{W}_i(\Delta^\ell_0 M) = \bar{W}_i(M) \) if and only if \( i = \pm 1 \).

In 1999, Mozyškis (see[24]) introduced the star duality of star bodies. Let \( M \in S^n_0 \) and \( i \) be the inversion of \( \mathbb{R}^n \backslash \{0\} \) with respect to \( S^{n-1} \):
\[ i(x) := \frac{x}{\|x\|^2}. \]
Then define the star duality \( M^\circ \) of \( M \) by
\[ M^\circ = cl(\mathbb{R}^n \backslash \{i(M)\}). \]
Meanwhile, Mozyškis (see[24]) proved that for \( M \in S^n_0 \) and any \( w \in S^{n-1} \), the star dual \( M^\circ \) satisfies
\[ \rho(M^\circ, w) = \frac{1}{\rho(M, w)}. \]  
(1.10)

With the emergence of this notion, a number of characterizations and inequalities were established about star bodies (see[15], [16], [17], [25], [47], [54]). Especially, \((-M)^\circ = -M^\circ \).

Next, according to the star duality of star bodies, we obtain the extremum of star dual form for general log radial bodies.

**Theorem 2.** If \( M \in S^n_0 \), \( i \in [-1, 1] \), then for arbitrary real number \( i \neq n \), we have
\[ \bar{W}_i(\Delta^\ell_0 M) \leq \bar{W}_i(\Delta^\ell_0 M) \leq \bar{W}_i(M^\circ). \]  
(1.11)
If \( M \notin S^n_0 \), then \( \bar{W}_i(\Delta^\ell_0 M) = \bar{W}_i(\Delta^\ell_0 M) \) if and only if \( \ell = 0 \); \( \bar{W}_i(\Delta^\ell_0 M) = \bar{W}_i(M^\circ) \) if and only if \( i = \pm 1 \).

Finally, the star dual form of Theorem 1.B is also established.

**Theorem 3.** Suppose \( M_1, M_2 \in S^n_0 \), then for every real number \( i < n \), we have
\[ \int_{S^{n-1}} \log \left( \frac{\rho(M_2, w)}{\rho(M_1, w)} \right) d\bar{W}_i(M_2) \leq \frac{1}{n-i} \log \frac{\bar{W}_i(M_2)}{\bar{W}_i(M_1)}, \]
with equality if and only if \( M_1 \) and \( M_2 \) are dilated of each other.

In this article, Section IV is dedicated to prove the Theorems 1.1-1.2. In last Section, we will give the proof of Theorem 1.3. In Section III, some properties of general log radial bodies will be obtained as well.

**II. PRELIMINARIES**

**A. The Radial Combination**

Let \( M, N \in S^n_0 \) and \( a, b \geq 0 \) \((a + b \neq 0)\), then the radial combination, \( a \cdot M + b \cdot N \in S^n_0 \), of \( M \) and \( N \) satisfies (20)
\[ \rho(a \cdot M + b \cdot N, i) = a \rho(M, i) + b \rho(N, i). \]

Here \( \cdot \) denotes the radial addition and \( \cdot \) denotes the radial scalar multiplication.

**B. Dual Quermassintegrals and Dual Mixed Quermassintegrals**

The dual quermassintegrals \( \bar{W}_i(M) \) of \( M \in S^n_0 \) is defined by (see[23])
\[ \bar{W}_i(M) = \frac{1}{n} \int_{S^{n-1}} \rho(M, w)^{n-i} dS(w), \quad \forall \, w \in S^{n-1}. \]

Here \( i \) is an arbitrary real number and \( S \) is the Lebesgue measure on \( S^{n-1} \). Particularly, \( \bar{W}_i(M) = V(M) \).

For \( M, N \in S^n_0 \) and every real number \( i \), the dual mixed quermassintegrals \( \bar{W}_i(M, N) \) is defined as (see[23])
\[ \bar{W}_i(M, N) = \frac{1}{n} \int_{S^{n-1}} \rho(M, w)^{n-i} \rho(N, w) dS(w), \]
\[ \forall \, w \in S^{n-1}. \]

By the Hölder’s integral inequality, a Minkowski inequality for dual mixed quermassintegrals was obtained easily.

**Theorem 2.B. Suppose \( M \in S^n_0 \) and \( i \in R \), then**
\[ \bar{W}_i(M) \cdot \bar{W}_i(M^\circ) \geq \kappa_n^2, \]  
(2.2)
with equality if and only if \( M \) is a centered ball. Here \( \kappa_n \) is the \( n \)-dimensional volume of the standard unit ball \( B \) in \( \mathbb{R}^n \).

**C. The Jessen’s Inequality and Dual Mixed Cone-Volume Probability Measure**

Let \( \mu \) is a probability measure on a space \( \Omega \) and \( h : \Omega \rightarrow D \subset \mathbb{R} \) is a \( \mu \)-integrable function, where \( D \) is a possibly infinite interval, Jessen’s inequality can be stated that if \( \varphi : D \rightarrow \mathbb{R} \) is a concave function, then
\[ \int_{\Omega} \varphi(h(z))d\mu(z) \leq \varphi \left( \int_{\Omega} h(z)d\mu(z) \right). \]  
(2.3)
When \( \varphi \) is strictly concave, the equality holds if and only if \( h(z) \) is a constant for \( \mu \)-almost every \( z \in \Omega \) (see[9]).

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If $M \in \mathcal{S}^{n}$, then the dual mixed cone-volume probability measure shall be written (see[35])
\[
\hat{V}_{i,M}(\cdot) = \frac{\rho(M, \cdot)\nu^{n-i}dS(\cdot)}{\nu_{W_{i}}(M)}.
\] (2.4)

Specially, the measure $\hat{V}_{0,0,M}(\cdot) = \hat{V}_{M}(\cdot)$ denotes the dual cone-volume probability measure while $i = 0$.

III. PROPERTIES OF GENERAL LOG RADIAL BODIES

The following properties for general log radial bodies will be established.

**Theorem 3.1.** If $M \in \mathcal{S}^{n}_{0}$, $\iota \in [-1, 1]$, then for $\phi \in GL(n)$, we have
\[
\widetilde{\Delta}_{0}^{\iota}(\phi M) = \phi \widetilde{\Delta}_{0}^{\iota}M.
\]

**Proof.** Combining with (1.1) and (1.4), it yields that for all $w \in \mathcal{S}^{n-1}$, we obtain
\[
\rho(\widetilde{\Delta}_{0}^{\iota}(\phi M), w) = \rho(\phi M, w)\gamma_{1}(\iota)\rho(-\phi M, w)\gamma_{2}(\iota)
= \rho(M, \phi^{-1}w)\gamma_{1}(\iota)\rho(-M, \phi^{-1}w)\gamma_{2}(\iota)
= \rho(\widetilde{\Delta}_{0}^{\iota}M, \phi^{-1}w)
= \rho(\phi \widetilde{\Delta}_{0}^{\iota}M, w).
\]

Therefore, $\widetilde{\Delta}_{0}^{\iota}(\phi M) = \phi \widetilde{\Delta}_{0}^{\iota}M$.

**Theorem 3.2.** Let $M \in \mathcal{S}^{n}_{0}$ and $\iota \in [-1, 1]$. If $\iota \neq 0$, then
\[
\widetilde{\Delta}_{0}^{\iota}M = \widetilde{\Delta}_{0}^{-\iota}M \iff M \in \mathcal{S}^{n}_{0}.
\]

**Proof.** Since
\[
\rho(\widetilde{\Delta}_{0}^{\iota}M, w) = \rho(M, w)\gamma_{1}(\iota)\rho(-M, w)\gamma_{2}(\iota).
\] (3.1)

According to (1.4) and (1.7), we obtain that for any $w \in \mathcal{S}^{n-1}$,
\[
\rho(\widetilde{\Delta}_{0}^{-\iota}M, w) = \rho(M, w)\gamma_{1}(-\iota)\rho(-M, w)\gamma_{2}(-\iota)
= \rho(M, w)\gamma_{2}(\iota)\rho(-M, w)\gamma_{1}(\iota).
\] (3.2)

If $M \in \mathcal{S}^{n}_{0}$, i.e. $M = -M$, then from (3.1) and (3.2), we have
\[
\rho(\widetilde{\Delta}_{0}^{\iota}M, w) = \rho(\widetilde{\Delta}_{0}^{-\iota}M, w).
\]

So $\widetilde{\Delta}_{0}^{\iota}M = \widetilde{\Delta}_{0}^{-\iota}M$.

On the contrary, if $\widetilde{\Delta}_{0}^{\iota}M = \widetilde{\Delta}_{0}^{-\iota}M$, then
\[
\rho(\widetilde{\Delta}_{0}^{\iota}M, w) = \rho(\widetilde{\Delta}_{0}^{-\iota}M, w).
\]

Using (3.1) and (3.2), we see
\[
\rho(M, w)\gamma_{1}(\iota)\rho(-M, w)\gamma_{2}(\iota) = \rho(M, w)\gamma_{2}(\iota)\rho(-M, w)\gamma_{1}(\iota).
\]

Then
\[
\left(\frac{\rho(M, w)}{\rho(-M, w)}\right)^{g_{1}(\iota)} = \left(\frac{\rho(-M, w)}{\rho(M, w)}\right)^{g_{2}(\iota)} = 1.
\]

Namely,
\[
\left(\frac{\rho(M, w)}{\rho(-M, w)}\right)^{g_{1}(\iota) - g_{2}(\iota)} = 1.
\]

From $\iota \neq 0$, it yields $g_{1}(\iota) - g_{2}(\iota) \neq 0$. Hence $\rho(M, w) = \rho(-M, w)$, i.e. $M \in \mathcal{S}^{n}_{0}$.

By Theorem 3.2, the following corollary is hold.

**Corollary 3.1.** Let $M \in \mathcal{S}^{n}_{0}$ and $\iota \in [-1, 1]$. If $M$ is not origin-symmetric, then
\[
\widetilde{\Delta}_{0}^{\iota}M = \widetilde{\Delta}_{0}^{-\iota}M \iff \iota = 0.
\]

**Theorem 3.3.** If $M \in \mathcal{S}^{n}_{0}$ and $\iota \in [-1, 1]$, then
\[
\widetilde{\Delta}_{0}^{-\iota}M = \widetilde{\Delta}_{0}^{-\iota}(-M) = -\widetilde{\Delta}_{0}^{\iota}M.
\] (3.3)

**Proof.** Using (3.2) and (1.4), for every $w \in \mathcal{S}^{n-1}$, we have
\[
\rho(\widetilde{\Delta}_{0}^{-\iota}M, w) = \rho(M, w)\gamma_{1}(-\iota)\rho(-M, w)\gamma_{2}(\iota)
= \rho(-M, w)\gamma_{1}(\iota)\rho(-(-M), w)\gamma_{2}(\iota)
= \rho(\widetilde{\Delta}_{0}^{\iota}(-M), w).
\]

So $\widetilde{\Delta}_{0}^{-\iota}M = \widetilde{\Delta}_{0}^{-\iota}(-M)$.

Furthermore, from the definition (1.4), it yields that for every $w \in \mathcal{S}^{n-1}$,
\[
\rho(-\widetilde{\Delta}_{0}^{-\iota}M, w) = \rho(\widetilde{\Delta}_{0}^{\iota}M, -w)
= \rho(M, -w)\gamma_{1}(\iota)\rho(-(-M), -w)\gamma_{2}(\iota)
= \rho(-M, w)\gamma_{1}(\iota)\rho(-(-M), w)\gamma_{2}(\iota)
= \rho(\widetilde{\Delta}_{0}^{\iota}(-M), -w).
\]

Hence $-\widetilde{\Delta}_{0}^{\iota}M = \widetilde{\Delta}_{0}^{-\iota}(-M)$. This gives (3.3).

**Theorem 3.4.** If $M \in \mathcal{S}^{n}_{0}$ and $\iota \in [-1, 1]$, then
\[
\widetilde{\Delta}_{0}^{\iota}M = M.
\] (3.4)

**Proof.** Since $M \in \mathcal{S}^{n}_{0}$, namely $M = -M$, By the definition (1.4) and (1.6), we see that for every $w \in \mathcal{S}^{n-1}$,
\[
\rho(\widetilde{\Delta}_{0}^{\iota}M, w) = \rho(M, w)\gamma_{1}(\iota)\rho(-M, w)\gamma_{2}(\iota) = \rho(M, w).
\]

This yields (3.4).

Theorem 3.4 immediately implies the following corollary.

**Corollary 3.2.** If $M_{1}, M_{2} \in \mathcal{S}^{n}_{0}$ and $\iota \in [-1, 1]$, then
\[
\widetilde{\Delta}_{0}^{\iota}M_{1} = \widetilde{\Delta}_{0}^{\iota}M_{2} \iff M_{1} = M_{2}.
\]

IV. THE EXTREMUM INEQUALITIES OF GENERAL LOG RADIAL BODIES

In this section, we will prove Theorems 1.1-1.2.

**Proof of Theorem 1.1.** According to the Theorem 1.A and (1.6), we infer that for all $\iota \in [-1, 1]$,
\[
\overline{W}_{i}(\widetilde{\Delta}_{0}^{\iota}M) = \overline{W}_{i}\left(\frac{g_{1}(\iota)}{M} \cdot M + g_{2}(\cdot) \cdot (-M)\right)
\leq \overline{W}_{i}(M)\gamma_{1}(\iota)\overline{W}_{i}(\cdot)\gamma_{2}(\iota)
= \overline{W}_{i}(M).
\] (4.1)

This yields the right hand side inequality of (1.9).

Apparently, (4.1) is an equation when $\iota = \pm 1$. So if $\iota \neq \pm 1$, then by the equality condition of Theorem 1.A, we know that the equality holds in (4.1) if and only if $M$ and $-M$ are dilated of each other. It shows $M = -M$, thereby $M \in \mathcal{S}^{n}_{0}$. 

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Therefore, in the right hand side inequality of (1.9), the equality holds if and only if $M \in S^n_{\alpha_0}$ or $\iota = \pm 1$. This implies that if $M \not\in S^n_{\alpha_0}$, then the equality holds in the right hand side inequality of (1.9) if and only if $\iota = \pm 1$.

Next, we complete the proof of left hand side inequality in (1.9). Combining with (1.4), (3.2) and (1.6), we obtain

$$\rho(\tilde{\Delta}_0 M, w)^{\frac{2}{\iota}} \rho(\tilde{\Delta}_0^{-1} M, w)^{\frac{1}{\iota}}$$

for arbitrary $w \in S^{n-1}$. Thus,

$$\tilde{\Delta}_0 M = \frac{1}{2} \tilde{\Delta}_0 M + \frac{1}{2} \tilde{\Delta}_0^{-1} M.$$  \hspace{1cm} (4.2)

From the Theorem 1.A and (3.3), it yields

$$\tilde{W}_i(\tilde{\Delta}_0 M) \leq \tilde{W}_i(\tilde{\Delta}_0 M)^{\frac{2}{\iota}} \tilde{W}_i(\tilde{\Delta}_0^{-1} M)^{\frac{1}{\iota}}$$

$$= \tilde{W}_i(\tilde{\Delta}_0 M)^{\frac{2}{\iota}} \tilde{W}_i(\tilde{\Delta}_0^{-1} M)^{\frac{1}{\iota}}$$

$$= \tilde{W}_i(\tilde{\Delta}_0^{\frac{2}{\iota}} M).$$ \hspace{1cm} (4.3)

This gives the left hand side inequality of (1.9).

Clearly, (4.3) is an identity when $\iota = 0$. Hence if $\iota \neq 0$, then by the condition of equality in Theorem 1.A, the equality holds in (4.3) if and only if $\tilde{\Delta}_0 M$ and $\tilde{\Delta}_0^{-1} M$ are dilates. This means $\tilde{\Delta}_0 M = \tilde{\Delta}_0^{-1} M$. So from Corollary 3.1, one can verify that if $M \not\in S^n_{\alpha_0}$, then the equality holds in the left hand side inequality of (1.9) if and only if $\iota = 0$.

By Theorem 1.1 and (2.2), we can easily deduce the following conclusion.

**Corollary 4.1.** If $M \in S^n_{\alpha_0}$ and $\iota \in [-1, 1]$, then for real number $i \neq n$, we have

$$\tilde{W}_i(M)\tilde{W}_i(\tilde{\Delta}_0 M) \geq \kappa_n^{\iota};$$  \hspace{1cm} (4.4)

and

$$\tilde{W}_i(M)\tilde{W}_i(\tilde{\Delta}_0^{-1} M) \geq \kappa_2^{\iota}. \hspace{1cm} (4.5)$$

In (4.4), the equality holds if and only if $\tilde{\Delta}_0 M$ is a centered ball; in (4.5), the equality holds if and only if $\tilde{\Delta}_0^{-1} M$ is a centered ball.

For the proof of Theorem 1.2, the following lemma is essential.

**Lemma 4.1.** Suppose $M_1, M_2 \in S^n_{\alpha_0}$ and $\lambda \in [0, 1]$, then for the real number $i \neq n$,

$$\tilde{W}_i\left(\left(1 - \lambda\right) \cdot M_1 \tilde{+}_0 \lambda \cdot M_2\right)^{\circ} \leq \tilde{W}_i(M_1^{\circ})^{1-\lambda} \tilde{W}_i(M_2^{\circ})^{\lambda}.$$ \hspace{1cm} (4.6)

When $0 < \lambda < 1$, the equality holds if and only if $M_1$ and $M_2$ are dilates.

**Proof.** Note that $\frac{1}{\iota^2} > 1$ when $\lambda \in (0, 1)$. By (1.10), (1.2) and the Hölder’s integral inequality, we have that for arbitrary $w \in S^{n-1}$,

$$\tilde{W}_i\left(\left(1 - \lambda\right) \cdot M_1 \tilde{+}_0 \lambda \cdot M_2\right)^{\circ}$$

$$= \frac{1}{n} \int_{S^{n-1}} \rho\left(\left(1 - \lambda\right) \cdot M_1 \tilde{+}_0 \lambda \cdot M_2, w\right)^{\iota - 1} dS(w)$$

$$= \frac{1}{n} \int_{S^{n-1}} \rho(1 - \lambda, M_1 \tilde{+}_0 \lambda \cdot M_2, w)^{\iota - 1} dS(w)$$

$$= \frac{1}{n} \int_{S^{n-1}} \rho(\tilde{M}_1, w)^{(1 - \lambda)(n - i)} \rho(\tilde{M}_2, w)^{\lambda(n - i)} dS(w)$$

$$= \frac{1}{n} \int_{S^{n-1}} \rho(\tilde{M}_1, w)^{(1 - \lambda)(n - i)} \frac{1}{\iota} dS(w)^{1-\lambda}$$

$$\leq \left[\frac{1}{n} \int_{S^{n-1}} \rho(\tilde{M}_1, w)^{(1 - \lambda)(n - i)} \frac{1}{\iota} dS(w)^{1-\lambda}\right]\frac{1}{n} \int_{S^{n-1}} \rho(\tilde{M}_2, w)^{\lambda(n - i)} dS(w)^{\lambda}$$

$$= \tilde{W}_i(M_1^{\circ})^{1-\lambda} \tilde{W}_i(M_2^{\circ})^\lambda.$$  \hspace{1cm} (4.7)

This obtains the right hand side inequality of (1.11).

If $\iota \neq \pm 1$, then from the equality condition of (4.6), the equality holds in (4.7) if and only if $M$ and $\tilde{M}$ are dilated of each other. It means $M = -M$, thereby $M \in S^n_{\alpha_0}$.

Therefore, in the right hand side inequality of (1.11), the equality holds if and only if $M \in S^n_{\alpha_0}$, or $\iota = \pm 1$. This implies that if $M \not\in S^n_{\alpha_0}$, then in the right hand side inequality of (1.11), the equality holds if and only if $\iota = \pm 1$.

The other side of the coin, from (4.2), (4.6) and (3.3), we obtain

$$\tilde{W}_i(\tilde{\Delta}_0 M) = \tilde{W}_i\left(\left(1 - \lambda\right) \cdot M_1 \tilde{+}_0 \lambda \cdot M_2\right)^{\circ}$$

$$\leq \tilde{W}_i\left(\tilde{\Delta}_0^{\circ} M\right)^{\frac{2}{\iota}} \tilde{W}_i\left(-\tilde{\Delta}_0^{-1} M\right)^{\frac{1}{\iota}}$$

$$= \tilde{W}_i(\tilde{\Delta}_0^{\circ} M).$$  \hspace{1cm} (4.8)
The left hand side inequality of (1.11) is proved. If $t \neq 0$, then by Lemma 4.1, the equality can be obtained in (4.8) if and only if $\tilde{x}_{i}M$ and $\tilde{x}_{i}^{-1}M$ are dilated of each other. It will show $\tilde{x}_{i}M = \tilde{x}_{i}^{-1}M$. So, the Corollary 3.1 tell us that if $M \notin S_{\text{int}}$, then in the left hand side inequality of (1.11), the equality holds if and only if $t = 0$.

V. THE STAR DUAL FORM OF THE DUAL LOG-MINKOWSKI INEQUALITY

In this section, we shall prove the star dual form of the dual log-Minkowski inequality.

Proof of Theorem 1.3. Notice that the logarithmic function $\log(\cdot)$ is concave and increasing on $(0, +\infty)$. So by (2.4), the Jessen’s inequality (2.3) and (2.1), we obtain

$$\int_{S^{n-1}} \log \left( \frac{\rho(M_2^o, w)}{\rho(M_1^o, w)} \right) dV_{1, M_2}(w) = \frac{1}{nW(M_2)} \int_{S^{n-1}} \log \left( \frac{\rho(M_2^o, w)}{\rho(M_1^o, w)} \right) \cdot \rho(M_2, w)^{n-i} dS(w) \leq \log \left( \frac{W_{1}(M_2)}{W_{1}(M_2)} \right) \cdot \frac{\rho(M_2, w)^{n-i}}{\rho(M_2, w)^{n-i}}$$

This yields the inequality (1.12).

Because $\log(\cdot)$ is strictly increasing, by the equality condition of Jessen’s inequality (2.3) and the Minkowski inequality (2.1), we find that the equality holds in (1.12) if and only if $M_1$ and $M_2$ are dilates.

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