A New Three-parameter Lifetime Distribution with Monotone or Non-monotone Hazard Rate Function

Tiefeng Zhu

Abstract—**This article proposes a new distribution for modeling complex life data. The beauty and importance of the new distribution lies in its ability to model both monotone and non-monotone hazard rates that are quite common in lifetime problems and reliability engineering. This new distribution has closed cumulative distribution function (CDF), closed probability distribution function (PDF) and closed hazard rate function (HRF), which are great convenient in practical application. We provide a comprehensive treatment of the mathematical properties of the new distribution and study its statistical properties such as moments, conditional moments and moment generating function etc.**

Index Terms—**Non-monotone, moments, residual life functions.**

I. Introduction

In reliability engineering and survival analysis, the systems with non-monotonic shaped HRF are common [1]. However, some well-known distributions like Rayleigh distribution [2] do not exhibit a non-monotonic shaped HRF and thus they can not be used to model the lifetimes of some complex systems. Hence, many authors like Peng and Yan [3], Roozegar and Jafari [4], Saboor et al. [5], Silva et al.[6], Basheer [7], Mukhtar et al.[8] and Hamed et al.[9] proposed new distributions with non-monotonic shaped hazard rate functions to overcome this shortage.

In this paper, we propose a new distribution and discuss its mathematical and statistical properties including moments, skewness, kurtosis, conditional moment etc. This new distribution has monotonic and non-monotonic hazard rate, which has great flexibility in modelling lifetime data. It also has closed CDF, PDF and HRF, which is convenient in practical application. The CDF, PDF and HRF of the proposed distribution are given by

$$G(x;\alpha,\beta,\gamma)=1-(1-e^{-\beta x^{-\alpha}})^{\gamma}, x>0;\alpha,\beta,\gamma>0, \quad (1)$$

$$g(x;\alpha,\beta,\gamma)=\alpha\beta\gamma e^{-\beta x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})^{\gamma-1}x^{-\alpha-1}, \quad (2)$$

and

$$h(x;\alpha,\beta,\gamma)=\frac{\alpha\beta\gamma e^{-\beta x^{-\alpha}}x^{-\alpha-1}}{1-e^{-\beta x^{-\alpha}}}, \quad (3)$$

respectively.

Manuscript received September 16, 2020; revised March 14, 2021. The research work is supported by the National Natural Science Foundation of China (11861049) and the Natural Science Foundation of Inner Mongolia (2020MS01001; 2020LH01002).

T. F. Zhu (corresponding author) is an Associate Professor of the School of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Hohhot, 010070, P. R. China, e-mail: (tfzhu2016@163.com).

II. Shapes of the PDF and HRF for the new distribution

Lemma 1. $\lim_{x\to 0} g(x;\alpha,\beta,\gamma)=0$.

Proof. By $\lim_{x\to 0} e^{-\beta x^{-\alpha}}=\lim_{x\to 0}\frac{1}{e^{\frac{\beta}{x^{\alpha}}}}=0$ and $\lim_{x\to 0} e^{-\beta x^{-\alpha}} x^{-\alpha-1}=\lim_{x\to 0}\frac{\frac{1}{x^{\alpha+1}}}{e^{\frac{\beta}{x^{\alpha}}}}=0$, then $\lim_{x\to 0} g(x;\alpha,\beta,\gamma)=\lim_{x\to 0}\alpha\beta\gamma e^{-\beta x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})^{\gamma-1}x^{-\alpha-1}=0$.

Lemma 2. $\lim_{x\to +\infty} g(x;\alpha,\beta,\gamma)=0$.

Proof. By $\lim_{x\to +\infty} e^{-\beta x^{-\alpha}}=\lim_{x\to +\infty}\frac{1}{e^{\frac{\beta}{x^{\alpha}}}}=1$ and $\lim_{x\to +\infty} x^{-\alpha-1}=\lim_{x\to +\infty}\frac{1}{x^{\alpha+1}}=0$, then $\lim_{x\to +\infty} g(x;\alpha,\beta,\gamma)=\lim_{x\to +\infty}\alpha\beta\gamma e^{-\beta x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})^{\gamma-1}x^{-\alpha-1}=0$.

The lemmas 1 and 2 imply that $g(x;\alpha,\beta,\gamma)$ converges to 0 as $x\to 0$ or $x\to +\infty$. Plots of the PDF are displayed in Fig.1 for selected parameter values. The plots in Fig.1 reveal how the parameters α, β and γ affect the new distribution's density function. We observe that $g(x;\alpha,\beta,\gamma)$ is always unimodal. It is not possible to compute the mode of $g(x;\alpha,\beta,\gamma)$ explicitly. The mode is defined as the maximal value of the PDF, denoted by x_{mode}, which can be obtained numerically by solving the following nonlinear equation

$$\alpha\beta x_{mode}^{-\alpha-1}-\alpha\beta(\gamma-1)e^{-\beta x_{mode}^{-\alpha}}(1-e^{-\beta x_{mode}^{-\alpha}})^{-1}x_{mode}^{-\alpha-1} -(\alpha+1)x_{mode}^{-1}=0.$$

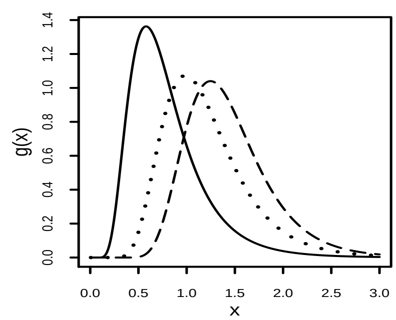


Fig. 1: Plots of the new distribution's density function for selected parameter values (solid line: $\alpha=0.9$, $\beta=2$, $\gamma=10$; dotted line: $\alpha=1.2$, $\beta=3$, $\gamma=9$; long dotted line: $\alpha=1.5$, $\beta=4$, $\gamma=8$.)

Plots of the HRF are displayed in Fig.2 for selected parameter values. The plots in Fig.2 reveal how the parameters α, β and γ affect the HRF. They indicate that the HRF of the new distribution can take the monotone and non-monotone forms.

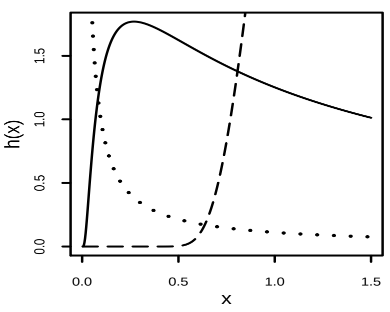


Fig. 2: Plots of the HRF for selected parameter values (solid line: $\alpha = 0.4$, $\beta = 2$, $\gamma = 10$; dotted line: $\alpha = 2$, $\beta = 3$, $\gamma = 12$; long dotted line: $\alpha = 0.1$, $\beta = 0.9$, $\gamma = 1.8$.)

III. General properties of the new distribution

In this section, we study some statistical properties of the new distribution. For the sake of convenience, here and henceforth, let X be a random variable following (2).

A. Moments

Some of the most important features and characteristics of a distribution can be studied through moments(e.g., skewness and kurtosis).

Theorem 1. The kth raw moment of the new distribution, denoted as $\mu_k^{'}$, $k = 1, 2, \cdots$, is given by

$$\begin{aligned} \mu_k^{'} &= E[X^k] \\ &= \alpha^2\gamma \sum_{n=0}^{\infty}(-1)^n \binom{\gamma-1}{n} \frac{\Gamma(-\alpha^2+2k)}{\beta(n-1)^{-\alpha^2+2k}}, \quad (4) \end{aligned}$$

here $\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx$.

Proof. By the definition of the kth raw moment,

$$\begin{aligned} \mu_k^{'} &= \int_0^{+\infty} x^k g(x;\alpha,\beta,\gamma)dx \\ &= \int_0^{+\infty} x^k \alpha\beta\gamma e^{-\beta x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})^{\gamma-1} x^{-\alpha-1}dx. \end{aligned}$$

Using the following expansion of $(1-e^{-\beta x^{-\alpha}})^{\gamma-1}$ given by

$$(1-e^{-\beta x^{-\alpha}})^{\gamma-1} = \sum_{n=0}^{\infty}(-1)^n\binom{\gamma-1}{n}e^{-n\beta x^{-\alpha}},$$

then we have

$$\begin{aligned} \mu_k^{'} &= \alpha\beta\gamma\int_0^{+\infty} x^k x^{-\alpha-1}e^{-\beta x^{-\alpha}}\sum_{n=0}^{\infty}(-1)^n\binom{\gamma-1}{n} \\ &\quad \times e^{-n\beta x^{-\alpha}}dx \\ &= \alpha\beta\gamma\sum_{n=0}^{\infty}(-1)^n\binom{\gamma-1}{n} \\ &\quad \times\int_0^{+\infty} x^{k-\alpha-1}e^{-\beta x^{-\alpha}(1-n)}dx. \end{aligned}$$

By

$$\int_0^{+\infty} x^{k-\alpha-1}e^{-\beta x^{-\alpha}(1-n)}dx = \alpha\frac{\Gamma(-\alpha^2+2k)}{\beta^2(n-1)^{-\alpha^2+2k}},$$

we have $\mu_k^{'} = \alpha^2\gamma\sum_{n=0}^{\infty}(-1)^n\binom{\gamma-1}{n}\frac{\Gamma(-\alpha^2+2k)}{\beta(n-1)^{-\alpha^2+2k}}$.

It is noted that $\mu_k^{'}$ are not easy to compute due to the sum and the gamma function including in the right part of the above formula. Hence, we can approximate the gamma function $\Gamma(a)$ using the Stirlings formula (Tian et al. [10]) as follows $\Gamma(a) \simeq \sqrt{2\pi}a^{a-1/2}e^{-a}$.

B. Quantile function, Skewness and Kurtosis.

The quantile function of X is determined by inverting (1) as

$$Q(u) = G^{-1}(u) = [-\frac{1}{\beta}\log(1-(1-u)^{\frac{1}{\gamma}})]^{-\frac{1}{\alpha}}. \quad (5)$$

The skewness and kurtosis measures can be calculated from the ordinary moments given in (4) using the following well-known expressions,

$$\begin{aligned} \text{skewness}(\beta_1) &= \frac{\mu_3^2}{\mu_2^3}, \quad (6) \\ \text{kurtosis}(\beta_2) &= \frac{\mu_4}{\mu_2^2}, \quad (7) \end{aligned}$$

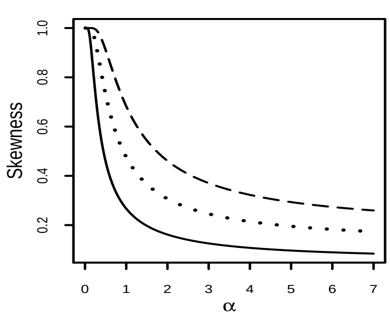
where $\mu_2 = [\mu_2^{'} - (\mu_1^{'})^2]$, $\mu_3 = [\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2(\mu_1^{'})^3]$ and $\mu_4 = [\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6(\mu_1^{'})^2\mu_2^{'} - 3(\mu_1^{'})^4]$. The shortcomings of the skewness (6) and kurtosis (7) measures are approximate computing. Hence, in this paper, we use Bowley's skewness and the Moors' kurtosis to compute skewness and kurtosis, respectively. The Bowley's skewness is based on quartiles

$$S = \frac{Q(\frac{3}{4}) - 2Q(\frac{2}{4}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

and the Moors' kurtosis is based on octiles

$$K = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})},$$

where $Q(\cdot)$ is given by (5). For fixed β and γ, the plots of the measures S and K as a function of α are shown in Fig. 3. These plots reveal that both the measures depend on the parameter α. The skewness and kurtosis tend to decrease as α increases.



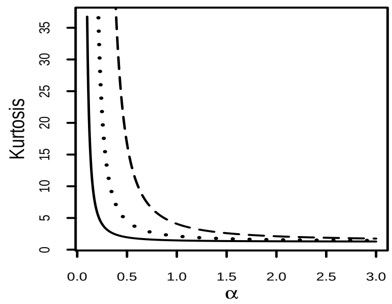


Fig. 3: Plots of the skewness and kurtosis of the new distribution for selected β and γ (solid line: $\beta = 2$, $\gamma = 3$; dotted line: $\beta = 2$, $\gamma = 1$; long dotted line: $\beta = 2$, $\gamma = 0.5$.)

C. The moment generating function

Theorem 2. The moment generating function of (2), denoted as $M_X(t)$, is given by

$$M_X(t) = \alpha^2\gamma \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^j}{j!}\binom{\gamma-1}{n}(-1)^n \frac{\Gamma(-\alpha^2+2j)}{\beta^2(n-1)^{-\alpha^2+2j}}.$$

Proof. By definition of the moment generating function of X, we have

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^{\infty} e^{tX} g(x;\alpha,\beta,\gamma)dx \\ &= \sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^{\infty} x^j g(x;\alpha,\beta,\gamma)dx. \end{aligned}$$

Substituting (4) into the above expression, the result is obtained.

We can easily obtain the characteristic function, $\phi_X(t) = E(e^{itX})$, by replacing t with it in $M_X(t)$. The characteristic function may be a more convenient tool.

D. Conditional moment and moment generating function

It is very important to obtain the conditional measures $E\{X^k \mid X > t\}(k = 1, 2, \cdots)$ and $E\{e^{tX} \mid X > x_0\}$ in reliability analysis.

Theorem 3. The conditional moment $E\{X^k \mid X > t\}$ and the conditional moment generating function $E\{e^{tX} \mid X > x_0\}$ are given by

$$E\{X^k \mid X > t\} = \frac{-\alpha^2\beta\gamma\sum_{n=0}^{\infty}\binom{\gamma-1}{n}(-1)^n\Upsilon(-\alpha^2+2k, t^{-\alpha})}{(1-e^{-\beta t^{-\alpha}})^{\gamma}},$$

and

$$E\{e^{tX} \mid X > x_0\} = \frac{-\alpha^2\beta\gamma\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^j}{j!}\binom{\gamma-1}{n}(-1)^n\Upsilon(-\alpha^2+2k, x_0^{-\alpha})}{(1-e^{-\beta x_0^{-\alpha}})^{\gamma}},$$

where $\Upsilon(a, x) = \int_0^x t^{a-1}e^{-t}dt$.

Proof. By the definitions

$$E\{X^k \mid X > t\} = \frac{1}{\overline{G}(t)}\int_t^{+\infty} x^k g(x)dx$$

and

$$E\{e^{tX} \mid X > x_0\} = \frac{1}{\overline{G}(x_0)}\int_{x_0}^{+\infty} e^{tx}g(x)dx,$$

the result is easily obtained, where $\overline{G}(t) = 1 - G(t)$.

E. Residual life functions with some reliability measures

In reliability, the residual life of non-negative random variable X is defined by $R_{(t)} := X - t \mid X > t$, the $R_{(t)}$ is interpreted as the remaining lifetime of a unit given that it has survival up to time t. The residual life plays a vital role in some areas like engineering, medical science, survival studies, economics and risk theory.

The survival function of the residual life $R_{(t)}$ of the new distribution is

$$S_{R_{(t)}}(x) = \frac{\overline{G}(x+t)}{\overline{G}(t)} = \frac{(1-e^{-\beta(x+t)^{-\alpha}})^{\gamma}}{(1-e^{-\beta t^{-\alpha}})^{\gamma}}, x > 0,$$

the associated PDF is

$$g_{R_{(t)}}(x) = \frac{g(x+t)}{\overline{G}(t)} = \frac{\alpha\beta\gamma e^{-\beta(x+t)^{-\alpha}}[(1-e^{-\beta(x+t)^{-\alpha}})^{\gamma-1}](x+t)^{(-\alpha-1)}}{(1-e^{-\beta t^{-\alpha}})^{\gamma}},$$

and the HRF of $R_{(t)}$ is

$$h_{R_{(t)}}(x) = \frac{g_{R_{(t)}}(x)}{S_{R_{(t)}}(x)} = \frac{\alpha\beta\gamma e^{-\beta(x+t)^{-\alpha}}[(1-e^{-\beta(x+t)^{-\alpha}})^{\gamma-1}](x+t)^{(-\alpha-1)}}{(1-e^{-\beta(x+t)^{-\alpha}})^{\gamma}}.$$

The mean and variance of residual lifetime have been studied in reliability, statistics and survival analysis. Many useful result have been derived. We refer the reader to Gupta and Kirmani [11].

Theorem 4. The mean and variance of residual life associated with X are given by

$$\begin{aligned} E(R_{(t)}) &= \frac{\alpha\beta\gamma}{(1-e^{-\beta t^{-\alpha}})^{\gamma}}\sum_{n=0}^{\infty}\binom{\gamma-1}{n}(-1)^n \\ &\quad \times\frac{-\alpha}{[-\beta(n+1)]^{\alpha+1}}\Upsilon(\alpha+1, t^{-\alpha}) - t \end{aligned}$$

and

$$\begin{aligned} Var(R_{(t)}) &= \frac{1}{\overline{G}(t)}\Big[E(X^2) - \alpha^2\gamma\sum_{n=0}^{\infty}\binom{\gamma-1}{n}(-1)^{n+1} \\ &\quad \times\frac{1}{-\beta(n+1)^{-\alpha^2+2\alpha}}\Gamma(-\alpha^2+2\alpha, t^{-\alpha})\Big] \\ &\quad -t^2 - 2tE(R_{(t)}) - [E(R_{(t)})]^2, \end{aligned}$$

where $\Gamma(a, x) = \int_x^{+\infty} t^{a-1}e^{-t}dt$.

Proof. By the definitions

$$E(R_{(t)}) = \frac{1}{\overline{G}(t)}\int_t^{+\infty} ug(u)du - t$$

and

$$\begin{aligned} Var(R_{(t)}) &= \frac{2}{\overline{G}(t)}\int_t^{+\infty} u\overline{G}(u)du - 2tE(R_{(t)}) \\ &\quad -[E(R_{(t)})]^2 \end{aligned}$$

$$= \frac{1}{\overline{G}(t)}[E(X^2) - \int_0^t u^2 g(u)du] \\ -t^2 - 2tE(R_{(t)}) - [E(R_{(t)})]^2,$$

the result is easily obtained.

F. Distribution of order statistics

The PDF and CDF of the ith order statistics of size n from (2), say $Y_{(i)}$, are given by

$$\begin{aligned} g_Y(y) &= \frac{n!}{(i-1)!(n-i)!}[G(y)]^{i-1}[1-G(y)]^{n-i}g(y) \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{l=0}^{n-i}\binom{n-i}{l}(-1)^l G^{i+l-1}(y) \\ &\quad \times g(y) \end{aligned}$$

and

$$\begin{aligned} G_Y(y) &= \sum_{j=i}^{n}\binom{n}{j}[G(y)]^j[1-G(y)]^{n-j} \\ &= \sum_{j=i}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l G^{j+l}(y), \end{aligned}$$

respectively, for $i = 1, 2, \cdots, n$.

Thus, the PDF and the CDF of the ith order statistics of (2) are obtained as

$$\begin{aligned} g_Y(y) &= \frac{\alpha\beta\gamma n!}{(i-1)!(n-i)!}\sum_{l=0}^{n-i}\binom{n-i}{l}(-1)^l \\ &\quad \times[1-(1-e^{-\beta y^{-\alpha}})^{\gamma}]^{i+l-1}e^{-\beta y^{-\alpha}} \\ &\quad \times(1-e^{-\beta y^{-\alpha}})^{\gamma-1}y^{-\alpha-1} \end{aligned}$$

and

$$\begin{aligned} G_Y(y) &= \sum_{j=i}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l \\ &\quad \times[1-(1-e^{-\beta y^{-\alpha}})^{\gamma}]^{j+l}. \end{aligned}$$

G. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have been used in economics to study income and poverty. Nowadays, these curves have many applications not only in economics to study income and poverty but also in other sciences including demography, insurance, medicine and engineering.

The Bonferroni curve $B_G[G(x)]$ for the new distribution is obtain as

$$\begin{aligned} B_G[G(x)] &= \frac{1}{E(X)G(x)}\int_0^x y g(y)dy \\ &= \frac{1}{E(X)G(x)}\int_0^x y\alpha\beta\gamma e^{-\beta y^{-\alpha}} \\ &\quad \times(1-e^{-\beta y^{-\alpha}})^{\gamma-1}y^{-\alpha-1}dy \\ &= \frac{\alpha\beta\gamma}{E(X)G(x)}\int_0^x e^{-\beta y^{-\alpha}}\sum_{n=0}^{+\infty}\binom{\gamma-1}{n} \\ &\quad \times(-1)^n e^{-n\beta y^{-\alpha}}y^{-\alpha}dy \\ &= \frac{\alpha\beta\gamma}{E(X)G(x)}\sum_{n=0}^{\infty}\binom{\gamma-1}{n}(-1)^n \\ &\quad \times\int_0^x e^{-(1+n)\beta y^{-\alpha}}y^{-\alpha}dy. \end{aligned}$$

Let $y^{-\alpha} = Y$, then

$$\begin{aligned} \int_0^x e^{-(1+n)\beta y^{-\alpha}}y^{-\alpha}dy &= -\int_{x^{-\alpha}}^{\infty} Y e^{-(1+n)\beta Y}\alpha Y^{\alpha-1}dY \\ &= -\frac{\alpha}{[(1+n)\beta]^{\alpha+1}}\Gamma(\alpha+1, x^{-\alpha}). \end{aligned}$$

Therefore,

$$\begin{aligned} B_G[G(x)] &= \frac{\alpha\beta\gamma}{E(X)G(x)}\sum_{n=0}^{+\infty}\binom{\gamma-1}{n}(-1)^{n+1} \\ &\quad \times\frac{\alpha}{[(1+n)\beta]^{\alpha+1}}\Gamma(\alpha+1, x^{-\alpha}). \end{aligned}$$

Consequently, the Lorenz curve of the new distribution is

$$\begin{aligned} L_G[G(x)] &= B_G[G(x)] \times G(x) \\ &= \frac{\alpha\beta\gamma}{E(X)}\sum_{n=0}^{+\infty}\binom{\gamma-1}{n}(-1)^{n+1} \\ &\quad \times\frac{\alpha}{[(1+n)\beta]^{\alpha+1}}\Gamma(\alpha+1, x^{-\alpha}). \end{aligned}$$

H. Measure of uncertainty

The entropy of a random variable Z is the measure of variation of uncertainty. There are various entropy measures available in statistics literature but one popular entropy measure is Renyi entropy. If Z is a continuous random variable having PDF $f(\cdot)$, then Renyi entropy is defined as

$$I_R(r) = \frac{1}{1-r}\log\Big\{\int f^r(z)dz\Big\},$$

where $r > 0$ and $r \neq 1$.

For the new distribution, the Renyi entropy is given by

$$\begin{aligned} I_R(r) &= \frac{1}{1-r}\log\Big\{\int_0^{\infty}[\alpha\beta\gamma e^{-\beta x^{-\alpha}} \\ &\quad \times(1-e^{-\beta x^{-\alpha}})^{\gamma-1}x^{-\alpha-1}]^r dx\Big\} \\ &= \frac{1}{1-r}\log\Big\{(\alpha\beta\gamma)^r\int_0^{\infty}e^{-\beta r x^{-\alpha}} \\ &\quad \times(1-e^{-\beta x^{-\alpha}})^{\gamma r-r}x^{-\alpha r-r}dx\Big\} \\ &= \frac{1}{1-r}\log\Big\{(\alpha\beta\gamma)^r\int_0^{\infty}e^{-\beta r x^{-\alpha}} \\ &\quad \times\sum_{n=0}^{\infty}\binom{\gamma r-r}{n}(-1)^n e^{-n\beta x^{-\alpha}}x^{-\alpha r-r}dx\Big\} \\ &= \frac{1}{1-r}\log\Big\{(\alpha\beta\gamma)^r\sum_{n=0}^{\infty}\binom{\gamma r-r}{n}(-1)^n \\ &\quad \times\int_0^{\infty}e^{-\beta(r+n)x^{-\alpha}}x^{-\alpha r-r}dx\Big\}. \end{aligned}$$

Let $x^{-\alpha} = u$, then

$$\begin{aligned} &\int_0^{\infty}e^{-\beta(r+n)x^{-\alpha}}x^{-\alpha r-r}dx \\ &= -\alpha\int_0^{\infty}u^{-\alpha^2 r-\alpha r+\alpha-1}e^{-\beta(r+n)u}du \\ &= \frac{1}{\beta(r+n)^{-\alpha^2 r-\alpha r+\alpha}}\int_0^{\infty}[\beta(r+n)u]^{-\alpha^2 r-\alpha r+\alpha-1} \end{aligned}$$

$$\begin{aligned} & \times e^{-\beta(r+n)u} d[\beta(r+n)u] \\ = & \frac{\Gamma(-\alpha^2 r - \alpha r + \alpha)}{\beta(r+n)^{-\alpha^2 r - \alpha r + \alpha}}. \end{aligned}$$

Therefore,

$$\begin{aligned} I_R(r) &= \frac{1}{1-r} \log \Big\{ (\alpha\beta\gamma)^r \sum_{n=0}^{\infty} \binom{\gamma r - r}{n} \\ & \times (-1)^n \frac{\Gamma(-\alpha^2 r - \alpha r + \alpha)}{\beta(r+n)^{-\alpha^2 r - \alpha r + \alpha}} \Big\}. \end{aligned}$$

IV. Concluding remarks

In this paper, we propose a new model for complex systems. The attractiveness of the proposed model is that it has closed cumulative distribution function, closed probability distribution function, closed hazard rate function, monotonic and non-monotonic hazard rate functions. Moreover, it has little number of parameters. These properties make it very useful to analyse lifetime data in engineering. Its mathematical and statistical properties have been discussed. Furthermore, explicit expressions for Bonferroni and Lorenz curves and Renyi entropy measure of the new distribution have been derived. We hope that the proposed model may attract wider applications in engineering use.

References

[1] E. T. Lee, and J. W. Wang, *Statistical Methods for Survival Data Analysis, 3rd edition.* Hoboken, New Jersey: John Wiley & Sons, Inc., 2013.

[2] A. V. Zakharov, O. V. Chernoyarov, A. V. Salnikova, and A. N. Faulgaber, “The distribution of the absolute maximum of the discontinuous stationary random process with Raileigh and Gaussian components,” *Engineering Letters*, vol. 27, no. 1, pp. 53-65, 2019.

[3] X. Y. Peng, and Z. Z. Yan, “Estimation and application for a new extended Weibull distribution,” *Reliability Engineering and System Safety*, vol. 121, pp. 34-42, 2014.

[4] R. Roozegar, and A. A. Jafari, “On bivariate exponentiated extended Weibull family of distributions,” *Statistics*, vol. 37, no. 2, pp. 564-576, 2015.

[5] A. Saboor, I. Elbatal, and M. C. Gauss, “The transmuted exponentiated Weibull geometric distribution: theory and applications,” *Hacettepe Journal of Mathematics and Statistics*, vol.45, no. 3, pp. 973-987, 2016.

[6] G. O. Silva, M. M. Ortega Edwin, and G. M. Cordeiro, “The beta modified Weibull distribution,” *Lifetime Data Analysis*, vol. 16, pp. 409-430, 2010.

[7] A. M. Basheer, “Alpha power inverse Weibull distribution with reliability application,” *Journal of Taibah University for Science*, vol. 13, no. 1, pp. 423-432, 2019.

[8] S. Mukhtar, A. Ali, and A. M. Alya, “Mc-donald modified Burr-III distribution: properties and applications,” *Journal of Taibah University for Science*, vol. 13, no. 1, pp. 423-432, 2019.

[9] A. Hamed, M. Alsharef, and R. K. Rao, “Analysis of energy and spectral efficiency in urban shadowing environment,” *IAENG International Journal of Computer Science*, vol. 43, no. 2, pp. 237-244, 2016.

[10] Y. Z. Tian, M. Z. Tian, and Q. Q. Zhu, “A new generalized linear exponential distribution and its applications,” *Acta Mathematicae Applicatae Sinica, English Series*, vol. 30, no. 4, pp. 1049-1062, 2014.

[11] R. C. Gupta, and S. N. U. A. Kirmani, “Residual coefficient of variation and some characterization results,” *Journal of Statistical Planning and Inference*, vol. 91, pp. 23-31, 2000.