An Introduction of a New Square Quadratic Proximal Point Scalarization Method for Multiobjective Programming

Ali Ou-yassine, *Member IAENG* and Salah-ddine Krit.

***Abstract*—Inspired by our finding in (Commun. Korean Math. Soc. 2019 Vol. 34, No. 2, 671—684), we suggest a new modified approach to proximal point scalarization methods (applied to multiobjetive programming problems). This method provides a new variant of φ-divirgence functions which is diffrent from the tow last variants used in [6] and [12].**
Our main contribution is the proposition of a new square root quadratic term in the regularized subproblem. In other words, the nonegative variables employed in the scalarization are placed in the square term instead of using the logarithmic term of Auslender et al. as regularization. Next, the unconstrained variables are introduced in the quadratic term. Finally, we prove that each limit point of the sequence generated by the method is a weak Pareto solution.

***Index Terms*—Multiobjective programming, proximal point algorithm, scalar representations, divirgence fonctions, square-quadratic regularization, logarithm-quadratic term.**

I. Introduction

This work considers the unconstrained multiobjective programming problem

$$min\{F(x): \quad x \in R^n\}, \tag{1}$$

where F is a convex mapping from R^n to R^m.
This class of problems has been addressed by Kaisa Miettinen at Ph.D. thesis in early 1999. Thereafter, it is included in the more general programming problems known as vector optimization [19], see Chinchuluun and Pardalos survey [7] for the other versions of multiobjective programming problems. This problem has received much attention in literature.

A bibliography of many multiobjective mathematical programming applications is presented by White at [24]. More information about real-life problems related to the economy, finance, industry and health services. It can be found at [21]. For some recent applications, see [25] and [13].

Evolutive algorithms [[9], [8] and [23]], and proximal point algorithms (PPA) [22] have been recognized as the most popular numerical methods and powerful algorithms solving the problem (1).

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Ali Ou-yassine is a Professor of Mathematics, Informatics and Management Department, Member of Engineering Sciences and Energies Laboratory, Polydisciplinary Faculty of Ouarzazate, B.P : 638, 45000, Ibn Zohr University, Morocco (e-mail: a.ouyassine@yahoo.com).

Salah-ddine Krit is a Professor of Mathematics, Informatics and Management Department, Head of Engineering Sciences and Energies Laboratory, Polydisciplinary Faculty of Ouarzazate, B.P : 638, 45000, Ibn Zohr University, Morocco (e-mail: salahddine.krit@gmail.com).

A classical method to solve (1) is the quadratic proximal point method for minimizing a closed proper convex function f in R^n generates a sequence $\{x^k\} \subset R^n$ such that, given x^0

$$\text{(PPA)} \qquad x^{k+1} \in argmin\{f(x) + \lambda_k q(x, x^k)\}, \tag{2}$$

where

$$q(x, x^k) = \frac{1}{2}\|x - x^k\|^2 \tag{3}$$

is a quadratic function of x.

Various efforts have been devoted to replacing the usual quadratic term with some kind of functions like-distances such as Bregman Distances and φ-divergences, see [10], [15] and [17].
Several interesting algorithms and condensed developments have been directed to extend the different versions of proximal point methods to a multiobjective case and vector optimization. For more details, check, Miettinem [21], Gopfert et al. [11], Bonnel et al. [4].

In 2010, Gregorio and Oliveira [12] developed the proximal point scalarization method with a variant of the logarithmic quadratic function of Auslender et al. [1] and [2]. More precisely, the scalar representation $\langle F(z), z\rangle$ has been used to convert problem (1) into a single objective optimization problems or a family of such problems with a real-valued objective function to be minimized, called scalarized problem, see [21]. Then, the unconstrained variables in the domain of F are introduced in the quadratic term. The nonegative variables employed in the scalarization are placed in the logarithmic term.

More recently, Castillo and Quintana [6] proposed a generalisation of the method in [12]. Moreover, they used two different scalar representations and two convex functions defined on the positive orthant for penalizing positive variables.

In this work, Instead of using the logarithmic quadratic regularization in [12] and the two convex functions defined on the positive orthant for penalizing positive variables intoduced in [6]. We introduce a new square root quadratic term in the regularized subproblem. Therefore, we propose the modified proximal point method based on scalar representation and over a new diffrent variant of the regularization. The next section presents preliminary results and basic concepts for a better understanding of the problem at hand.
In Sec. 3, we present the φ-divergence function used in

the proximal method as variant of the logarithm-quadratic regularization introduced in [12] and the inverse barrier term used in [6], convergence results are established.

II. Preliminaries

We list some fundamental basic concepts that are useful in the consequent analysis.

Definition 1 Let $y, \bar{y} \in R^m$ be vectors. We have,

$$y \leq \bar{y} \Leftrightarrow y_i \leq \bar{y}_i, \qquad i = 1, ..., m,$$

$$y < \bar{y} \Leftrightarrow y_i \leq \bar{y}_i, \qquad i = 1, ..., m,$$

with the strict inequality assured at least one indice and

$$y \ll \bar{y} \Leftrightarrow y_i < \bar{y}_i, \qquad \forall i = 1, ..., m.$$

It is easy to see that $\leq$ satisfies the axioms of partial order relation in R^m. In a more general closed convex pointed cone K on R^m we can build a partial order relation $\leq_K$ assuming that
$y \leq_K \bar{y} \quad if \quad \bar{y} - y \in K \ (y <_K \bar{y} \quad if \quad \bar{y} - y \in int(K))$.

Definition 2 We say that $a \in R^n$ is a local pareto solution to the problem (1) if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ satisfying $F(x) < F(a)$.

Definition 3 $a \in R^n$ is known as weak local pareto solution if there is a disc $B_\delta(a) \subset R^n$, with $\delta > 0$, such that there is no $x \in B_\delta(a)$ satisfying $F(x) \ll F(a)$.

We will denote by $argmin\{F(x)|x \in R^n\}$ and $argmin_w\{F(x)|x \in R^n\}$ the local pareto solution and the local weak pareto solution set to the problem (1). It is easy to see that
$argmin\{F(x)|x \in R^n\} \subset argmin_w\{F(x)|x \in R^n\}$. More details about pareto optimality and multiobjective optimization can be found in Chinchuluun et al. [7].

A. Scalar representation

Scalarization is a concept in vector optimization that plays a fundamental role in the development of methods to solve this class of problems and it is also employed as a tool to get the convergence of other algorithms, such as, for example, the proximal point method presented by Göpfert et al. [4] and Bonnel et al. [5].

Definition 4 A real valued function $f : R^n \to R$ is said to be a strict scalar representation of a map $F : R^n \to R^m$ when given $x, \bar{x} \in R^n$

$$F(x) \leq F(\bar{x}) \Rightarrow f(x) \leq f(\bar{x}),$$

and

$$F(x) \ll F(\bar{x}) \Rightarrow f(x) < f(\bar{x}).$$

Futhermore, we say that f is a weak scalar representation of F if

$$F(x) \ll F(\bar{x}) \Rightarrow f(x) < f(\bar{x}).$$

It is obvious that all strict scalar representations are weak scalar representations. The next result shows an interesting form to get scalar representation for maps.

Proposition 1 Let $f : R^n \to R$ be a function. f is a strict scalar representation of F if, and only if f is a composition of F with a strictly increasing function $g : F(R^n) \to R$.

Proof. Suppose that g is a strictly increasing function. Given $x, y \in R^n$,

$$F(x) \leq F(y) \quad implies \quad g \circ F(x) \leq g \circ F(y).$$

$$F(x) \ll F(y) \quad implies \quad g \circ F(x) < g \circ F(y).$$

We have that $f = g \circ F$ is a strict scalar representation of F.
Now, suppose that f is a strict scalar representation. We must build a function g. Given $z \in F(R^n)$, we put $g(z) = f(x)$, for any $x \in R^n$, such that $F(x) = z$. We can see that f is well defined. Of course, if exists $y \in R^n$ with $F(y) = z$ we have that $F(x) \leq F(y)$ and $F(y) \leq F(x)$. The definition of scalar representation implies $f(x) = f(y)$. This fact shows that $g(z)$ independs from the choice of $x \in R^n$, such that $F(x) = z$. It is easy to see that g is an strictly increasing function. In fact, given $z, w \in F(R^n)$, i.e., $z = F(x)$ and $w = F(y)$, for any $x, y \in R^n$, if $z \leq w$ (respectively, $z \ll w$) then $g(z) = f(x) \leq f(y) = g(w)(respectively, g(z) < g(w))$.
Our argumentation employs the next result to establish the convergence of the method proposed in the Sect. 5. Note that we search for a weak pareto solution for the multiobjective optimization problem (1).

Proposition 2 Let $f : R^n \to R$ be a weak scalar representation of a map $F : R^n \to R^m$ and $argmin\{f(x)|x \in R^n\}$ the local minimizer set of f. We have the inclusion

$$argmin\{f(x)|x \in R^n\} \subset argmin_w\{F(x)|x \in R^n\}.$$

Proof. The Proposition follows immediately from the Definition 4.

B. Scalar representation and convexity

Definition 5 We say that $F : R^n \to R^m$ is a convex map if, and only if, for every $x, y \in R^n$ and $\lambda \in (0, 1)$,

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y). \qquad (1)$$

Under this assumption, the problem (1) is said convex. The inequality (2) also implies that F is a convex map if, and only if, each component $F_i : R^n \to R, i = 1, ..., m$, is a convex function. The relevance of the convexity in multiobjective programming is due to the fact that every local (weak) pareto solution is also a global (weak) pareto solution for unconstrained or constrained multiobjective optimization problems. This result is discussed in Theorem 2.2.3, in Miettinen [3].
The Proposition 1 establishes necessary and sufficient conditions to build strict scalar representations of F. According to the Proposition 2.9 in Luc [1], to get the convexity of the scalar problem we must choose a convex increasing function g from $F(R^n)$ to R. The function $g_z : R^m \to R$ given by $g_z(y) = \langle y, z \rangle$, with $z \in R^m_+ \ \{0\}$ fixed, is an example of convex increasing function that we can compose with F to get a convex escalar strict representation f of F.

Bonnel et al. [5] employ this representation to get the convergence of the classical proximal point algorithm extended to vector optimization. Göpfert et al. [4] also present a scalar proximal point algorithm with Bregman distances in the same lines for vector optimization on spaces of finite dimension. Note that $g_z \circ F(x)$ is convex just when z is fixed. If z is a variable incorporated to the problem we can not guarantee the convexity of the function $f(x,z) = \langle F(x), z\rangle$.
In our case, we assume the existence of a convex function $f : R^n \times R^m_+ \to R$ that satisfies the following properties
$(P1)$ f is bounded below for any $\alpha \in R$, i.e, $f(x,z) \geq \alpha$ for every $(x,z) \in R^n \times R^m_+$;
$(P2)$ f is convex in $Rn \times R^m_+$, i.e., given $(x_1,z_1),(x_2,z_2) \in R^n \times R^m_+$ and $\lambda \in (0,1)$

$$f(\lambda(x_1,z_1)+(1-\lambda)(x_2,z_2)) \leq \lambda f(x_1,z_1)+(1-\lambda)f(x_2,z_2);$$

$(P3)$ f is a strict scalar representation of F, with respect to x, i.e.,

$$F(x) \leq F(y) \Rightarrow f(x,z) \leq f(y,z),$$

and

$$F(x) \ll F(y) \Rightarrow f(x,z) < f(y,z),$$

for every $x,y \in R^n$ and $z \in R^m_+$;
$(P4)$ f is differentiable, with respect to z and

$$\frac{\partial}{\partial z} f(x,z) = h(x,z),$$

where $h(x,z) = (h_1(x,z),...,h_m(x,z))^T$ is a continuous map from $R^n \times R^m$ to R^m_+, i.e, $h_i(x,z) \geq 0$ for all $i = 1,...,m$.
The set of functions that satisfy these properties is nonempty, for example, we have

(1) (see [12])

$$f(x,z) = \sum_{i=1}^{m} \exp(z_i + F_i(x)) \tag{2}$$

(2) (see [12]) $f(x,z) = \sum_{i=1}^{m}[z_i + h(F_i(x))]$ with

$$h(x) = \begin{cases} \frac{1}{2-x} & x \leq 1 \\ x^2 & x > 1 \end{cases} \tag{3}$$

(3) (see [6])

$$f(x,z) = \sum_{i=1}^{m} g(z_i + h(F_i(x))), \tag{4}$$

where $g(w) = w + \frac{w+\sqrt{1+w^2}}{2}$, such that $g \in G : R \to R_{++}$ is a crescent function satisfying some proprieties of the family functions G. For more informations about the family functions G, see [6]. The function h is given by

$$\begin{cases} \frac{x}{1-x} & x \leq 0 \\ x^3 & x > 0, \end{cases} \tag{5}$$

or

$$\begin{cases} \frac{x}{1-x} & x \leq 0 \\ x^2 & x > 0. \end{cases} \tag{6}$$

Note that the assumption $(P1)$ is not a strong hypothesis because the definition of f implies that there is no point $(x,z) \in R^n \times R^m_+$ such that $f(x,z) = \infty$ (we use this notation to mean that there is no point $(x,z) \in R^n \times R^m_+$ with $\lim_{k\to+\infty} |f(x^k,z^k)| = +\infty$ for all sequence $\{(x^k,z^k)\}_{k\in N} \subset R^n \times R^m_+$ satisfying

$$\lim_{k\to+\infty}(x^k,z^k) = (x,z).$$

On the other hand, we do not exclude the possibility of z vanishing. The set of functions satisfying those properties is not empty. As an example it is easy to show that $f(x,z) = \sum_{i=1}^m \exp(z_i + F_i(x))$ satisfies $(P1)$ to $(P4)$.
In this work, we search for a $x^* \in argmin\{f_{\bar z}(x) = f(x,\bar z) | x \in R^n\}$, for any $\bar z \in R^m_+$. Therefore, by $(P3)$ and Proposition 2 we conclude that x^* is a weak pareto solution for the unconstrained multiobjective optimization problem (1).

Before the introduction of the proximal point scalarisation method, we give in the next subsection some basis about the family of φ-divergence functions that we will use in our approach..

C. Family of φ-divergence functions

Consider $\varphi : R \to R_+$ a closed proper convex function and it must hold the following properties,

1) φ is twice continuously differentiable on $int(R^n) = (0,+\infty)$.
2) φ is strictly convex on its domain.
3) $\lim_{x\to 0^+} \frac{d\varphi(x)}{dx} = -\infty$.
4) $\varphi(1) = \frac{d\varphi(1)}{dx} = 0$ and $\frac{d^2\varphi(1)}{dx^2} > 0$.
5) There exists $\nu \in \left(\frac{1}{2}\frac{d^2\varphi(1)}{dx^2}, \frac{d^2\varphi(1)}{dx^2}\right)$ such that $\forall t > 0$
$$\left(1-\frac{1}{t}\right)\left(\frac{d^2\varphi(1)}{dx^2} + \nu(t-1)\right) \leq \frac{d\varphi(t)}{dx} \leq \frac{d^2\varphi(1)}{dx^2}(t-1).$$

The following few examples of φ functions enjoys many attractive properties for developing efficient algorithms to solve Multiobjective programming problems.

$$\varphi_1(t) = t - log(t) - 1 \tag{7}$$

$$\varphi_2(t) = tlog(t) - t + 1 \tag{8}$$

$$\varphi_3(t) = bt + at^{-b} - (a+b), \quad with \quad a \geq 1, b > 0 \tag{9}$$

$$\varphi_4(t) = (\sqrt{t} - 1)^2. \tag{10}$$

III. Proximal point scalarisation method

Consider the φ-divergence function φ, $F : R^n \to R^m$ and f satisfying $(P1)$ to $(P4)$. Given $(x^0,z^0) \in R^n \times R^m_{++}$, the bounded sequences $\beta^k, \mu^k > 0, k = 0,1,...$, the proximal point scalarization method generates sequences $\{x^k\} \subset R^n, \{z^k\} \subset R^m_+$, with x^{k+1} and z^{k+1} solving the problem

$$(x^{k+1},z^{k+1}) \in argmin\{f(x,z) + \beta^k \sum_{i=1}^{m} \varphi(\frac{z_i}{z_i^k}) + \frac{\mu^k}{2}\|x - x^k\|^2\}, \tag{1}$$

where $x^k \in \Omega^k, z \in R^m_{++}$ and $\Omega^k = \{x \in R^n | F(x) \leq F(x^k)\}$.
Note that φ involves only the variable z and the quadratic term involves the variable x.

In [12], R. Gregorio, P. Oliveira et al. have used $\varphi_1(t)$ as a variant of the logarithmic-quadratic regularisation introduced by Auslender in [2].

Later on, R. Castillo, C. Quintana et al. [6], have proposed a new modified proximal point scalarisation method by using the inverse barrier $\varphi_3(t)$ as φ-divergence function (with $a = b = 1$).

Based on our finding published in [12], this manuscript proposes the φ-divergence function $\varphi_4(t)$.

A. Square Quadratic Proximal Point scalarisation Method

In our proposed method, we considered the φ function φ_4, and (1) is equivalent to solve the problem
$F : R^n \to R^m$ and $f : R^n \times R^m_+ \to R$ verifies the properties (P1) to (P4). Given $x^0 \in R^n$, $z^0 \in R^m_{++}$ and the sequences $\beta^k, \mu^k > 0, k = 0, 1, ...,$ the square root quadratic proximal point scalarization method generates sequences $\{x^k\} \subset R^n$, $z^k \subset R^m_+$, where x^{k+1} and z^{k+1} solve the problem

$$(x^{k+1}, z^{k+1}) \in argmin \quad \theta^k(x, z), \tag{2}$$

with

$$\theta^k(x, z) = f(x, z) + \beta^k \sum_{i=1}^{m} \left(\sqrt{\frac{z_i}{z_i^k}} - 1\right)^2 + \frac{\mu^k}{2}\|x - x^k\|^2,$$

where $x^k \in \Omega^k, z \in R^m_{++}$ and $\Omega^k = \{x \in R^n | F(x) \leq F(x^k)\}$.
$\frac{z}{z^k}$ and $\sqrt{\frac{z}{z^k}}$ are the vectors whose ith-components are given by $\frac{z_i}{z_i^k}$ and $\sqrt{\frac{z_i}{z_i^k}}$ and $e \in R^m$ is the vector with all components equal to 1.

Lemma 1 (Well-posedness) Let $F : R^n \to R^m$ be a convex function and $f : R^n \times R^m_+ \to R$ a convex function verifying properties $(P1)$ to $(P4)$. Then, for each $k \in N$, there exists only one solution (x^{k+1}, z^{k+1}) for problem (SQPS) problem characterized by

$$\mu^k(x^k - x^{k+1}) \in \{\partial_{x^{k+1}} f(x^{k+1}, z^{k+1})\}, \tag{3}$$

and

$$\frac{1}{z_i^{k+1}}\sqrt{\frac{z_i^{k+1}}{z_i^k}} - \frac{1}{z_i^k} \quad = \quad \frac{h_i(x^{k+1}, z^{k+1})}{\beta^k} \tag{4}$$

$$for \quad i \in \{1, ..., m\}, \quad x^{k+1} \in \Omega^k \quad and \quad z^{k+1} \in R^m_{++}.$$

Proof. The convexity of F implies its continuity and the convexity of Ω^k. It is followed from the continuity of F that Ω^k is closed. Therefore, $\Omega^k \times R^m_+$ is a closed convex set. Taking $t = \frac{z_i}{z_i^k}$, we have that the function $g : R_{++} \to R$, defined by

$$g(t) = (\sqrt{t} - 1)^2$$

is strictly convex ($g''(t) = \frac{1}{2\sqrt{t^3}} > 0$, for every $t > 0$), with its minimum at $g(1) = 0$. We conclude that $g(t) \geq 0$, for every $t > 0$. This implies that the function $\left\langle \left(\sqrt{\frac{z}{z^k}} - e\right)^2, e\right\rangle$ is strictly convex and nonegative. On the other hand,

$$\begin{aligned} f(x,z) \quad &+ \quad \beta\left\langle \left(\sqrt{\frac{z}{z^k}} - e\right)^2, e\right\rangle + \frac{\mu}{2}\|x - x^k\|^2 \\ &= \quad f(x,z) + \beta\left\langle \left(\sqrt{\frac{z}{z^k}} - e\right)^2, e\right\rangle \\ &+ \quad \frac{\mu}{2}(\|x\|^2 - 2\langle x, x^k\rangle + \|x^k\|^2) \\ &\geq \quad \alpha + \frac{\mu}{2}(\|x\| - \|x^k\|)^2 \\ &+ \quad \beta\left(\left\|\left(\sqrt{\frac{z}{z^k}} - e\right)^2\right\|_1\right), \end{aligned}$$

where $\| \bullet \|_1$ is the 1–norm on R^m defined by $\|z\|_1 = \sum_{i=1}^m |z_i|$. The inequality above is given by the Cauchy–Schawrz inequality.

Now, define $\|(x, z)\| = \|x\| + \|z\|$ and suppose that $\|(x, z)\| \to +\infty$. This implies that $\|x\| \to +\infty$ or $\|z\| \to +\infty$. In the first case it is obvious that

$$\begin{aligned} \theta^k(x,z) \quad &= \quad f(x,z) + \beta\left\langle \left(\sqrt{\frac{z}{z^k}} - e\right)^2, e\right\rangle \\ &+ \quad \frac{\mu}{2}\|x - x^k\|^2 \to +\infty. \end{aligned}$$

Suppose that $\|z\| \to +\infty$. Since all norms are equivalent on R^m, without loss of generality, we assume that $\|z\|_1 \to +\infty$. This implies that $|z_l| \to +\infty$ for some $1 \leq l \leq m$.

Since $z_i \geq 0$ for all $i = 1, ..., m$, we have for this indice l that $z_l \to +\infty$.
On the other hand, $\left\|\left(\sqrt{\frac{z}{z^k}} - e\right)^2\right\|_1 = \sum_{i=1}^m \left(\sqrt{\frac{z_i}{z_i^k}} - 1\right)^2$.
Therefore, the part of the sum associated to the component z_l satisfies $\left(\sqrt{\frac{z_l}{z_l^k}} - 1\right)^2 \to +\infty$.

Notice that $g(t)$ is a strictly increasing function for $t > 1$ ($g'(t) > 0 \quad for \quad every \quad t > 1$). This is sufficient to have $\theta^k(x, z) \to +\infty$ when $\|z\|_1 \to +\infty$. Therefore, θ^k is coercive. We also have that θ^k is a sum of a convex function f with two strictly convex regularization. This implies that θ^k is strictly convex.
Then, the first part of the lemma follows. Take the optimality conditions for the square quadratic proximal scalarization problem and we have the relation (3) and the Eq. (4).

To finish this proof we must show that
$z_i^{k+1} > 0, i = 1, ..., m$. By induction, $z^0 > 0$ by the initialization of the method. Let $z^k > 0$. Then, by Eq. (4) we obtain that

$$z_i^{k+1} = \frac{1}{\frac{h_i(x^{k+1}, z^{k+1})}{\beta^k} + \frac{1}{z_i^k}}\sqrt{\frac{z_i^{k+1}}{z_i^k}},$$

since h satisfies (P4), we conclude that

$$z_i^{k+1} > 0, \forall i = 1, ..., m.$$

Note that each function h_i is not necessarily separable, i. e., we do not need to have $h_i(x, z) = \xi(x) + \eta(z)$ as we are going to see in the following.

In our work we can establish the same stopping rule as in Bonnel et al. [4], i.e., if

$$(x^{k+1}, z^{k+1}) = (x^k, z^k), \tag{5}$$

then x^k is a weak pareto solution for the unconstrained multiobjective optimization problem (1).

Now, we can prove the convergence of our method if the stopping rule never applies.

Theorem 1 (Convergence) Let $F : R^n \to R^m$ be a convex map and $f : R^n \times R^m_+ \to R$ be a function verifying the properties (P1) to (P4). Suppose that $\Omega^0 = \{x \in R^n | F(x) \leq F(x^0)\}$ is bounded. If $\{\mu^k\}_{k\in N}$ and $\{\beta^k\}_{k\in N}$ are sequences of real positive numbers, with $\{\mu^k\}_{k\in N}$ bounded, then the sequence $\{(x^k, z^k)\}_{k\in N}$ generated by the square root quadratic proximal point scalarization method is bounded and each cluster point of $\{x^k\}_{k\in N}$ is a weak pareto solution for the unconstrained multiobjective optimization problem.

Proof. Similar to [[12], Theorem 1].

Proposition 3 (Stop criterion) Let $\{(x^k, z^k)\}_{k\in N}$ be the sequence generated by the (SQPS) method. If $(x^{k+1}, z^{k+1}) = (x^k, z^k)$ for any integer k then x^k is a weak pareto solution for the unconstrained multiobjective optimization problem (1).

Proof. Similar to [[12], Proposition 3].

IV. Conclusion

The contribution of this manuscript is twofold.

First, for solving multiobjective programming problems, the researchers adopted just two divergence functions in the regularized subproblem by using a logarithmic term or the inverse barrier term.
For this reason, we thought to propose a new modified proximal point method based on scalar representation and over a new square root quadratic term as regularization.

Second, this manuscript demonstrates theoretically that the convergence can be proved under mild assumptions.

The suggested work offers :

1. A basic and reference article for all future works that seeks to develop our approach.
2. A new direction for researchers in their scientific research concerning the development of new alternative and concurrent methods to solve multiobjective programming problems.

In summary, we proposed a new class of scalarization algorithms for solving multiobjective programming problems using a square root quadratic proximal term. This work revolves around convergence and the basic theory of the new method. The upcoming research will focus on the numerical results portion and comparison with other similar recent algorithms.

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